RIMS Workshop

Introduction to Idealistic Filtration Program

An approach to resolution of singularities in positive characteristics

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Lecture 2

Algorithm

for

local resolution of singularities

of

an idealistic filtration with boundary

in $\mathrm{char}(k) = 0$

via $(\sigma, \widetilde{\mu}, s)$-method

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9 Example (31)

1 Basic Definitions

Resolution of singularities

of an idealistic filtration with boundary

Global Construct a sequence of transformations

$$
\begin{array}{l}
(W, \mathbb{I}, E) \\
\quad \| \\
(W_0, \mathbb{I}_0, E_0) \leftarrow \cdots \\
\qquad (W_i, \mathbb{I}_i, E_i) \stackrel{\pi_{i+1}}{\longleftarrow} (W_{i+1}, \mathbb{I}_{i+1}, E_{i+1}) \\
\qquad\qquad \cdots \leftarrow (W_l, \mathbb{I}_l, E_l)
\end{array}
$$

s.t.

$$\mathrm{Supp}(\mathbb{I}_l) = \emptyset.$$

Local Given $P = P_0 \in W = W_0$.

Objects $(W_i, \mathbb{I}_i, E_i)$ defined only as algebraically local germs around P_i.

We (or the devil) pick up an arbitrary point P_{i+1} in the fiber of P_i after blowing up the center C_i.

Analytically Local (or Formal) Change "algebraically local" above into "analytically local".

Transformation

$$(W_i, \mathbb{I}_i, E_i) \overset{\pi_{i+1}}{\longleftarrow} (W_{i+1}, \mathbb{I}_{i+1}, E_{i+1})$$

of an idealistic filtration with boundary

(1) $W_i \overset{\pi_{i+1}}{\longleftarrow} W_{i+1}$ blowup with center C_i

s.t.

$$\begin{cases} C_i \subset \mathrm{Supp}(\mathbb{I}_i), \\ C_i \text{ nonsingular}, \\ C_i \text{ transversal to } E_i. \end{cases}$$

Note: An idealistic filtration $\mathbb{I}$ is of i.f.g. type

$\overset{\text{def}}{\Longleftrightarrow} \mathbb{I} = G(\{(\mathcal{I}_\lambda, a_\lambda); a_\lambda \in \mathbb{Z}_{\geq 0}\}_{\lambda \in \Lambda, \#\Lambda < \infty})$

(2) Idealistic filtration of i.f.g. type transforms

from $\mathbb{I}_i = \{\mathcal{I}_{i,a}\}_{a \in \mathbb{R}}$ to $\mathbb{I}_{i+1} = \pi_{i+1}^{\sharp}(\mathbb{I}_i) = \{\mathcal{I}_{i+1,a}\}_{a \in \mathbb{R}}$

where

for $a \in \mathbb{Z}_{>0}$

$$\mathcal{I}_{i+1,a} = \pi_{i+1}^{-1}(\mathcal{I}_{i,a})\mathcal{O}_{W_{i+1}} \cdot I(\pi_{i+1}^{-1}(C_i))^{-a},$$

for general $a \in \mathbb{R}$

$$\begin{cases} \mathcal{I}_{i+1,a} = \mathcal{O}_{W_{i+1}}, & a \leq 0 \\ \mathcal{I}_{i+1,a} = \mathcal{I}_{i+1,\lceil a \rceil}, & a > 0. \end{cases}$$

(3) $E_{i+1} = E_i \cup \pi_{i+1}^{-1}(C_i)$.

2 General reduction steps

Res. sing. of an idealistic filtration with boundary

$\Downarrow$

Res. sing. of a basic object

$\Downarrow$

Embedded resolution

$\Downarrow$ + **functoriality**

Res. sing. of an abstract algebraic variety over k

3 Plan of our lectures

Lecture 1 (Kawanoue) Philosophy and framework of IFP

Lecture 2 Present

an algorithm for local resolution of singularities of an idealistic filtration with boundary in $\mathrm{char} = 0$ via $(\sigma, \widetilde{\mu}, s)$-method

($\Longrightarrow$ Local uniformization theorem in $\mathrm{char} = 0$; well-known)

Lecture 3

Question Can we translate the $(\sigma, \widetilde{\mu}, s)$-algorithm in $\mathrm{char} = 0$ into the one in $\mathrm{char} = p > 0$?

Answer Yes !

Question Does the translation work as a real algorithm ?

Answer No !

Present some BAD examples.

Analyze why they are bad.

$\longrightarrow$

Introduction of invariant $\widetilde{\nu}$.

Observe how $\widetilde{\nu}$ overcomes the difficulties caused by the bad examples.

Lecture 4 Present

a candidate of an algorithm for analytically local resolution of singularities of an idealistic filtration with boundary in $\mathrm{char} = p > 0$ via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method

($\Longrightarrow$ Analytically local uniformization conjecture in $\mathrm{char} = p > 0$; unknown)

∘ Emphasize the difference between

$(\sigma, \widetilde{\mu}, s)$-method in $\mathrm{char} = 0$ &

$(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method in $\mathrm{char} = p > 0$.

∘ Discuss how to deal with

Anomalies in the MONOMIAL CASE.

Lecture 5 (Kawanoue) Going

from "analytically local" to "(algebraically) local"

4 Outline of the algorithm in char $= 0$ via $(\sigma, \widetilde{\mu}, s)$-method

Basic structure

Weaving of the strand & construction of the modification

In year i, we construct

the strand of invariants "inv" and

the modifications $(W_i^j, \mathbb{I}_i^j, E_i^j)$

of the transformation $(W_i, \mathbb{I}_i, E_i) = (W_i^0, \mathbb{I}_i^0, E_i^0)$.

$$\mathrm{inv}(P) = (\sigma, \widetilde{\mu}, s)(\sigma, \widetilde{\mu}, s)\cdots$$

$(\mathbb{I}_i, E_i) = (\mathbb{I}_i^0, E_i^0)$ $\quad$ $(\mathbb{I}_i^1, E_i^1)$

$(\sigma_i^1, \widetilde{\mu}_i^1, s_i^1)$ $\quad$ $(\sigma_i^2, \widetilde{\mu}_i^2, s_i^2)$

$\cdots$ $\quad$ $\cdots$

$\cdots$ $(\mathbb{I}_i^{j-1}, E_i^{j-1})$ $\quad$ $(\mathbb{I}_i^j, E_i^j)$

$(\sigma_i^j, \widetilde{\mu}_i^j, s_i^j)$

$\cdots$ $(\mathbb{I}_i^{m_i-1}, E_i^{m_i-1})$ $\quad$ $(\mathbb{I}_i^{m_i}, E_i^{m_i})$

$(\sigma_i^{m_i-1}, \widetilde{\mu}_i^{m_i-1}, s_i^{m_i-1})$ $\quad$ $\begin{cases} (\sigma_i^{m_i}, \infty, 0) \text{ or} \\ (\sigma_i^{m_i}, 0, 0, \Gamma) \end{cases}$

Note: In the classical setting,

$$\mathrm{inv}_{\mathrm{classic}}(P) = (w, s)(w, s)(w, s)\cdots.$$

Weaving goes

$$(\mathcal{I}_i, a) = (\mathcal{I}_i^0, a_i^0) \quad (\mathcal{I}_i^1 = \{\mathcal{I}_i^0\}'|_{H_i^1}, a_i^1)$$
$$(w_i^1, s_i^1) \qquad (w_i^2, s_i^2)\cdots$$

Termination in the horizontal direction

$$(\sigma_i^1, t_i^0) > (\sigma_i^2, t_i^1) > \cdots$$
$$\cdots \qquad > (\sigma_i^j, t_i^{j-1} = \#E_i^{j-1}) > (\sigma_i^{j+1}, t_i^j) > \cdots$$

$+ \{(\sigma, t)\}$ satisfies the descending chain condition

$\Longrightarrow$

In a fixed year i, weaving of the strand "inv" ends after finitely many stages.

Induction on σ (and t)

Enlargement of the idealistic filtration & shrinking of the boundary

$$\mathbb{I}_i^0 \subset \mathbb{I}_i^1 \subset \cdots \subset \mathbb{I}_i^{j-1} \subset \mathbb{I}_i^j \subset \cdots \subset \mathbb{I}_i^{m_i}$$
$$E_i^0 \supset E_i^1 \supset \cdots \supset E_i^{j-1} \supset E_i^j \supset \cdots \supset E_i^{m_i}$$

Choice of the center

$$C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i}).$$

Termination in the vertical direction

$$\mathbf{inv}(P_0)$$
$$\vee$$
$$\mathbf{inv}(P_1)$$
$$\vee$$
$$\ldots$$
$$\mathbf{inv}(P_{i-1})$$
$$\vee$$
$$\mathbf{inv}(P_i)$$
$$\vee$$
$$\ldots$$

There is NO such strictly decreasing and infinite sequence.

$\Longrightarrow$

Our algorithm ends after finitely many years.

A closer look at the inductive weaving of the strand & construction of the modifications

Assume inductively we have already woven "inv" and constructed its associated modifications up to year $(i-1)$.

Assume also inductively we have already woven "inv" and constructed its associated modifications up to stage $(j-1)$ in year i.

year $(i-1)$ ↓

year i — — — ⟶

stage $(j-1)$

$$(\mathbb{I}_i, \boldsymbol{E}_i) = (\mathbb{I}_i^0, \boldsymbol{E}_i^0) \quad (\mathbb{I}_i^1, \boldsymbol{E}_i^1) \quad \cdots (\mathbb{I}_i^{j-1}, \boldsymbol{E}_i^{j-1})$$

$$\mathrm{inv}^{\leq j-1}(\boldsymbol{P}_i) = \quad (\boldsymbol{\sigma}_i^1, \widetilde{\boldsymbol{\mu}}_i^1, \boldsymbol{s}_i^1) \quad \cdots (\boldsymbol{\sigma}_i^{j-1}, \widetilde{\boldsymbol{\mu}}_i^{j-1}, \boldsymbol{s}_i^{j-1})$$

Want to construct

$$\& \ (\mathbb{I}_i^j, \boldsymbol{E}_i^j)$$

$$(\boldsymbol{\sigma}_i^j, \widetilde{\boldsymbol{\mu}}_i^j, \boldsymbol{s}_i^j)$$

Summary of the construction

Case: $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

$$\begin{cases} \sigma_i^j &= \sigma\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ \widetilde{\mu}_i^j &= \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ s_i^j &= \# E_{i,\mathrm{aged}}^{j-1} = \# E_i^{j-1} \end{cases}$$

and

$$\begin{cases} \mathbb{I}_i^j & = \mathrm{Bd}\left(\mathrm{Comp}(\mathbb{I}_i^{j-1})\right) \\ \textbf{with} & \mathrm{Comp}(\mathbb{I}_i^{j-1}) = \mathrm{Cpc}(\mathbb{I}_i^{j-1}) \\ E_i^j & = E_i^{j-1} \setminus E_{i,\mathrm{aged}}^{j-1} = E_i^{j-1} \setminus E_i^{j-1} = \emptyset. \end{cases}$$

Case: $\mathrm{inv}^{\leq j-1}(P_i) = \mathrm{inv}^{\leq j-1}(P_{i-1})$

$$\begin{cases} \sigma_i^j &= \sigma_{i-1}^j \\ \widetilde{\mu}_i^j &= \mu_{\mathbb{H}, E_{i,\mathrm{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ s_i^j &= \# E_{i,\mathrm{aged}}^{j-1} \end{cases}$$

and

$$\begin{cases} \mathbb{I}_i^j & = \mathrm{Bd}\left(\mathrm{Comp}(\mathbb{I}_i^{j-1})\right) \\ \textbf{with} & \mathrm{Comp}(\mathbb{I}_i^{j-1}) \\ & = \begin{cases} \mathrm{Cpc}(\mathbb{I}_i^{j-1}) & \textbf{if } \widetilde{\mu}_i^j < \widetilde{\mu}_{i-1}^j \\ \mathfrak{D}_{E_{i,\mathrm{young}}}^{j-1}\left(\pi^\sharp(\mathrm{Comp}(\mathbb{I}_{i-1}^{j-1}))\right) & \textbf{if } \widetilde{\mu}_i^j = \widetilde{\mu}_{i-1}^j \end{cases} \\ E_i^j & = E_i^{j-1} \setminus E_{i,\mathrm{aged}}^{j-1}. \end{cases}$$

5 Detail of the inductive weaving and construction

Case: $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

$(\sigma_i^j, \widetilde{\mu}_i^j, s_i^j)$

Start with $\mathbb{I}_i^{j-1}$.

Take the $\mathfrak{D}$-saturation $\mathfrak{D}(\mathbb{I}_i^{j-1})$.

Set

$$\begin{cases} \sigma_i^j & = \sigma\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ & \mathbb{H}; \text{ an LGS of } \mathfrak{D}(\mathbb{I}_i^{j-1}) \\ \widetilde{\mu}_i^j & = \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right). \end{cases}$$

Lemma

$\mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)$ is independent of the choice of $\mathbb{H}$.

Therefore, $\widetilde{\mu}_i^j$ is well-defined.

Also set

$$\begin{cases} s_i^j & = \#\text{ of irred. comp. in } E_{i,\text{aged}}^{j-1} \\ & \quad (\text{passing through } P_i) \\ & = \#\text{ of irred. comp. in } E_i^{j-1} \text{ in this case.} \end{cases}$$

$(\mathbb{I}_i^j, E_i^j)$

- **Companion Modification**

$$\mathrm{Comp}(\mathbb{I}_i^{j-1}) := \mathrm{Cpc}(\mathbb{I}_i^{j-1})$$

Construction of "Cpc" and idea behind it

Take

$$\begin{cases} \mathbb{H} = \{(h_1, 1), \cdots, (h_l, 1)\} \text{ LGS} \\ (x_1 = h_1, \cdots, x_l = h_l, x_{l+1}, \cdots, x_d) \\ \qquad\qquad\qquad \text{reg. sys. of parameters} \end{cases}$$

Power Series Expansion: Given $f \in \widehat{\mathcal{O}_{W_i,P_i}}$,

$$f = \sum c_B(f) H^B \text{ where } \begin{cases} H^B = h_1^{b_1} \cdots h_l^{b_l} \\ c_B(f) \in k[[x_{l+1}, \cdots x_d]] \end{cases}$$

Observe

$$\widetilde{\mu}_i^j = \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)$$

$$= \inf\{\mathrm{ord}\left(c_{\mathbb{O}}(f)\right)/a; (f, a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}\}.$$

Want to add

$$\{(c_{\mathbb{O}}(f), \widetilde{\mu}_i^j \cdot a); (f, a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}\}.$$

MAIN MECHANISM OF INDUCTION

Take

$$(f, a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}$$

with

$$\operatorname{ord}(c_{\mathbb{O}}(f))/a \underset{\text{exactly}}{=} \widetilde{\mu}_i^j.$$

Then

$$\begin{cases} (c_{\mathbb{O}}(f), \widetilde{\mu}_i^j \cdot a) = (c_{\mathbb{O}}(f), \operatorname{ord}(c_{\mathbb{O}}(f)) \\ c_{\mathbb{O}}(f) \in k[[x_{l+1}, \cdots, x_d]]. \end{cases}$$

At the next $(j+1)$-th stage, we have

δ: an appropriate diff. operator

of degree $\operatorname{ord}(c_{\mathbb{O}}(f)) - 1$

such that

$$\begin{cases} (\delta(c_{\mathbb{O}}(f)), 1) \in \mathbb{H}_i^{j+1} \subset \mathfrak{D}(\mathbb{I}_i^{j+1}) \\ (\delta(c_{\mathbb{O}}(f)), 1) \notin \mathbb{H}_i^j = \mathbb{H}. \end{cases}$$

$\Longrightarrow$

$$\boxed{\sigma_i^j > \sigma_i^{j+1}}$$

Naive candidate for "Cpc"

$$\mathrm{NaiveCpc}(\mathbb{I}_i^{j-1}) = G\left(\mathfrak{D}(\mathbb{I}_i^{j-1}) \cup \left\{ \begin{array}{l} (c_{\mathbb{O}}(f), \widetilde{\mu}_i^j \cdot a); \\ (f,a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0} \end{array} \right\}\right)$$

Technical Requirements

∘ "Cpc" should be independent of the choice of $\mathbb{H}$ and reg. sys. of parameters $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$.

∘ "Cpc" should be an idealistic filtration of i.f.g. type.

Real Construction for "Cpc"

$$\mathrm{Cpc}(\mathbb{I}_i^{j-1}) = G\left[IL\left\{\mathfrak{D}\left(\mathrm{NaiveCpc}(\mathbb{I}_i^{j-1})\right)\right\}\right]$$

where

IL: the operator of taking the elements

at the Integral Level

Note: Description above is at the analytic level.

$\mathrm{char} = 0$ Can be done at the algebraic level.

$\mathrm{char} = p > 0$ See Lecture 5 by Kawanoue.

Lemma $\mathrm{Cpc}(\mathbb{I}_i^{j-1})$ **is independent of the choice of** $\mathbb{H}$ **and** $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$.

∘ **Boundary Modification**

$$\mathrm{Bd}\left(\mathrm{Comp}(\mathbb{I}_i^{j-1})\right) = G\left(\mathrm{Comp}(\mathbb{I}_i^{j-1}) \cup \left\{\begin{array}{l}(f_\lambda, 1); \\ F_\lambda \subset E_{i,\mathrm{aged}}^{j-1}\end{array}\right\}\right)$$

Case: $\mathrm{inv}^{\leq j-1}(P_i) = \mathrm{inv}^{\leq j-1}(P_{i-1})$

MAIN POINTS

- Use of "History"
- Use of "Logarithmic Differentiation"
- Adjustment of the notion of LGS

Go back in "history" to year i_{aged}

when the value $\mathrm{inv}^{\leq j-1}(P_i)$ first started;

$$\begin{aligned}\mathrm{inv}^{\leq j-1}(P_i) &= \mathrm{inv}^{\leq j-1}(P_{i-1}) \\ &\cdots \\ &= \mathrm{inv}^{\leq j-1}(P_{i_{\text{aged}}}) \\ &< \mathrm{inv}^{\leq j-1}(P_{i_{\text{aged}}-1})\end{aligned}$$

Decomposition of the boundary

$$E_i^{j-1} = E_{i,\text{young}}^{j-1} \sqcup E_{i,\text{aged}}^{j-1}$$

where

$$\begin{cases} E_{i,\text{young}}^{j-1} = \text{the collection of the exceptional divisors created after year } i_{\text{aged}} \\ E_{i,\text{aged}}^{j-1} = E_i^{j-1} \setminus E_{i,\text{young}}^{j-1}. \end{cases}$$

Notion of LGS adjusted

$$\begin{cases} V & := \cap_{F_\lambda \subset E^{j-1}_{i,\text{young}}} F_\lambda \\ \mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i) & = \mathfrak{D}_{E^{j-1}_{i,\text{young}}}\text{-saturation of } \mathbb{I}^{j-1}_i \\ \left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V & = \text{ its restriction to } V \end{cases}$$

Lemma $\left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V$ is $\mathfrak{D}$-saturated.

$$\begin{array}{ccc} \mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i) & \overset{\text{surjection}}{\longrightarrow} & \left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V \\ & & \mathfrak{D}\text{-saturated} \\ \cup & & \cup \\ \mathbb{H} & \overset{\text{lift}}{\longrightarrow} & \mathbb{H}_V;\ \text{an LGS} \end{array}$$

Definition $\mathbb{H}$ is an LGS of $\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)$.

$$\begin{cases} \sigma_V & := \sigma\left(\left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V\right) \\ c & := \operatorname{codim}_{W_i} V \\ \sigma^j_{i,\log} & := \sigma_V + c^{\mathbb{Z}_{\geq 0}} = (\sigma_{V,e} + c)_{e \in \mathbb{Z}_{\geq 0}} \end{cases}$$

Lemma $\sigma^j_{i,\log} = \sigma^j_{i-1,\log} = \sigma^j_{i-1}$.

$(\sigma_i^j, \widetilde{\mu}_i^j, s_i^j)$

Set

$$\begin{cases} \sigma_i^j = \sigma_{i,\log}^j = \sigma_{i-1,\log}^j = \sigma_{i-1}^j \\ \widetilde{\mu}_i^j = \mu_{\mathbb{H},E_{i,\text{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ \quad = \mu_{\mathbb{H}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) - \sum_{F_\lambda \subset E_{i,\text{young}}^{j-1}} \mu_\lambda \\ s_i^j = \#\text{ of irred. comp. in } E_{i,\text{aged}}^{j-1} \\ \quad \text{(passing through } P_i\text{)} \end{cases}$$

where

$$\begin{cases} \mu_{\mathbb{H}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ \quad = \inf\left\{\operatorname{ord}\left(c_{\mathbb{O}}(f)\right)/a; (f,a) \in \mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}), \right. \\ \qquad \left. a \in \mathbb{Z}_{>0}, f = \sum c_B(f)H^B\right\} \\ \mu_\lambda = \inf\left\{n/a; c_{\mathbb{O}}(f) \text{ divisible by } f_\lambda^n, \right. \\ \qquad (f,a) \in \mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}, \\ \qquad \left. f = \sum c_B(f)H^B\right\} \end{cases}$$

Lemma $\mu_{\mathbb{H},E_{i,\text{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right)$ **is independent of the choice of** $\mathbb{H}$ **(or** $\mathbb{H}_V$**).**

Therefore, $\widetilde{\mu}_i^j$ **is well-defined.**

$(\mathbb{I}_i^j, E_i^j)$

$$\begin{cases} \mathbb{I}_i^j & = \mathrm{Bd}\left(\mathrm{Comp}(\mathbb{I}_i^{j-1})\right) \\ \text{with} & \mathrm{Comp}(\mathbb{I}_i^{j-1}) \\ & = \begin{cases} \mathrm{Cpc}(\mathbb{I}_i^{j-1}) & \text{if } \widetilde{\mu}_i^j < \widetilde{\mu}_{i-1}^j \\ \mathfrak{D}^{j-1}_{E_{i,\mathrm{young}}}\left(\pi^\sharp(\mathrm{Comp}(\mathbb{I}_{i-1}^{j-1}))\right) & \text{if } \widetilde{\mu}_i^j = \widetilde{\mu}_{i-1}^j \end{cases} \\ E_i^j & = E_i^{j-1} \setminus E_{i,\mathrm{aged}}^{j-1} \end{cases}$$

Description of "Cpc" in case $\widetilde{\mu}_i^j < \widetilde{\mu}_{i-1}^j$

Consider

BlackBox $=$

$$\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}\left[G\left(\begin{array}{l} \mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1}) \cup \\ \Big\{(c_{\mathbb{O}}(f) \otimes_k \{(\prod f_\lambda^{\mu_\lambda})^a\}^{-1}, \widetilde{\mu}_i^j \cdot a); \\ \qquad (f, a) \in \mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}, \\ \qquad f = \sum c_B(f) H^B \Big\} \cup \\ \Big\{(f_\lambda \otimes (f_\lambda^{\frac{q}{r_\lambda}})^{-1}, 0); \\ \qquad q = 1, \cdots, r_\lambda - 1, \\ \qquad r_\lambda;\ \text{the denominator of } \mu_\lambda, \\ \qquad F_\lambda \subset E_{i,\mathrm{young}}^{j-1} \Big\} \end{array}\right)\right]$$

$IPIL$; the operator to take the elements being at the integral levels as well as having only integral powers in $\otimes_k$__.

$IPIL(\boxed{\text{BlackBox}})$

$\downarrow$ Eliminate $\otimes_k$ by turning it into the real multiplication

Image

$$\mathrm{Cpc}(\mathbb{I}_i^{j-1}) := \mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathrm{Image}).$$

Lemma $\mathrm{Cpc}(\mathbb{I}_i^{j-1})$ is indepenedent of the choice of $\mathbb{H}$ (or $\mathbb{H}_V$) and $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$.

Note: We take $(x_{l+1}, \cdots, x_d)$ to contain $\{f_\lambda; F_\lambda \subset E_{i,\mathrm{young}}^{j-1}\}$.

6 Termination in the horizontal direction (revisited)

$$(\sigma_i^1, t_i^0) > (\sigma_i^2, t_i^1) > \cdots$$

In fact, we have

$$(\sigma_i^j, t_i^{j-1}) > (\sigma_i^{j+1}, t_i^j),$$

since

$$\begin{cases} \widetilde{\mu}_i^j \neq \infty \text{ or } 0 \\ \quad \to \sigma_i^j > \sigma_i^{j+1} \\ \widetilde{\mu}_i^j = \infty \text{ or } 0 \ \& \ s_i^j \neq 0 \\ \quad \to \sigma_i^j \geq \sigma_i^{j+1} \ \& \ t_i^{j-1} > t_i^j \\ \widetilde{\mu}_i^j = \infty \text{ or } 0 \ \& \ s_i^j = 0 \\ \quad \to \text{End of weaving.} \end{cases}$$

$\{(\sigma, t)\}$ satisfies the descending chain condition.

$\Longrightarrow$

In a fixed year i, weaving of the strand "inv" ends after finitely many years.

Main mechanism of induction on σ (and t)

7 Choice of the center (revisited)

Choose $C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i})$.

Case: $(\mathbb{I}_i^{m_i-1}, E_i^{m_i-1}) \quad (\mathbb{I}_i^{m_i}, E_i^{m_i})$

$$(\sigma_i^{m_i}, \infty, 0)$$

(i) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ **nonsingular.**

$\because$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) = \{h_1 = \cdots = h_l = 0\}$$

where

$\mathbb{H} = \{(h_\alpha, 1)\}_{\alpha=1}^{l}$ **an LGS for** $\mathbb{I}_i^{m_i} = \mathfrak{D}_{E_{i,\mathrm{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1})$

$||$

(in this case $\mathfrak{D}(\mathbb{I}_i^{m_i-1})$**)**

Note: In the framework of IFP,

$$\left\{\begin{array}{l} \mathbb{I}_i^{m_i} = \mathfrak{D}(\mathbb{I}_i^{m_i-1}) \\ \quad ;\ \mathfrak{D}\text{-saturated} \\ \mu_{\mathbb{H}} = \infty \end{array}\right. \quad \Longrightarrow \quad \mathrm{Supp}(\mathbb{I}_i^{m_i}) \text{ nonsingular.}$$

Nonsingularity Principle

(ii) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ transversal to E_i.

$\because$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp E_i^{m_i} = E_i^{m_i-1} \setminus E_{i,\mathrm{aged}}^{m_i-1} = E_{i,\mathrm{young}}^{m_i-1}$$

by construction

and

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \subset \cap_{F_\lambda \subset E_{i,\mathrm{aged}}^0 \cup \cdots \cup E_{i,\mathrm{aged}}^{m_i-1}} F_\lambda$$

by construction of "Bd"

$\therefore$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp \underbrace{\left(E_{i,\mathrm{aged}}^0 \cup \cdots \cup E_{i,\mathrm{aged}}^{m_i-1}\right) \cup E_{i,\mathrm{young}}^{m_i-1}}_{\substack{\| \\ E_i}}$$

Case: $(\mathbb{I}_i^{m_i-1}, E_i^{m_i-1}) \quad (\mathbb{I}_i^{m_i}, E_i^{m_i})$

$$(\sigma_i^{m_i}, 0, 0)$$

This is the MONOMIAL CASE.

We introduce the invariant $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$

where

$$\begin{cases} \Gamma_1 = -\min\{n; \exists(\lambda_1, \cdots, \lambda_n) \\ \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\ \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \\ \Gamma_2 = \max\{\mu_{\lambda_1} + \cdots + \mu_{\lambda_n}; \exists(\lambda_1, \cdots, \lambda_n) \\ \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\ \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n} \\ \quad -n = \Gamma_1\} \\ \Gamma_3 = \max\{(\lambda_1, \cdots, \lambda_n); \\ \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\ \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \\ \quad -n = \Gamma_1, \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} = \Gamma_2\} \end{cases}$$

We replace

$(\sigma_i^{m_i}, 0, 0)$ the original m_i-th unit

with

$(\sigma_i^{m_i}, 0, 0, \Gamma)$ the new m_i-th unit.

We also replace

$\mathbb{I}_i^{m_i} = \mathrm{Bd}(\mathrm{Comp}(\mathbb{I}_i^{m_i-1})) = \mathfrak{D}_{E_{i,\mathrm{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1})$

the original m_i-th modification

with

$$\mathbb{I}_i^{m_i} = G\left(\begin{array}{l}\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1}) \cup \\ \{(f_{\lambda_1}, 1), \cdots, (f_{\lambda_n}, 1); (\lambda_1, \cdots, \lambda_n) = \Gamma_3\}\end{array}\right)$$

the new m_i-th modification.

(i) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ nonsingular.

$\because$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) = \{h_1 = \cdots = h_l = 0\} \cap \{f_{\lambda_1} = \cdots = f_{\lambda_n} = 0\}$$

where

$\mathbb{H} = \{(h_\alpha, 1)\}_{\alpha=1}^{l}$ an LGS for $\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1})$

Question **What should be the statement of Nonsingularity Principle in the MONOMIAL CASE in the framework of IFP ?**

Note: Answer given only via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method.

(ii) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ transversal to E_i.

$\because$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp E_i^{m_i} = E_i^{m_i-1} \setminus E_{i,\mathrm{aged}}^{m_i-1} = E_{i,\mathrm{young}}^{m_i-1}$$

by construction

and

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \subset \cap_{F_\lambda \subset E_{i,\mathrm{aged}}^0 \cup \cdots \cup E_{i,\mathrm{aged}}^{m_i-1}} F_\lambda$$

by construction of "Bd"

$\therefore$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp \underbrace{\left(E_{i,\mathrm{aged}}^0 \cup \cdots \cup E_{i,\mathrm{aged}}^{m_i-1}\right) \cup E_{i,\mathrm{young}}^{m_i-1}}_{\substack{\parallel \\ E_i}}$$

8 Termination in the vertical direction (revisited)

Crucial Claim **The strand of invariants "inv" never increases after blowup, i.e.,**

$$\mathrm{inv}(P_i) \leq \mathrm{inv}(P_{i-1}).$$

Proof of the crucial claim is not trivial.

Claim **The strand of invariants "inv" actually strictly decreases after blowup, i.e.,**

$$\mathrm{inv}(P_i) < \mathrm{inv}(P_{i-1}).$$

Proof of the claim using Crucial Claim

Observe

(i) $P_i \in \mathrm{Supp}(\mathbb{I}_i^j)\ \forall j$

(ii) $\mathrm{inv}^{\leq j}(P_i) = \mathrm{inv}^{\leq j}(P_{i-1})$

$$\Longrightarrow \mathfrak{D}_{E^j_{i,\mathrm{young}}}(\mathbb{I}_i^j) = \mathfrak{D}_{E^j_{i,\mathrm{young}}}(\pi^\sharp(\mathbb{I}_{i-1}^j)).$$

Suppose

$$\begin{array}{ccc} \mathrm{inv}(P_i) & = & \mathrm{inv}(P_{i-1}) \\ \| & & \| \\ \mathrm{inv}^{\leq m_i}(P_i) & & \mathrm{inv}^{\leq m_{i-1}}(P_{i-1}) \end{array} \quad \text{with } m_i = m_{i-1}.$$

Then by (ii) with $j = m_i$, we have

$$\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i}}(\mathbb{I}_i^{m_i}) = \mathfrak{D}_{E_{i,\mathrm{young}}^{m_i}}(\pi^\sharp(\mathbb{I}_{i-1}^{m_i})).$$

On the other hand,

$$\mathrm{Supp}\left(\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i}}(\pi^\sharp(\mathbb{I}_{i-1}^{m_i}))\right) = \mathrm{Supp}\left(\pi^\sharp(\mathbb{I}_{i-1}^{m_i})\right) = \emptyset,$$

since

the last (m_i-th) modification has

the distinguished feature that

its transformation after blowup has NO support.

But then by (i)

$$P_i \in \mathrm{Supp}\left(\mathbb{I}_i^{m_i}\right) = \mathrm{Supp}\left(\mathfrak{D}_{E_{i,\mathrm{young}}^{m_i}}(\mathbb{I}_i^{m_i})\right) = \emptyset,$$

a contradiction !

Last Claim The strictly decreasing sequence

$$\mathrm{inv}(P_0) > \mathrm{inv}(P_1) > \cdots$$
$$\cdots \quad > \mathrm{inv}(P_{i-1}) > \mathrm{inv}(P_i) > \cdots$$

stops after finitely many years.

Caution No descending chain condition

for the value set of "inv", since

denominators of $\widetilde{\mu}$ and Γ_2 in $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$

are NOT a priori bounded.

Proof of the last claim

Suppose inductively "$\mathrm{inv}^{\leq j}$" stabilizes, i.e.,

$\exists i_j$ s.t. $\mathrm{inv}^{\leq j}(P_i) = \mathrm{inv}^{\leq j}(P_{i_j})\ \forall i \geq i_j$.

Then

$$\begin{aligned}
&\mathfrak{D}_{E^j_{i,\mathrm{young}}}(\mathbb{I}^j_i) \\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp(\mathbb{I}^j_{i-1})\right) \\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\left(\mathfrak{D}_{E^j_{i-1,\mathrm{young}}}(\mathbb{I}^j_{i-1})\right)\right) \\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\left(\mathfrak{D}_{E^j_{i-1,\mathrm{young}}}\left(\pi^\sharp(\mathbb{I}^j_{i-2})\right)\right)\right) \\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\pi^\sharp(\mathbb{I}^j_{i-2})\right) \cdots \\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\pi^\sharp \cdots \pi^\sharp(\mathbb{I}^j_{i_j})\right)
\end{aligned}$$

$\Longrightarrow$

Denominators of $\widetilde{\mu}_i^j$ are uniformly bounded by the number determined by the levels of the generators of $\mathbb{I}_{i_j}^j$.

(Similarly denominators of Γ_2 are uniformly bounded.)

$\Longrightarrow$

"$\mathrm{inv}^{\leq j+1}$" stabilizes after finitely many years.

$\Longrightarrow$

"inv" stabilizes after finitely many years. Q.E.D.

Note: We can NOT extend "inv" infinitely in the horizontal direction (i.e., can NOT increase "j" infinitely), since the set $\{(\sigma, t)\}$ satisfies the descending chain condition !

9 Example

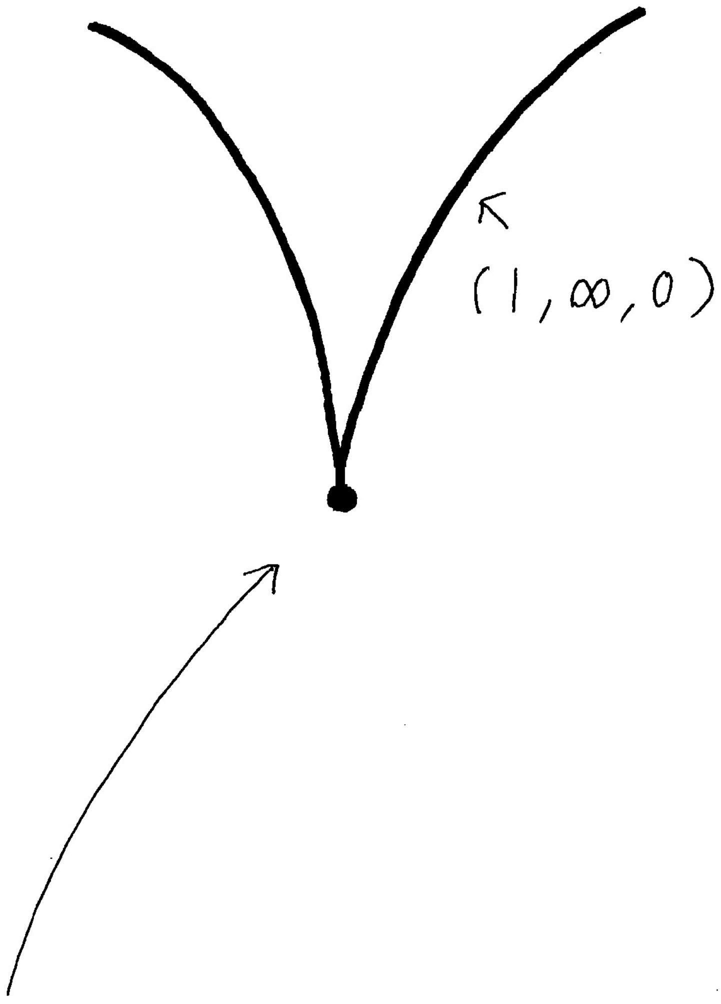
Res. sing. of the idealistic filtration with boundary

$$(W, \mathbb{I}, E) = (\mathbb{A}^2, G(\{(x^2 - y^3, 1)\}), \emptyset)$$

Year 0

$(\mathbb{I}_0^0, E_0^0 = \emptyset)$ $(x^2 - y^3, 1)$

$(2, 2, 0)$



$(\mathbb{I}_0^1, E_0^1 = \emptyset)$ $(x^2 - y^3, 2)$

$(2x, 1)$

$(3y^2, 1)$

$(2, 2, 0)(1, \frac{3}{2}, 0)$

$(\mathbb{I}_0^2, E_0^2 = \emptyset)$ $(y^3, 3)$

$(3y^2, 2)$

$(6y, 1)$

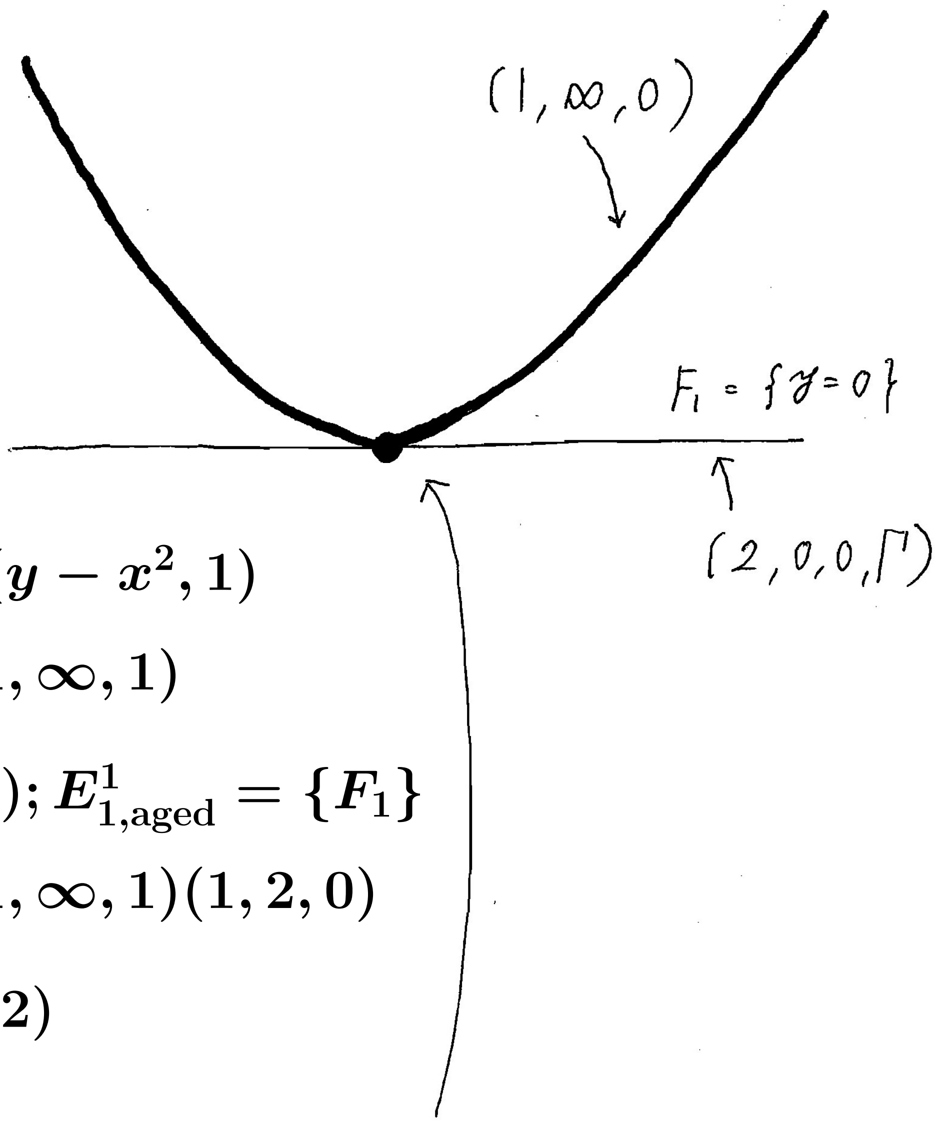
$(2, 2, 0)(1, \frac{3}{2}, 0)(0, \infty, 0)$

Blowup with center (x, y)

Year 1

$(\mathbb{I}_1^0, E_1^0 = \{F_1\})\ (y(y - x^2), 1)$

$(2, 1, 0)$



$(\mathbb{I}_1^1, E_1^1 = \{F_1\})\ (y - x^2, 1)$

$(2, 1, 0)(1, \infty, 1)$

$(\mathbb{I}_1^2, E_1^2 = \emptyset)\ (y, 1);\ E_{1,\text{aged}}^1 = \{F_1\}$

$(2, 1, 0)(1, \infty, 1)(1, 2, 0)$

$(\mathbb{I}_1^3, E_1^3 = \emptyset)\ (x^2, 2)$

$(2x, 1)$

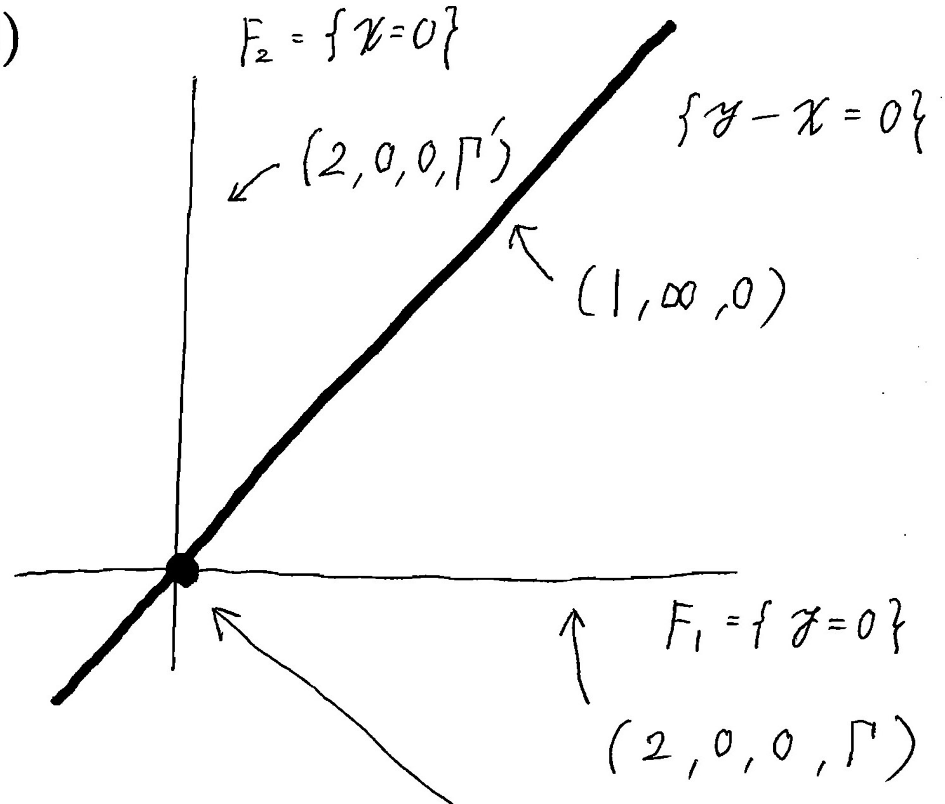
$(2, 1, 0)(1, \infty, 1)(1, 2, 0)(0, \infty, 0)$

Blowup with center (x, y)

Year 2

$$(\mathbb{I}_2^0, E_2^0 = \{F_1, F_2\}) \ (xy(y-x), 1)$$

$$(2, 1, 0)$$



$$(\mathbb{I}_2^1, E_2^1 = \{F_1, F_2\}) \ (y-x, 1)$$

$$(2, 1, 0)(1, \infty, 1)$$

$$(\mathbb{I}_2^2, E_2^2 = \{F_2\}) \ (y, 1); E_{2,\text{aged}}^1 = \{F_1\}$$

$$(2, 1, 0)(1, \infty, 1)(1, 0, 1)$$

$$(\mathbb{I}_2^3, E_2^3 = \emptyset) \ (x, 1); E_{2,\text{aged}}^2 = \{F_2\}$$

$$(2, 1, 0)(1, \infty, 1)(1, 0, 1)(0, \infty, 0)$$

Blowup with center (x, y)

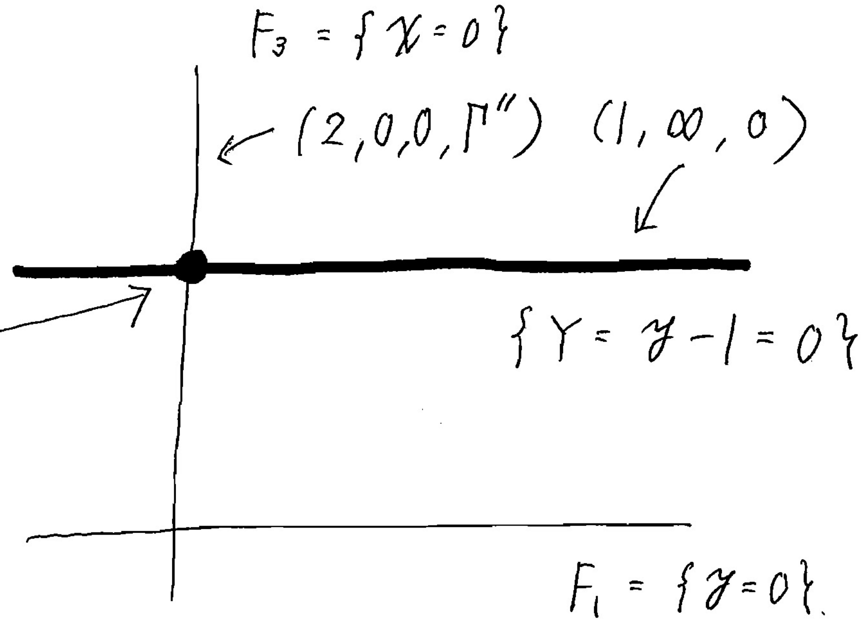
Year 3

$$(\mathbb{I}_3^0, E_3^0 = \{F_1, F_2, F_3\}) \ (x^2 y(y-x), 1)$$

$$\| $$

$$(x^2(Y+1)Y, 1)$$

$$(2, 1, 0)$$



$$(\mathbb{I}_3^1, E_3^1 = \{F_1, F_2, F_3\}) \ (Y, 1)$$

$$(2, 1, 0)(1, \infty, 0)$$

$$(\mathbb{I}_3^2, E_3^2 = \{F_1, F_2, F_3\})$$

$$C_3 = \mathrm{Supp}(\mathbb{I}_3^2) = \{Y = 0\}$$

Center of blowup in year 3

Observation

$$C_3 = \mathrm{Supp}(\mathbb{I}_3^2) \neq \mathrm{MaxLocus}(\mathrm{inv})$$

because

"Comp" fails to separate "MaxLocus" when $\widetilde{\mu} = 1$

Anomaly when $\widetilde{\mu} = 1$

$\Longrightarrow$

Our algorithm (even in $\mathrm{char} = 0$) is only local

(for the moment).