RESOLUTION OF SINGULARITIES OF THREEFOLDS IN POSITIVE CHARACTERISTIC
A fiew obstructions to embedded resolution.

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INTRODUCTION

The purpose of this conference is to give some hints about the proof of the theorem cited below and to show on two examples that the notion of maximal contect fails completely in the case of the positive characteristic.

Theorem. (Cossart et Piltant) [CP1,CP2]. *Let k be a field of positive characteristic which is differentially finite over a perfect field k_0 and Z/k be a reduced quasiprojective scheme of dimension three with singular locus Σ. There exists a projective morphism $\pi: \tilde{Z} \to Z$, such that*
(i) $\tilde{Z}$ is regular.
(ii) π induces an isomorphim $\tilde{Z}\backslash\pi^{-1}(\Sigma) \simeq Z\backslash\Sigma$.
(iii) $\pi^{-1}(\Sigma) \subset \tilde{Z}$ is a divisor with strict normal crossings.

I Main reduction.

In [CP1], we prove that the proof may be reduced to the proof of this theorem below.

Theorem. *Let k be a field of positive characteristic which is differentially finite over a perfect field k_0, i.e. Ω^1_{k/k_0} has finite dimension.*

Let S be a regular local ring of dimension three, essentially of finite type over k and such that $K := QF(S)$ has transcendance degree 3 over k. Let $\bar{R}$ be an Artin-Schreier or purely inseparable singularity of dimension three over S.

Let $K := QF(S)$ and $L := QF(\bar{R})$ (in particular L/K is a finite field extension).

Then, each k-valuation μ of L dominating $\bar{R}$ and satisfying properties (i) and (ii) below has a local uniformization:
(i) μ has rank one and $\kappa(\mu)/\kappa(S)$ is algebraic;
(ii) μ is the unique extension of its restriction to K.

I.1 Notations.

We let $R := S[X]_{(X,m_S)}$, X_0 =Spec$(R/(h)$, x_0 his closed point, $\mathfrak{M} = (X, m_S)$, and $k(x_0) = R/\mathfrak{M}$ is a finitely generated field extension of k. We denote by (u_1, u_2, u_3) a regular system of parameters (r.s.p. for short) of S, so $\mathfrak{M} = (X, u_1, u_2, u_3)$. We assume all along this text that h is irreducible over $S[X]$, i.e. that f is not of the form $\theta^p - \theta g^{p-1}$ for any $\theta \in S$.

If $g \neq 0$, such a singularity is called "Artin-Schreier", if $g = 0$, it is called "purely inseparable".

I.2 Purely inseparable case.

Let us concentrate on the purely inseparable case which already contains enough difficulties.

To simplify, we take

$$h = X^p + f(u_1, u_2, u_3),\ f \in k[u_1, u_2, u_3].$$

We suppose that there is an exceptional divisor E which contains locally the singular locus of $h = 0$ and such that I(E) divises $u_1u_2u_3$. This can be achieved easily (see [CP1]). We suppose that $\mathrm{ord}_{x_0}(f) \geqslant p$ (else, the singularity is eqasily solved). We define $\mathcal{J}(f, E)$, the ideal generated by the coefficients of $df \in \Omega^1_{S/k_0}(\log E)$. We defined $H(x) = \prod_{div(u_i)\subset E} U_i^{a(i)}$, where $a(i) = \mathrm{ord}_{u_i}(\mathcal{J}(f,E))$, then set

$$J(f,E) := H(x)^{-1}\mathcal{J}(f,E).$$

The main invariant is:

$$\omega(x) := \mathrm{ord}_x(J(f,E))$$

which, obviously does not depend on X.

I.2.1 Important remark. The case $\omega(x_0) = 0$ is easily solved: see [CP2] **II.4.4** to **II.4.7**.

It can be shown that

$$\omega(x_0) = 0 \Leftrightarrow f = \gamma M \mathrm{mod} S^p,\ M = \prod_{div(u_i)\subset E} u_i^{a(i)},$$

with γ invertible or parameter orthogonal to E and, if γ invertible, then M is not a p-power.

Villamayor's example. In his conference, O. Villamayor prosed the following example as a "terminal case".

$$X^2 + u^2 v,\ p = 2,\ E = \mathrm{div}(u).$$

The computations give:

$$\mathcal{J}(E) = (u^2),\ H(x) = u^2,\ J(E) = (1),\ \omega(x_0) = 0.$$

We agree with Orlando: it is a terminal case.

All this leads to:

I.2.2 Second reduction.

Let $W := \mathrm{Spec}S$, find W', some iterated blowing up of W, where $x \in W'$ the center of the restriction of μ in W' verifies:

$$\omega(x) < \omega(x_0).$$

The problem is reduced to a problem of moniomalization modulo p-powers in a smooth variety: the problem is more difficult, but the dimension of the ambient space drops of one.

I.2.3 Permissible centers.

Now we forget X_0 and we work in W, E is a normal crossing divisor in W. We do not write here the definitions [CP2, chapter 1, II.5] of the permissible centers. We just recall that, if Y is permissible at x, them it is smooth at x and locally normal crossing with E, furthermore, **closed points are permissible**. Bad news: **permissible centers are not necessarily contained in the locus where ω is maximal.** We have the following propositions.

I.2.3.1 Proposition. *If Y is irreducible of generic point ξ and permissible at x, then $\omega(\xi) \geqslant \omega(x) - 1$.*

I.2.3.2 Transformation laws. *Let Y be as above and $\pi : X' \longrightarrow X'$ be the blowing up centred at Y. Then $J(f',E')$ is the weak transform of $J(f,E,Y)$ (defined below), where f' is the strict transform of f and E' the total transform of E.*

I.2.3.3 Jacobian adapted to Y.

Let

$$\mathcal{D}(E) := \{D \in \mathrm{Der}_{k_0}\mathcal{O}_W \mid D(I(E)) \subset I(E)\},$$

$$\mathcal{D}(E,Y) := \{D \in \mathcal{D}(E) \mid D(I(Y)) \subset I(Y)\}.$$

Then

$$J(f,E,Y) := H(x)^{-1}(D(f), D \in \mathcal{D}(\mathcal{E},\mathcal{Y}).$$

I.2.3.4 Example.

In the case where $Y = x$ and E=div(u_1u_2), $k_0 = k$, we get

$$\mathcal{D}(E,Y) = (\frac{u_1\partial}{\partial u_1}, \frac{u_2\partial}{\partial u_2}, \mathfrak{M}\frac{\partial}{\partial u_3}).$$

I.2.3.5 Proposition. *With notations as above, if $x' \in W'$ is above x, then*

$$\omega(x') \leqslant \omega(x).$$

II No maximal contact for ω.

It is very well known that there is no maximal contact for the Hilbert-samuel function in characteristic $p > 0$ (see section **III**).

Optimist people may think that there may be maximal contact for the couple (HS, ω) where HS is the Hilbert-Samuel function. This is wrong. The following example shows that there cannot exist in X a surface Σ such that the Hilbert-Samuel function the strict transforms of Σ is constant at the points x_i above x with $(HS(x_0), \omega(x_0)) = (HS(x_i), \omega(x_i))$, here $HS(x)$ is just the local multiplicity which is equal to p.

II.1. Example in dimension 3. (joint work with O. Piltant).

With the notations of **I.2**. We have a singularity in dimension 3 of equation:

$$h = X^p + u_1^a u_2^b (v^{p^e} + (u_2 - u_1)^{p^e} u_1^{p+2} + extra).$$

$extra \in S = k[u_1, u_2, v]$, of very big order, $E = \mathrm{div}(u_1u_2)$, $a + b = 0 \bmod p$, $ab \neq 0 \bmod p$, $p \neq 2$. We suppose $k = k_0$ algebraically closed.

Computations give:

$$H(x_0) = u_1^a u_2^b,\ J(E) = (v^{p^e}) \bmod \mathfrak{M}^{1+p^e},\ \omega(x_o) = p^e.$$

We blow up along the origin.

We take as new origin the point x_1 of parameters

$$(X/u_1, u_1, w := (u_2 - u_1)/u_1, v/u_1)$$

that we note (X, u_1, w, v), using an usual convention.

We get

$$h_1 = X^p + u_1^{a+b-p}(w+1)^b (v^{p^e} + w^{p^e} u_1^{p+2} + u_1^A extra')$$

where h_1 is the strict transform of h, $A \in \mathbb{N}$, A very big. The reader sees that the set of points x' above x_0 with $(HS(x_0), \omega(x_0)) = (HS(x'), \omega(x'))$ is exactly Proj(X, V), where $V :=$in$_{x_0}(v)$, this implies that, **if Σ exists, its directrix [CJS,1.20] at x_0 is $\mathbf{V}(X, V)$.**

We change one variable: let $Y := X + u_1^{\frac{a+b-p}{p}}$, then

$$h_1 = Y^p + u_1^{a+b-p}(\gamma v^{p^e} w + w^{p^e} u_1^{p+2} + u_1^A extra'),$$

$$E_1 = \mathrm{div}(u_1),\ h(x_1) = u_1^{a+b-p},\ J(E_1) = (v^{p^e}, w^{p^e} u_1^{p+2}) \bmod (u_1^A),\ \omega(x') = p^e.$$

We go on: we look at the sequences of blowing ups centered at closed points on the strict transform of (Y, v, w). We use the usual convention i.e. we note the parameters of x_{i+1} (Y, u_1, v, w) instead of $(Y/u_1^i, u_1, v/u_1^i, w/u_1^i)$.

We get at x_2:

$$h_2 = X^p + u_1^{something}(w+1)^b(v^{p^e} + w^{p^e} u_1 p + 1 + u_1^A extra_2)$$

$$E_2 = \mathrm{div}(u_1),\ h(x_2) = u_1^{something},\ J(E_2) = (v^{p^e}, w^{p^e} u_1^{p+1}) \bmod (u_1^A),\ \omega(x') = p^e.$$

We get at x_{p+3}:

$$h_{p+3} = X^p + u_1^{something}(w+1)^b(v^{p^e} + w^{p^e} + u_1^A extra_3)$$

$$E_{p+3}\mathrm{div}(u_1),\ h(x_3) = u_1^{something},\ J(E_3) = (v^{p^e}, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

Things are looking well: the initial part of $J(E_{p+3})$ is (v^{p^e}, w^{p^e}), the directrix of the ideal $J(E_{p+3})$ has dimension 1: the equations are $v = w = 0$, the dimension was 2 for $J(E_i)$, $i < p+3$, the equation was $v = 0$.

We get at x_{p+4}:

$$h_{p+4} = X^p + u_1^{something}(w+1)^b(v^{p^e} u_1 + w^{p^e} + u_1^A extra_4)$$

$$E_{p+4} = \mathrm{div}(u_1),\ h(x_4) = u_1^{something},\ J(E_4) = (v^{p^e} u_1, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

The initial part of $J(E_{p+4})$ is (w^{p^e}), the directrix of the ideal $J(E_{p+4})$ has dimension 2: the equation is $w = 0$.

We get at x_{p+3+N}:

$$h_{p+3+N} = X^p + u_1^{something}(w+1)^b(v^{p^e} u_1^N + w^{p^e} + u_1^A extra_{p+3+N})$$

$$E_{p+3+N} = \mathrm{div}(u_1),\ h(x_{p+3+N}) = u_1^{something},\ J(E_{p+3+N}) = (v^{p^e} u_1^N, w^{p^e}) \bmod (u_1^A),\ \omega(x') = p^e.$$

The initial part of $J(E_{p+3+N})$ is (w^{p^e}), the directrix of the ideal $J(E_{p+3+N})$ has dimension 2: the equation is $w = 0$.

Let us blow up along x_{p+3+N}: it is easily seen that the set of points x' above x_{p+3+N} with $(HS(x_0), \omega(x_0)) = (HS(x'), \omega(x'))$ is exactly $\mathrm{Proj}(X, W)$, where $W := \mathrm{in}_{x_{p+3+N}}(w)$, this implies that, **if Σ exists, the directrix of its strict transform Σ_{p+3+N} at x_{p+3+N} is $\mathbf{V}(X, W)$.** This is impossible, by [CJS,12.1,12.3,13.3], its ideal should be $(X, V) \bmod (U_1)$.

II.2 Conclusion

This kills the hope to have a maximal contact in "Giraud's sense" for (HS, ω), i.e. to find a surface Σ such that the strict drop of its local Hilbert-Samuel function would imply a strict drop of some invariant of the original singularity.

The question is: define another invariant, finer than (HS, ω) such that, for this invariant, you get a maximal contact in "Giraud's sense"... For the moment, there is no answer.

III No maximal contact along a valuation. (joint work with U. Jannsen and S. Saito).

III.1. Recall of the algorithm, dimension 2, hypersurface case.

We follow Hironaka in [CGO, appendix]. X is a singular surface embedded in a 3-dimensional smooth k-variety Z, k is an algebraic closed field of characteristic $p > 0$. We suppose that the worse HS-stratum is a finite union of closed points, in this case the worse HS-stratum is the locus of maximal multiplicity $\mu(x)$, we call it $\mathrm{HS}(X)$

The algorithm is: blow up the locus of multiplicity $\geqslant \mu(x)$. This will stop.

More precisely, in an open neighbourhood $U \subset X$ of $x \in \mathrm{HS}(X)$, you blow up X, Z along x:

$$X_1 \subset Z_1 \longrightarrow X \subset Z.$$

One can show that there are three different cases [CJS, section 2].
(i) Either there is no point in X_1 near to x (no point with same multiplicity): STOP, the maximum multiplicity dropped strictly above x.
(ii) Either there are a finite number of closed points in X_1 near to x, then above U, blow up X_1, Z_1 along these points:

$$X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1.$$

(iii) Or the set of points in X_1 near to x is a projective line D_1 then above U, blow up X_1, Z_1 along D_1:

$$X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1,$$

either there is no point in X_2 near to x, either there are a finite number of closed points in X_2 near to x or the set of points in X_2 near to x is a projective line D_2 which projects isomorphically on D_1, then above U, blow up X_2, Z_2 along D_2.

In the latter case, above U, the algorithm creates a "fundamental sequence" [CJS, section 5], i.e. a finite sequence of blowing ups

$$X_m \subset Z_m \longrightarrow X_{m-1} \subset Zm-1 \longrightarrow \cdots X_2 \subset Z_2 \longrightarrow X_1 \subset Z_1 \longrightarrow X \subset Z,$$

where the center of the blowing up $X_i \subset Z_i \longrightarrow X_{i-1} \subset Z_{i-1}$ is a projective line D_{i-1} which projects isomorphically on D_1, $2 \leqslant i \leqslant m$ and either there is no point in X_m near to x, or there are a finite number of closed points in X_m near to x.

III.2 Maximal contact along a valuation

The example given in [CJS, section 15] shows that there is no maximal contact in positive characteristic. What is new in this paper is that there is no maximal contact along a valuation. Let us recall the definition of maximal contact [CJS, section 15].

III.2.1 Definition: hypersurface of maximal contact.

Let Z be an excellent regular scheme and $X \subset Z$ be a closed subscheme.

A closed subscheme $W \subset Z$ is called to have maximal contact with X at $x \in X$ if the following conditions are satisfied:
(i) $x \in W$.
(ii) Take any sequence of permissible blowups [...]:

$$\begin{array}{ccccccccccc} Z = & Z_0 & \overset{\pi_1}{\leftarrow} & Z_1 & \overset{\pi_2}{\leftarrow} & Z_2 & \leftarrow \ldots \leftarrow & Z_{n-1} & \overset{\pi_n}{\leftarrow} & Z_n & \leftarrow \cdots \\ & \cup & & \cup & & \cup & & \cup & & \cup & \\ X = & X_0 & \overset{\pi_1}{\leftarrow} & X_1 & \overset{\pi_2}{\leftarrow} & X_2 & \leftarrow \ldots \leftarrow & X_{n-1} & \overset{\pi_n}{\leftarrow} & X_n & \leftarrow \cdots \end{array} \quad (1)$$

where for any $n \geqslant 0$

$$\begin{array}{ccccc} Z_{n+1} & = & \mathrm{Bl}_{D_n}(Z_n) & \stackrel{\pi_{n+1}}{\leftarrow} & Z_n \\ & & \cup & & \cup \\ X_{n+1} & = & \mathrm{Bl}_{D_n}(X_n) & \stackrel{\pi_{n+1}}{\leftarrow} & X_n \end{array} \tag{2}$$

and $D_n \subset X_n$ is permissible. Assume that there exists a sequence of points $x_n \in D_n$ $(n = 0, 1, \dots)$ such that $x_0 = x$ and x_n is near to x_{n-1} for all $n \geqslant 1$. Then $D_n \subset W_n$ for all $n \geqslant 0$, where W_n is strict transform W in Z_n.

Some optimist people asked us:

"Your definition of maximal contact is weaker than Hironaka's, but still too strong, for the uniformization problem, you just need the definition below. Have you an example where there is no maximal contact along a valuation?"

We found one just before Kyoto conference.

III.2.2 Definition: maximal contact along a valuation. *In the definition above suppose X is a projective variety over a field k, let ν be a k-valuation, then on each X_n, ν has a center x_n.*

We say that a closed subscheme $W \subset Z$ has maximal contact with X along ν if, for every sequence (1) where $x_0 = x$ and x_n is near to x_{n-1} for all $n \geqslant 1$, then $x_n \in W_n$ for all $n \geqslant 0$, where W_n is strict transform W in Z_n.

Indeed if for any k-valuation ν there could exist a smooth $W(\nu)$ satisfying **III.2.2**, life would be much easier (as says the guru Abhyankar) in desingularization theory.

III.2.3 The example.

$$p = 3$$

No exeptional divisor E, $E = \varnothing$, take

$$f := y^3 + u_2^2[(u_1^3 - u_2^2)(u_1^3 + u_2^2)^3 + u_1^N],\ N \neq 0 \bmod (p),\ N >> 0.$$

Let us recall some definitions.

III.2.3.1 Hironaka's polyhedrons. [H1] or [CJS, section 7] *In the case where*

$$f = y^m + \sum_{i,a,b,a+b>i,0\leqslant i\leqslant m} \lambda_{i,a,b} y^{m-i} u_1^a u_2^b,\ \lambda_{i,a,b} \in k,$$

$\Delta(f, u, y)$ is the convex hull spanned by $\{(\frac{a}{i}, \frac{b}{i}), \lambda_{i,a,b} \neq 0\} + {\mathbb{R}_{\geqslant 0}}^2$. Hironaka defines $\Delta(f, u)$ as:

$$\Delta(f, u) = \cap_{y, in_{\mathfrak{M}}(f)=Y^m} \Delta(f, u, y).$$

III.2.3.2 Notations. $\delta(f, u, t)$ =inf$\{a + b, (a, b) \in \Delta(f, u, t)\}$, $\delta(f, u)$ =inf$\{a + b, (a, b) \in \Delta(f, u)\}$.

We write sometimes $\delta(x)$ instead of $\delta(f, u)$ where x is the point of parameters (y, u_1, u_2), indeed, one can prove that $\delta(f, u)$ does not depend on (u_1, u_2).

In our example, we get

$$\delta(x) = 3 + 1/3$$

III.2.3.3 $\epsilon(x)$**.** *With the notations and hypotheses of* **III.2.3.1***, assume that there is an exceptional divisor E with components smooth and orthogonal to* $y = 0$*, then assume that*

$$E \subset \operatorname{div}(u_1 u_2),$$

(we say E has "new components"), we define $A_1 = inf\{a | (a,b) \in \Delta(f,u)\}$*, mutatis mutandis, we define* A_2*. Then*

$$\epsilon(x) := \delta(x) - \sum_{\operatorname{div}(u_i) \subset E} A_i.$$

In our example, $E = \varnothing$, we get

$$\epsilon(x) = \delta(x) = 3 + 1/3$$

III.2.3.4 Change of $\epsilon(x)$**.** In the example, to avoid useless denominators, we will replace $\epsilon(x)$ by $3\epsilon(x)$ that we still denote by $\epsilon(x)$. From now on:

$$\epsilon(x) = 10.$$

III.2.3.5. *In the example, x is isolated in the HS-stratum.*

Indeed

$$\frac{\partial f}{\partial u_1} = u_1^{N-1} u_2^2.$$

So, if a curve is contained in the Hs-stratum at the beginning, $u_1^{N-1} u_2^2$ has order at least 2 along this curve which is contained in $\operatorname{div}(u_1)$ or $\operatorname{div}(u_2)$

(i) if it is contained in $\operatorname{div}(u_2)$, as $f = y^3 \bmod (u_2)$, the only possibility is the curve $V(y, u_2)$ which does not fit

(ii) if it is contained in $\operatorname{div}(u_1)$, it should be $V(z, u_1)$ with $y^3 + u_2^{10} \in (z, u_1)^3$

$$\frac{\partial y^3 + u_2^{10}}{\partial u_1} = u_2^9 \in (z, u_1)^2$$

so $z = u_2$, which does not fit.

III.2.3.6 Let us start Hironaka's algorithm. We blow up along the origin and take the point x_1 above of parameters

$y/u_1, u_1, u_2/u_1$ $\epsilon(x_1) = 6$

${y^{(1)}}^3 + u_1^7 {u_2^{(1)}}^2 [(u_1 - {u_2^{(1)}}^2)(u_1 + {u_2^{(1)}}^2)^3 + u_1^{N-8}]$

The exceptional divisor is $\operatorname{div}(u_1)$ "new component". Following Hironaka, we make the "fundamental sequence": we get the point x_3 above of parameters

$$(y^{(3)}, u_1^{(3)}, u_2^{(3)}) := (y^{(1)}/u_1^2, u_1, u_2^{(1)}),$$

$f_3 := {y^{(3)}}^3 + u_1^{(3)} {u_2^{(3)}}^2 [(u_1^{(3)} - {u_2^{(3)}}^2)(u_1^{(3)} + {u_2^{(3)}}^2)^3 + {u_1^{(3)}}^{N-8}]$ $\epsilon(x_3) = 6$

We make again the "fundamental sequence": we first blow up along x_3, we look at the point x_4 above of parameters

$$(y^{(4)}, u_1^{(4)}, u_2^{(4)}) := (y^{(3)}/u_2^{(3)}, u_1^{(3)}/u_2^{(3)}, u_2^{(3)})$$

$f_4 := {y^{(4)}}^3 + u_1^{(4)} {u_2^{(4)}}^4 [(u_2^{(4)} - u_2^{(4)})(u_2^{(4)} + u_2^{(4)})^3 + {u_1^{(4)}}^{N-8} {u_2^{(4)}}^{N-13}]$, $\epsilon(x_4) = 4$

$\mathrm{div}(u_1^{(4)} u_2^{(4)})$ is the exceptional divisor, both components are "new": we end this second "fundamental sequence" we look at the point x_5 above of parameters

$$(y^{(5)}, u_1^{(5)}, u_2^{(5)}) := (y^{(4)}/u_2^{(4)}, u_1^{(4)}, u_2^{(4)})$$

$f_5 := {y^{(5)}}^3 + u_1^{(5)} u_2^{(5)} [(u_1^{(5)} - u_2^{(5)})(u_1^{(5)} + u_2^{(5)})^3 + {u_1^{(5)}}^{N-8} {u_2^{(5)}}^{N-12}]$, $\epsilon(x_5) = 4$.

Following the algorithm, we blow up along the origin, above at the point of parameters

$z := y^{(5)}/u_1^{(5)}, v_1 := u_1^{(5)}, v := u_2^{(5)}/u_1^{(5)} + 1$, let us see that the ϵ-invariant increases strictly: $\epsilon(x_6) = 5$ (kangaroo point as defined by H. Hauser [HH]).

III.2.3.7 Surprising computation.

Exercise for the reader: compute $\omega(x_5)$ and $\omega(x_6)$ (cf. **I.2**). The main point is that $z^3 + v_1^3(v-1)[(v+1)v^3 + v_1^{2N-24}(v-1)^{N-12}]$ and $(2,1)$ a *solvable* vertex of $\Delta(f, z, v_1, v)$.

Let us solve it.

$$f = z^3 + v_1^3[(v^2-1)v^3 + v_1^{N-12}(v-1)^{N-11}]$$

$$w := z - v_1^2 v.$$

This gives

$$f = w^3 + v_1^3[v^5 + v_1^{N-12}(v-1)^{N-11}]$$

As $N \neq 0 \bmod (3)$, $\Delta(f, w, v_1, v)$ has two non solvable vertices (non integer coordinates)

$$(1, 5/3),\ (N/3 - 4, 0),$$

$\mathrm{div}(v_1)$ is the new component: $\epsilon(x_6) = 5 > \epsilon(x_5) = 4$.

III.2.3.8 No maximal contact on this example.

I claim that, in this example, if we end the fundamental sequence at x_6 and add another fundamental sequence, there will be a point $x_9 \in X_9$ such that there exists no $t = y - \gamma$, $\gamma \in k[[u_1, u_2]]$ such that the x_i are on the strict transform of $\mathrm{div}(t)$, $0 \leqslant i \leqslant 8$. One can see that it implies that there is no smooth hypersurface $W \subset Z$ such that the x_i are on the strict transform of $W \subset Z$. We define x_7 as the point on the strict transform of $\mathrm{div}(v)$ in the bu along x_6. These points x, x_1,....,x_8,x_9 are near to each other: there is a valuation ν of the function field whose center on X_i is x_i, $0 \leqslant i \leqslant 9$: there is no maximal contact ALONG THE VALUATION ν. So no hope of maximal contact, even for the uniformization problem or Hironaka's game.

Suppose a smooth hypersurface $W \subset Z$ has maximal contact along ν, let us call $t = 0$ its equation in a neighbourhood of x.

III.2.4 Proposition [CJS, section 15] *Suppose $x = x_0$ isolated in its HS-stratum, then if there exists a smooth hypersurface $t = 0$ such that along a fundamental sequence starting at $x = x_0$ the x_j, $j \geqslant 0$ (resp. $j \geqslant i$) are on the strict transform of div(t), then*

$$t = y - \gamma,\ \gamma \in k[[u_1, u_2]],\ \lfloor \delta(f, u, t) \rfloor = \lfloor \delta(f, u) \rfloor.$$

As $\delta(x) = 3 + 1/3$, by **III.2.4**, $\gamma \in (u_1, u_2)^3$.

$$\gamma = P_3(u_1,u_2) + P_4(u_1,u_2) + \rho,\ \rho \in (u_1,u_2)^5,$$

$P_i(u_1,u_2) \in k[u_1,u_2]$, homogeneous of degree i or $=0$, $i=3,4$.

$$f := t^3 + u_2^2[(u_1^3 - u_2^2)(u_1^3+u_2^2)^3 + u_1^N] + P_3^3 + P_4^3 + \rho^3.$$

$$f_1 = {t^{(1)}}^3 + u_1^7 {u_2^{(1)}}^2[(u_1 - {u_2^{(1)}}^2)(u_1+{u_2^{(1)}}^2)^3 + u_1^{N-8}] + u_1^9 P_3(1,u_2^{(1)})^3 + u_1^9 P_4(1,u_2^{(1)})^3 + u_1^{12}\rho'.$$

$f_3 := {t^{(3)}}^3 + u_1^{(3)} {u_2^{(3)}}^2[(u_1^{(3)} - {u_2^{(3)}}^2)(u_1^{(3)} + {u_2^{(3)}}^2)^3 + {u_1^{(3)}}^{N-8}] + P_3(1,u_2^{(3)})^3 + {u_1^{(3)}}^3 P_4(1,u_2^{(3)})^3 + {u_1^{(3)}}^6 {\rho'}^3$,

$\delta(x_3) = 2 + 1/3$, so by **III.2.4**, $P_3(1,u_2^{(3)}) \in (u_2^{(3)})^2$, let us denote $P_3(1,u_2^{(3)}) = a{u_2^{(3)}}^2 + b{u_2^{(3)}}^3$, $a,b\in k$.

$f_4 := {t^{(4)}}^3 + u_1^{(4)} {u_2^{(4)}}^4[(u_2^{(4)} - u_2^{(4)})(u_2^{(4)} + u_2^{(4)})^3 + {u_2^{(4)}}^{N-8}{u_2^{(4)}}^{N-13}] + a^3{u_2^{(4)}}^3 + b^3{u_2^{(4)}}^6 + {u_1^{(4)}}^3 P_4(1,u_2^{(4)})^3 + {u_1^{(4)}}^6{u_2^{(4)}}^3{\rho''}^3$.

As $\mathrm{V}(y^{(4)},u_2^{(4)})$ is permissible, it is contained in $\mathrm{div}(t^{(4)})$, so $\mathrm{V}(y^{(4)},u_2^{(4)}) = \mathrm{V}(t^{(4)},u_2^{(4)})$

$$P_4(1,u_2^{(4)}) = \lambda u_2^{(4)} + c{u_2^{(4)}}^2 + d{u_2^{(4)}}^3 + e{u_2^{(4)}}^4,\ \lambda,c,d,e\in k.$$

$f_5 := {t^{(5)}}^3 + u_1^{(5)} u_2^{(5)}[(u_1^{(5)} - u_2^{(5)})(u_1^{(5)} + u_2^{(5)})^3 + {u_1^{(5)}}^{N-8}{u_2^{(5)}}^{N-12}] + a^3 + b^3{u_2^{(5)}}^3 + {u_1^{(5)}}^3(\lambda^3 + c^3{u_2^{(5)}}^3 + d^3{u_2^{(5)}}^6 + e^3{u_2^{(5)}}^9) + {u_1^{(5)}}^9{\rho''}^3$,

$\delta(f_5,u) = 2$, by **III.2.4**,

$$a = b = \lambda = 0,\ \rho' \text{ not invertible.}$$

$f_7 := {t^{(6)}}^3 + {v_1}^3(v-1)[(v+1)v^3 + v_1^{N-12}(v-1)^{N-12}] + {v_1}^3(c^3(v-1)^3 + d^3(v-1)^6{v_1}^3 + e^3(v-1)^9{v_1}^6) + {v_1}^6 \times \textit{something}$

$$f_7 = w^3 + v_1^3[v^5 + v_1^{N-12}(v-1)^{N-11}].$$

$\delta(x_7) = 2 + 2/3$.

We end the fundamental sequence, we get

$f_8 = t^3 + (v-1)[(v+1)v^3 + v_1^{N-12}(v-1)^{N-12}] + c^3(v-1)^3 + d^3(v-1)^6{v_1}^3 + e^3(v-1)^9{v_1}^6 + {v_1}^3 \times \textit{something}$

$$f_8 = w^3 + v^5 + v_1^{N-12}(v-1)^{N-11}$$

$\delta(x_8) = 5/3$, $c=0$, else there is no point on the strict transform of $\mathrm{div}(t)$ and x_8 is, furthermore by **III.2.4**,

$$d = 0.$$

We go on: we blow up x_8 and we look at the near point x_9 on the strict transform of $v=0$.

$$f_9 = w^3 + u^2v^5 + u^{N-15}(uv-1)^{N-11}$$

As x_9 is supposed to be on the strict transform of $\mathrm{div}(t)$,

$$\textit{something} \text{ is not invertible.}$$

$f_9 = t^3 + (uv-1)[(uv+1)v^3 + u^{N-15}(v-1)^{N-12}] + e^3(uv-1)^9u^3 + u^3 \times \textit{something}'$, $\textit{something}'$ is invertible or divisible by u^3.

In any case, as $\delta(x_8) = 7/3 > 2$, the monomial $(uv-1)(uv+1)v^3$ gives a contradiction with **III.2.4**.

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