SINGULARITIES IN POSITIVE CHARACTERISTIC II

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Outline

OUTLINE

Theorem (Stage A)

If $\tau_{\mathcal{G}} \geq e$ there is a well defined sequence of permissible transformations:

$$\begin{array}{ccccc}
(V^{(d)}, \mathcal{G}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d)}, \mathcal{G}_r) \\
\downarrow \beta & & & & \downarrow \beta_r \\
(V^{(d-e)}, \mathcal{G}^{(d-e)}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d-e)}, \mathcal{G}_r^{(d-e)})
\end{array}$$

such that $\mathrm{Sing}(\mathcal{G}_r) = \emptyset$ or $\mathcal{G}_r^{(d-e)}$ is monomial:

$$\mathcal{G}_r^{(d-e)} \sim \mathcal{O}_{V_r^{(e)}}[(I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r})W^s].$$

$\beta : V^{(d)} \longrightarrow V^{(d-\tau)}$ smooth locally at x, $\mathcal{G}$, $\tau_{\mathcal{G},x} \geq \tau$.
Assume β is a composition of smooth morphisms

$$V^{(d)} \longrightarrow V^{(d-1)} \longrightarrow \dots \longrightarrow V^{(d-\tau)}$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$\mathrm{Sing}(\mathcal{G}) = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

$$X = V(f_b),\ F_b \text{ the set of } b\text{-fold points of } X$$

Define

$$\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_b W^b] \text{ and } F_b = \mathrm{Sing}(\mathcal{G}),$$

set $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ generic and Z transversal to β, then $f_b = f_b(Z)$ monic,

$$\mathcal{G}' = \mathcal{O}_{V^{(d)}}[f_b(Z)W^b, \Delta^\alpha(f_b(Z))W^{b-\alpha}]_{1\leq\alpha\leq b-1}$$

$$\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(\mathcal{G}').$$

$\mathcal{G}'$ is relative differential. Then, there exists an elimination algebra, $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d-1)}}[W]$ and

$$\beta^*(\mathcal{R}_{\mathcal{G},\beta}) \subset \mathcal{G}'.$$

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UNIVERSAL ELIMINATION ALGEBRA

$$F_n(Z) = (Z - Y_1)\dots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$

$$k[Y_i - Y_j] \subset k[Z - Y_1, \dots, Z - Y_n]$$

$$\begin{array}{ccc} k[Y_i - Y_j]^{S_n} & \subset & k[Z - Y_1, \dots, Z - Y_n]^{S_n} \\ \| & & \| \\ k[H_1, \dots, H_r] & \subset & k[F_n(Z), \Delta^{\alpha}(F_n)]_{1 \leq \alpha \leq n-1} \end{array}$$

$$k[H_1 W^{d_1}, \dots, H_r W^{d_r}] \subset k[F_n(Z) W^n, \Delta^{\alpha}(F_n(Z)) W^{n-\alpha}]_{1 \leq \alpha \leq n-1}$$

Universal elimination algebra

$$F_n(Z) = (Z - Y_1)\dots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$

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$$\boxed{k[H_1 W^{d_1}, \dots, H_r W^{d_r}] \subset k[F_n(Z)W^n, \Delta^{\alpha}(F_n(Z))W^{n-\alpha}]_{1 \leq \alpha \leq n-1}}$$

SPECIALIZATION FOR ONE POLYNOMIAL

$$\boxed{k[H_1 W^{d_1}, \dots, H_r W^{d_r}] \subset k[F_n(Z)W^n, \Delta^{\alpha}(F_n(Z))W^{n-\alpha}]_{1\leq\alpha\leq n-1}}$$

Fix $A \xrightarrow{\beta^*} A[Z]$ and

$$f_n(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n \in A[Z].$$

By specialization

$$\begin{array}{ccc} \mathcal{G} = A[Z][f_n(Z)W^n, \Delta^{\alpha}(f_n(Z))W^{n-\alpha}]_{1\leq\alpha\leq n-1} & \subset & A[Z][W] \\ \cup & & \cup \\ \mathcal{R}_{\mathcal{G},\beta} & \subset & A[W] \end{array}$$

Specialization for several polynomials

$F_{n_1}(Z), \dots, F_{n_s}(Z)$ universal polynomials.

$$\boxed{k[H_1 W^{d_1}, \dots, H_{r_s} W^{d_{r_s}}] \subset k[F_{n_i}(Z) W^n, \Delta^{\alpha}(F_{n_i}(Z)) W^{n_i-\alpha}]_{1 \leq \alpha \leq n_i - 1}}$$

Fix $A \stackrel{\beta^*}{\longrightarrow} A[Z]$ and

$$f_{n_i}(Z) = Z^{n_i} + a_1^i Z^{n_i-1} + \cdots + a_{n_i}^i \in A[Z] \quad 1 \leq i \leq s$$

There is a universal algebra for s polynomials which specializes to

$$\mathcal{G} = A[f_{n_i} W^{n_i}, \Delta^{\alpha_i}(f_{n_i}) W^{n_i - \alpha_i}]_{1 \leq \alpha_i \leq n_i - 1,\ 1 \leq i \leq s}$$

and a universal elimination algebra which specializes to

$$\beta^*(\mathcal{R}_{\mathcal{G},\beta}) \subset \mathcal{G} \text{ (free of Z).}$$

Fix $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ and $\mathcal{G}$ relative differential to β so that $\tau_{\mathcal{G}} \geq 1$, then

- $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{n_1}W^{n_1}, \dots, f_{n_s}W^{n_s}]$ with $\operatorname{ord}(f_{n_i}) = n_i$.
- $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{n_i}W^{n_i}, \Delta^{\alpha_i}(f_{n_i})W^{n_i-\alpha_i}]_{1\leq i\leq s,\ 1\leq\alpha_i\leq n_i-1}$.
- There is an inclusion $\beta^*(\mathcal{R}_{\mathcal{G},\beta}) \subset \mathcal{G}$.

Theorem

There exists a local relative presentation

$$\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{\mathbf{n_1}}W^{\mathbf{n_1}}, \Delta^{\alpha}(f_{\mathbf{n_1}})W^{\mathbf{n_1}-\alpha}]_{1\leq\alpha\leq\mathbf{n_1}-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta})$$

Remark

The smallest n_1 is of the form p^e for $e \geq 0$

Theorem

$$\tau_{\mathcal{R}_{\mathcal{G},\beta}} = \tau_{\mathcal{G}} - 1 \quad (\mathcal{G} \textit{ differential})$$

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Local presentation

Fix $x \in \text{Sing}(\mathcal{G})$ and $\tau_{\mathcal{G},x} \geq r$, $V^{(d)} \xrightarrow{\beta} V^{(d-r)}$ smooth and generic. There is a factorization

$$\begin{array}{ccccccccc} V^{(d)} & \xrightarrow{\beta_1} & V^{(d-1)} & \longrightarrow & \dots & \longrightarrow & V^{(d-r+1)} & \xrightarrow{\beta_r} & V^{(d-r)} \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d-1)} & & & & \mathcal{G}^{(d-r+1)} & & \mathcal{G}^{(d-r)} \\ f_0 W^{p^{e_1}} & & f_1 W^{p^{e_2}} & & & & f_{r-1} W^{p^{e_r}} & & \end{array}$$

- $\mathcal{G}^{(d-i)} \sim \mathcal{O}_{V^{(d-i)}}[f_i W^{p^{e_i}}, \Delta^{\alpha_i}(f_i) W^{p^{e_i}-\alpha_i}]_{1 \leq \alpha_i \leq p^{e_i}-1} \odot \beta_i^*(\mathcal{G}^{(d-i-1)})$.
- $\mathcal{G}^{(d)} \sim \mathcal{O}_{V^{(d)}}[f_i W^{p^{e_i}}, \Delta^{\alpha_i}(f_i) W^{p^{e_i}-\alpha_i}]_{1 \leq \alpha_i \leq p^{e_i}-1, 1 \leq, i \leq r} \odot \beta^*(\mathcal{G}^{(d-r)})$, with $0 \leq e_1 \leq e_2 \leq \dots \leq e_r$.
- $\text{Sing}(\mathcal{G}^{(d)}) = \text{Sing}(\mathcal{O}_{V^{(d)}}[f_i W^{p^{e_i}}]_{1 \leq i \leq r} \odot \beta^*(\mathcal{G}^{(d-r)}))$.

Local presentation

Fix $x \in \mathrm{Sing}(\mathcal{G})$ and $\tau_{\mathcal{G},x} \geq r$, $V^{(d)} \xrightarrow{\beta} V^{(d-r)}$ smooth and generic. There is a factorization

$$\begin{array}{ccccccccc} V^{(d)} & \xrightarrow{\beta_1} & V^{(d-1)} & \longrightarrow & \dots & \longrightarrow & V^{(d-r+1)} & \xrightarrow{\beta_r} & V^{(d-r)} \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d-1)} & & & & \mathcal{G}^{(d-r+1)} & & \mathcal{G}^{(d-r)} \\ f_0 W^{p^{e_1}} & & f_1 W^{p^{e_2}} & & & & f_{r-1} W^{p^{e_r}} & & \end{array}$$

- $\mathcal{G}^{(d-i)} \sim \mathcal{O}_{V^{(d-i)}}[f_i W^{p^{e_i}}, \Delta^{\alpha_i}(f_i) W^{p^{e_i}-\alpha_i}]_{1 \leq \alpha_i \leq p^{e_i}-1} \odot \beta_i^*(\mathcal{G}^{(d-i-1)})$.
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LOCAL PRESENTATION

Fix $x \in \mathrm{Sing}(\mathcal{G})$ and $\tau_{\mathcal{G},x} \geq r$, $V^{(d)} \xrightarrow{\beta} V^{(d-r)}$ smooth and generic. There is a factorization

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- $\mathcal{G}^{(d-i)} \sim \mathcal{O}_{V^{(d-i)}}[f_i W^{p^{e_i}}, \Delta^{\alpha_i}(f_i) W^{p^{e_i}-\alpha_i}]_{1 \leq \alpha_i \leq p^{e_i}-1} \odot \beta_i^*(\mathcal{G}^{(d-i-1)})$.
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- $\mathcal{G}^{(d)} \sim \mathcal{O}_{V^{(d)}}[f_i W^{p^{e_i}}, \Delta^{\alpha_i}(f_i) W^{p^{e_i}-\alpha_i}]_{1 \leq \alpha_i \leq p^{e_i}-1\ ,1 \leq, i \leq r} \odot \beta^*(\mathcal{G}^{(d-r)})$, with $0 \leq e_1 \leq e_2 \leq \dots \leq e_r$.
- $\mathrm{Sing}(\mathcal{G}^{(d)}) = \mathrm{Sing}(\mathcal{O}_{V^{(d)}}[f_i W^{p^{e_i}}]_{1 \leq i \leq r} \odot \beta^*(\mathcal{G}^{(d-r)}))$.

$\beta : V^{(d)} \longrightarrow V^{(d-\tau)}$ smooth locally at x
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$$\mathrm{Sing}(\mathcal{G}) = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

Local relative presentation and Permissible transformations

There is a commutative diagram

$$\begin{array}{ccccc} \mathcal{G} & V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \mathcal{G}_1 \\ & \downarrow \beta & & \downarrow \beta_1 & \\ \mathcal{R}_{\mathcal{G},\beta} & V^{(d-1)} & \xleftarrow{\pi_1'} & V_1^{(d-1)} & (\mathcal{R}_{\mathcal{G},\beta})_1 \end{array}$$

$$\text{If } \mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_n W^n, \Delta^{\alpha}(f_n) W^n] \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}),$$

$$\text{then } \mathcal{G}_1 \sim \mathcal{O}_{V_1^{(d)}}[f_n^{(1)} W^n, \Delta^{\alpha}(f_n^{(1)}) W^n] \odot \beta_1^*((\mathcal{R}_{\mathcal{G},\beta})_1) \qquad (f_n^{(1)} \text{ s.t.})$$

STABILITY OF LOCAL RELATIVE PRESENTATION

OUTLINE

THE MONOMIAL CASE

A. Benito and O. Villamayor, 'Monoidal transformations of singularities in positive characteristic'
http://arXiv.org/abs/0811.4148 26 November 2008.

V smooth, $E = \{H_1, \dots, H_r\}$ smooth hypersurfaces with n.c.

- Monomial ideal supported on E:

$$\mathcal{M} = I(H_1)^{\alpha_1} \cdot I(H_2)^{\alpha_2} \cdots I(H_r)^{\alpha_r}.$$

- Monomial algebra:

Rees algebra $\mathcal{O}_V[\mathcal{M} W^s]$ for $s \in \mathbb{Z}_{>0}$.

Theorem (Stage A)

Assume $\tau_{\mathcal{G}} \geq 1$There is a sequence of permissible transformations

$$
\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_{C_1}} & V_1^{(d)} & \longleftarrow & \cdots & \xleftarrow{\pi_{C_r}} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_{\beta(C_1)}} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi'_{\beta(C_r)}} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}
$$

$$\operatorname{Sing}(\mathcal{G}_r) = \emptyset \text{ or } (\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r} W^s$$

- $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[fW^{p^e}, \Delta^{\alpha}(f)W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta})$
- $\mathcal{G}_i \sim \mathcal{O}_{V_i^{(d)}}[f^{(i)}W^{p^e}, \Delta^{\alpha}(f^{(i)})W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)$

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\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_{\beta(C_1)}} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi'_{\beta(C_r)}} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}$$

$$\mathrm{Sing}(\mathcal{G}_r) = \emptyset \text{ or } (\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s$$

- $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[fW^{p^e}, \Delta^{\alpha}(f)W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta})$
- $\mathcal{G}_i \sim \mathcal{O}_{V_i^{(d)}}[f^{(i)}W^{p^e}, \Delta^{\alpha}(f^{(i)})W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)$

THEOREM (STAGE A)

Assume $\tau_{\mathcal{G}} \geq 1$There is a sequence of permissible transformations

$$
\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_{C_1}} & V_1^{(d)} & \longleftarrow & \cdots & \xleftarrow{\pi_{C_r}} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_{\beta(C_1)}} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi'_{\beta(C_r)}} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}
$$

$$\mathrm{Sing}(\mathcal{G}_r) = \emptyset \ \text{ or } \ (\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s$$

- $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[fW^{p^e}, \Delta^{\alpha}(f)W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta})$
- $\mathcal{G}_i \sim \mathcal{O}_{V_i^{(d)}}[f^{(i)}W^{p^e}, \Delta^{\alpha}(f^{(i)})W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)$

DEFINITION

$V^{(d)}$ smooth, $E = \{H_1, \ldots, H_r\}$ smooth hypersurfaces with n.c.
$\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$, $x \in \mathrm{Sing}(\mathcal{G})$, $\tau_{\mathcal{G},x} \geq 1$, $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ generic.
$\mathcal{O}_{V^{(d-1)}}[\mathcal{M}W^s]$ has monomial contact with $\mathcal{G}$ if locally there is a smooth section defined by $z \in \mathcal{O}_{V^{(d)},x}$ so that

$$\mathcal{G} \subset \langle z \rangle W \odot \mathcal{M}W^s.$$

Monomial contact in local coordinates

$$\mathcal{G} \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

There is a r.s.p. $\{y_1, \ldots, y_{d-1}\}$ in $\mathcal{O}_{V^{(d-1)},\beta(x)}$, then z is such that:

(I) $\{z, y_1, \ldots, y_{d-1}\}$ is a r.s.p. at $\mathcal{O}_{V^{(d)},x}$.

(II) $\mathcal{G} \subset \langle z \rangle W \odot \langle y_1^{h_1} \ldots y_j^{h_j} \rangle W^s$ (locally at x), where $y_1^{h_1} \ldots y_j^{h_j}$ generates the monomial ideal $\mathcal{M}$ at $\mathcal{O}_{V^{(d-1)},\beta(x)}$.

$$zW \text{ ``}\in\text{''} \mathcal{G} \odot \mathcal{M} W^s$$

LEMMA

Fix $\mathcal{G} \subset \mathcal{O}_V[W]$. If $\tau_{\mathcal{G}} \geq 1$ and codimension of $\mathrm{Sing}(\mathcal{G})$ is 1 in V, then there exists $Z(\subset V)$ smooth hypersurface so that

$$\mathcal{G} \sim \mathcal{O}_V[I(Z)W].$$

Set

$$\mathcal{G}_i\,|_{H_i^{(d)}} := \mathcal{O}_{H_i^{(d)}}[\overline{f^{(i)}}W^{p^e}, \overline{\Delta^{\alpha}(f^{(i)})}W^{p^e-\alpha}]_{1\leq\alpha\leq p^e-1} \odot \overline{\beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)}$$

We say that $H_i^{(d)}$ satisfies condition (CD) if

(CD) $\quad \mathrm{Sing}(\mathcal{G}_i|_{H_i^{(d)}})$ is of pure codimension 1 in $H_i^{(d)}(\subset V_i^{(d)})$.

Let $H_i'^{(d)} \subset V_r^{(d)}$, then $H_i^{(d)}$ satisfies (CD) iff $\mathrm{Sing}(\mathcal{G}_r|_{H_i'^{(d)}})$ is of pure codimension one in $H_i'^{(d)} \subset V_r^{(d)}$

LEMMA

Fix $\mathcal{G} \subset \mathcal{O}_V[W]$. If $\tau_{\mathcal{G}} \geq 1$ and codimension of $\mathrm{Sing}(\mathcal{G})$ is 1 in V, then there exists $Z(\subset V)$ smooth hypersurface so that

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$$\mathcal{G}_i \mid_{H_i^{(d)}} := \mathcal{O}_{H_i^{(d)}}[\overline{f^{(i)}}W^{p^e}, \overline{\Delta^\alpha(f^{(i)})}W^{p^e-\alpha}]_{1\leq\alpha\leq p^e-1} \odot \overline{\beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)}$$

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Lemma

Fix $\mathcal{G} \subset \mathcal{O}_V[W]$. If $\tau_{\mathcal{G}} \geq 1$ and codimension of $\mathrm{Sing}(\mathcal{G})$ is 1 in V, then there exists $Z(\subset V)$ smooth hypersurface so that

$$\mathcal{G} \sim \mathcal{O}_V[I(Z)W].$$

Set

$$\mathcal{G}_i \mid_{H_i^{(d)}} := \mathcal{O}_{H_i^{(d)}}[\overline{f^{(i)}}W^{p^e}, \overline{\Delta^{\alpha}(f^{(i)})}W^{p^e-\alpha}]_{1\leq\alpha\leq p^e-1} \odot \overline{\beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)}$$

We say that $H_i^{(d)}$ satisfies condition (CD) if

(CD) $\mathrm{Sing}(\mathcal{G}_i|_{H_i^{(d)}})$ is of pure codimension 1 in $H_i^{(d)}(\subset V_i^{(d)})$.

Let $H_i'^{(d)} \subset V_r^{(d)}$, then $H_i^{(d)}$ satisfies (CD) iff $\mathrm{Sing}(\mathcal{G}_r|_{H_i'^{(d)}})$ is of pure codimension one in $H_i'^{(d)} \subset V_r^{(d)}$

LEMMA

Fix $\mathcal{G} \subset \mathcal{O}_V[W]$. If $\tau_{\mathcal{G}} \geq 1$ and codimension of $\mathrm{Sing}(\mathcal{G})$ is 1 in V, then there exists $Z(\subset V)$ smooth hypersurface so that

$$\mathcal{G} \sim \mathcal{O}_V[I(Z)W].$$

Set

$$\mathcal{G}_i \mid_{H_i^{(d)}} := \mathcal{O}_{H_i^{(d)}}[\overline{f^{(i)}} W^{p^e}, \overline{\Delta^{\alpha}(f^{(i)})} W^{p^e-\alpha}]_{1\leq\alpha\leq p^e-1} \odot \overline{\beta_i^*((\mathcal{R}_{\mathcal{G},\beta})_i)}$$

We say that $H_i^{(d)}$ satisfies condition (CD) if

(CD) $\quad \mathrm{Sing}(\mathcal{G}_i|_{H_i^{(d)}})$ is of pure codimension 1 in $H_i^{(d)}(\subset V_i^{(d)})$.

Let $H_i'^{(d)} \subset V_r^{(d)}$, then $H_i^{(d)}$ satisfies (CD) iff $\mathrm{Sing}(\mathcal{G}_r|_{H_i'^{(d)}})$ is of pure codimension one in $H_i'^{(d)} \subset V_r^{(d)}$

Proposition

Condition (CD) holds for H_i.

$$\Updownarrow$$

The τ-invariant of $\mathcal{G}_{i-1}$ is 1 along closed points of C_i.

$$\Updownarrow$$

$\overline{f^{(i)}}$ is a p^e-th power of a regular parameter.

Outline

Main Theorem: Stage B'

Set as in Theorem Stage A:

$$
\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow & \cdots & \xleftarrow{\pi_r} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi_1'} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi_r'} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}
$$

$(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s = \mathcal{N} W^s$ (monomial algebra).

THEOREM (STAGE B')

There is a monomial $\mathcal{M}$ of "tight" contact s.t.

(i) $\mathcal{M} = I(H_1)^{h_1} \dots I(H_r)^{h_r}$ and $0 \leq h_i \leq \alpha_i$ for $i = 1, \dots, r$.

(ii) $\mathcal{M} W^s$ has monomial contact relative to β_r with $\mathcal{G}_r$ locally at any closed point in $\mathrm{Sing}(\mathcal{G}_r)$, i.e.,

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

Recall

$$\mathcal{G}_i \sim \mathcal{O}_{V_i^{(d)}}[f_{p^e}^{(i)} W^{p^e}, \Delta^{\alpha}(f_{p^e}^{(i)}) W^{p^e-\alpha}]_{1\leq\alpha\leq p^e-1} \odot \mathcal{R}_{\mathcal{G},\beta}$$

If $\mathcal{G}_i \subset \langle z \rangle W \odot (y_1^{h_1} \dots y_i^{h_i}) W^s$ $(h_j \leq \alpha_j)$, then

$$f_{p^e}^{(i)} = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e}$$

with $a_j W^j \in \langle y_1^{h_1} \dots y_i^{h_i} \rangle W^s$.

$$h_\ell \geq 1 \iff f_{p^e}^{(i)}|_{y_\ell=0} = z^{p^e} \iff H_\ell = \{y_\ell = 0\} \text{ satisfies (CD)}$$

OUTLINE

Fix $x \in \mathrm{Sing}(\mathcal{G})$ a center $I(C) = \langle z, y_1, \dots, y_r \rangle$ and $f = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z]$ from a local relative presentation,

$$\begin{array}{ccccc} \mathcal{G} & V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \mathcal{G}_1 \\ & \downarrow \beta & & \downarrow \beta_1 & \\ \mathcal{R}_{\mathcal{G},\beta} & V^{(d-1)} & \xleftarrow{\pi_1'} & V_1^{(d-1)} & (\mathcal{R}_{\mathcal{G},\beta})_1 \end{array}$$

$x' \in H_1^{(d)} = \{y_1 = 0\}$ mapping to x and $f_1 = z_1^{p^e} + a_1^1 z_1^{p^e-1} + \cdots + a_{p^e}^1 \in \mathcal{O}_{V_1^{(d-1)}}[z_1]$ s.t. of f.
May assume $\frac{z}{y_1}$ non-invertible at x'. For a suitable z_1, $\overline{a_i} = 0$ (condition (CD) holds). So

$$\mathcal{G}_1 \subset \langle z_1 \rangle W \odot \langle y_1 \rangle W^s.$$

Define h_1: highest so that $\mathcal{G}_1 \subset \langle z_1 \rangle W \odot \langle y_1^{h_1} \rangle W^s$ for all x'.

Outline

Proposition

Given $x \in \mathrm{Sing}(\mathcal{G}_i) \subset V_i^{(d)}$ and $\mathcal{G}_i \subset \langle z \rangle W \odot \left(y_{j_1}^{h_{j_1}} \dots y_{j_\ell}^{h_{j_\ell}} \right) W^s$, if C_i is a permissible center so that $\tau = 1$ along C_i.
At $x' \in \mathrm{Sing}(\mathcal{G}_{i+1}) \cap H_{i+1}$ there is an expression

$$\mathcal{G}_{i+1} \subset \langle z_1 \rangle W \odot \left(\left(\frac{y_{j_1}}{y_{i+1}} \right)^{h_{j_1}} \dots \left(\frac{y_{j_\ell}}{y_{i+1}} \right)^{h_{j_\ell}} y_{i+1}^{r_{i+1}} \right) W^s$$

so that y_{i+1} defines H_{i+1} and $r_{i+1} \geq 1$

Definition

Define h_{i+1}: the highest r_{i+1} so that the previous inclusion holds.

$$\mathcal{G}_i \subset \langle z \rangle W \odot \left(y_{j_1}^{h_{j_1}} \dots y_{j_\ell}^{h_{j_\ell}} \right) W^s \quad (*)$$

$$\mathcal{G}_{i+1} \subset \langle z_1 \rangle W \odot \left(y_{j_1}^{h_{j_1}} \dots y_{j_\ell}^{h_{j_\ell}} y_{i+1}^{r_{i+1}} \right) W^s$$

KEY POINT

We can assume that

- $\langle z \rangle \subset I(C_i)$ in $(*)$.
- $\mathrm{In}_x(\langle f_{p^e}^{(i)} \rangle)$ is a p^e-th power of $\mathrm{In}_x(z)$ at $gr_x(\mathcal{O}_{V_i^{(d)},x})$.

Notice that

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

implies that

$$\mathrm{Sing}(\langle z \rangle W \odot \mathcal{M} W^s) \subset \mathrm{Sing}(\mathcal{G}_r).$$

A resolution of $\langle z \rangle W \odot \mathcal{M} W^s$,

$$V_r^{(d)} \longleftarrow V_{r+1}^{(d)} \longleftarrow \cdots \longleftarrow V_R^{(d)}$$

induces a sequence of transformations of $\mathcal{G}_r$,

$$\begin{array}{ccccccc} \mathcal{G}_r & & \mathcal{G}_{r+1} & & & & \mathcal{G}_R \\ V_r^{(d)} & \longleftarrow & V_{r+1}^{(d)} & \longleftarrow & \cdots & \longleftarrow & V_R^{(d)}. \end{array}$$

Notice that

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

implies that

$$\operatorname{Sing}(\langle z \rangle W \odot \mathcal{M} W^s) \subset \operatorname{Sing}(\mathcal{G}_r).$$

A resolution of $\langle z \rangle W \odot \mathcal{M} W^s$,

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induces a sequence of transformations of $\mathcal{G}_r$,

$$\begin{array}{ccccccc} \mathcal{G}_r & & \mathcal{G}_{r+1} & & & & \mathcal{G}_R \\ V_r^{(d)} & \longleftarrow & V_{r+1}^{(d)} & \longleftarrow & \cdots & \longleftarrow & V_R^{(d)}. \end{array}$$

OUTLINE

Example

$$V^{(4)}$$

$$\mathcal{G}: \quad (T^2 + XYZ)W^2$$

$$\mathcal{R}_{\mathcal{G}}: \ (XY, XZ, YZ)W^1$$

Example

$$V^{(4)} \xleftarrow{\pi_{\overline{0}}} V_1^{(4)} \supset U_X$$

$$\mathcal{G}: \quad (T^2 + XYZ)W^2 \qquad (T^2 + XYZ)W^2$$

$$\mathcal{R}_{\mathcal{G}}: \; (XY, XZ, YZ)W^1 \qquad (XY, XZ)W^1$$

Example

$$\begin{array}{cccccc} & V^{(4)} & \xleftarrow{\pi_{\overline{0}}} & V_1^{(4)} \supset U_X & \xleftarrow{\pi_{\overline{0}}} & V_2^{(4)} \supset U_Y \\ \mathcal{G}: & (T^2+XYZ)W^2 & & (T^2+XYZ)W^2 & & (T^2+XYZ)W^2 \\ \mathcal{R}_{\mathcal{G}}: & (XY, XZ, YZ)W^1 & & (XY, XZ)W^1 & & (XY)W^1 \end{array}$$

MONOMIAL CASE

Example

$$V^{(4)} \xleftarrow{\pi_{\overline{0}}} V_1^{(4)} \supset U_X \xleftarrow{\pi_{\overline{0}}} V_2^{(4)} \supset U_Y$$

$$\mathcal{G}: \quad (T^2+XYZ)W^2 \qquad (T^2+XYZ)W^2 \qquad (T^2+XYZ)W^2$$

$$\mathcal{R}_{\mathcal{G}}: \quad (XY, XZ, YZ)W^1 \qquad (XY, XZ)W^1 \qquad (XY)W^1$$

$$\mathcal{G}_1 \subset \langle T\rangle W \odot I(H_1)W^2$$

$$V^{(4)} \xleftarrow{\pi_{\overline{0}}} V_1^{(4)} \supset U_X \xleftarrow{\pi_{\overline{0}}} V_2^{(4)} \supset U_Y$$

$$\mathcal{G}: \quad (T^2 + XYZ)W^2 \qquad (T^2 + XYZ)W^2 \qquad (T^2 + XYZ)W^2$$

$$\mathcal{R}_{\mathcal{G}}: \quad (XY, XZ, YZ)W^1 \qquad (XY, XZ)W^1 \qquad (XY)W^1$$

$$\mathcal{G}_1 \subset \langle T \rangle W \odot I(H_1)W^2$$

$$\mathcal{G}_2 \subset \langle T \rangle W \odot I(H_1)I(H_2)W^2$$

Example

$$\begin{array}{lccc} & V^{(4)} \xleftarrow{\pi_{\overline{0}}} & V_1^{(4)} \supset U_X \xleftarrow{\pi_{\overline{0}}} & V_2^{(4)} \supset U_Y \\ \mathcal{G}: & (T^2+XYZ)W^2 & (T^2+XYZ)W^2 & (T^2+XYZ)W^2 \\ \mathcal{R}_{\mathcal{G}}: & (XY,XZ,YZ)W^1 & (XY,XZ)W^1 & (XY)W^1 \end{array}$$

$$\mathcal{G}_1 \subset \langle T\rangle W\odot \subset I(H_1)W^2$$

$$\mathcal{G}_2 \subset \langle T\rangle W\odot \subset I(H_1)I(H_2)W^2$$

TIGHT MONOMIAL ALGEBRA: $\mathcal{M}W^2 = I(H_1)I(H_2)W^2$

A resolution of $\mathcal{M}W^2$ is achieved by a blow-up at

$$I(C') = \langle X, Y \rangle \subset \mathcal{O}_{V^{(3)}}.$$

This induces a blow-up at $I(C) = \langle T, X, Y \rangle \subset \mathcal{O}_{V^{(4)}}$, i.e.,

$$\begin{array}{lccc}
 & V_2^{(4)} & \xleftarrow{\pi_C} & V_3^{(4)} \supset U_X \\
\mathcal{G}: & (T^2 + XYZ)W^2 & & (T^2 + XY)W^2 \\
\mathcal{R}_{\mathcal{G}}: & (XY)W^1 & & (XY)W^1 \\
 & & & \mathcal{M}_1 W^2 = YW^2
\end{array}$$

and the τ-invariant of $\mathcal{G}_3$ has increased.