RIMS Workshop

Introduction to Idealistic Filtration Program

An approach to resolution of singularities in positive characteristics

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Lecture 4

Algorithm

for

analytically local resolution of singularities

of

an idealistic filtration with boundary

in $\mathrm{char}(k) = p > 0$

via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method

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Contents

1 Outline of the algorithm in char $p > 0$ via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method

Basic structure

Weaving of the strand

& construction of the modification

In year i, we construct

the strand of invariants "inv" and

the modifications $(W_i^j, \mathbb{I}_i^j, E_i^j)$

of the transformation $(W_i, \mathbb{I}_i, E_i) = (W_i^0, \mathbb{I}_i^0, E_i^0)$.

$$\mathrm{inv}(P) = (\sigma, \widetilde{\mu}, \widetilde{\nu}, s)(\sigma, \widetilde{\mu}, \widetilde{\nu}, s) \cdots$$

$(\mathbb{I}_i, E_i) = (\mathbb{I}_i^0, E_i^0)$ $\qquad (\mathbb{I}_i^1, E_i^1)$

$(\sigma_i^1, \widetilde{\mu}_i^1, \widetilde{\nu}_i^1, s_i^1)$ $(\sigma_i^2, \widetilde{\mu}_i^2, \widetilde{\nu}_i^2, s_i^2)$

$\cdots\cdots$

$\cdots (\mathbb{I}_i^{j-1}, E_i^{j-1})$ $\qquad (\mathbb{I}_i^j, E_i^j)$

$(\sigma_i^j, \widetilde{\mu}_i^j, \widetilde{\nu}_i^j, s_i^j)$

$\cdots$ $\qquad (\mathbb{I}_i^{m_i-1}, E_i^{m_i-1})$ $\qquad (\mathbb{I}_i^{m_i}, E_i^{m_i})$

$$(\sigma_i^{m_i-1}, \widetilde{\mu}_i^{m_i-1}, \widetilde{\nu}_i^{m_i-1}, s_i^{m_i-1}) \qquad \begin{cases} (\sigma_i^{m_i}, \infty, \infty, 0) \text{ or} \\ (\sigma_i^{m_i}, 0, 0, 0, \Gamma) \end{cases}$$

Termination in the horizontal direction

$$(\sigma_i^1, t_i^0) > (\sigma_i^2, t_i^1) > \cdots$$

$$\cdots > (\sigma_i^j, t_i^{j-1} = \#E_i^{j-1}) > (\sigma_i^{j+1}, t_i^j) > \cdots$$

+ $\{(\sigma, t)\}$ satisfies the descending chain condition

$\Longrightarrow$

In a fixed year i, weaving of the strand "inv" ends after finitely many stages.

Induction on σ (and t)

Enlargement of the idealistic filtration & shrinking of the boundary

$$\mathbb{I}_i^0 \subset \mathbb{I}_i^1 \subset \cdots \subset \mathbb{I}_i^{j-1} \subset \mathbb{I}_i^j \subset \cdots \subset \mathbb{I}_i^{m_i}$$

$$E_i^0 \supset E_i^1 \supset \cdots \supset E_i^{j-1} \supset E_i^j \supset \cdots \supset E_i^{m_i}$$

Choice of the center

$$C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i}).$$

Termination in the vertical direction

$$\mathbf{inv}(P_0)$$
$$\vee$$
$$\mathbf{inv}(P_1)$$
$$\vee$$
$$\dots$$
$$\mathbf{inv}(P_{i-1})$$
$$\vee$$
$$\mathbf{inv}(P_i)$$
$$\vee$$
$$\dots$$

There is NO such strictly decreasing and infinite sequence.

$\Longrightarrow$

Our algorithm ends after finitely many years.

A closer look at the inductive weaving of the strand & construction of the modifications

Assume inductively we have already woven "inv" and constructed its associated modifications

up to year $(i-1)$.

Assume also inductively we have already woven "inv" and constructed its associated modifications

up to stage $(j-1)$ in year i.

year $(i-1)$ ↓

year i $- - - \longrightarrow$

stage $(j-1)$

$$(\mathbb{I}_i, \boldsymbol{E}_i) = (\mathbb{I}_i^0, \boldsymbol{E}_i^0) \quad (\mathbb{I}_i^1, \boldsymbol{E}_i^1) \quad \cdots (\mathbb{I}_i^{j-1}, \boldsymbol{E}_i^{j-1})$$

$$\mathrm{inv}^{\leq j-1}(\boldsymbol{P}_i) = (\boldsymbol{\sigma}_i^1, \widetilde{\boldsymbol{\mu}}_i^1, \widetilde{\boldsymbol{\nu}}_i^1, \boldsymbol{s}_i^1) \cdots (\boldsymbol{\sigma}_i^{j-1}, \widetilde{\boldsymbol{\mu}}_i^{j-1}, \widetilde{\boldsymbol{\nu}}_i^{j-1}, \boldsymbol{s}_i^{j-1})$$

Want to construct

$$\& \ (\mathbb{I}_i^j, \boldsymbol{E}_i^j)$$

$$(\boldsymbol{\sigma}_i^j, \widetilde{\boldsymbol{\mu}}_i^j, \widetilde{\boldsymbol{\nu}}_i^j, \boldsymbol{s}_i^j)$$

Summary of the construction

Case: $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

$$\begin{cases} \sigma_i^j & = \sigma\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ \widetilde{\mu}_i^j & = \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ \widetilde{\nu}_i^j & = \nu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ s_i^j & = \# E_{i,\mathrm{aged}}^{j-1} = \# E_i^{j-1} \end{cases}$$

and

$$\begin{cases} \mathbb{I}_i^j & = \mathrm{Bd}\left(\mathrm{Comb}(\mathbb{I}_i^{j-1})\right) \\ \textbf{with} & \mathrm{Comp}(\mathbb{I}_i^{j-1}) = \mathrm{Cbc}(\mathbb{I}_i^{j-1}) \\ E_i^j & = E_i^{j-1} \setminus E_{i,\mathrm{aged}}^{j-1} = E_i^{j-1} \setminus E_i^{j-1} = \emptyset. \end{cases}$$

Case: $\mathrm{inv}^{\leq j-1}(P_i) = \mathrm{inv}^{\leq j-1}(P_{i-1})$

$$\begin{cases} \sigma_i^j &= \sigma_{i-1}^j \\ \widetilde{\mu}_i^j &= \mu_{\mathbb{H}, E_{i,\mathrm{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ \widetilde{\nu}_i^j &= \nu_{\mathbb{H}, E_{i,\mathrm{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ s_i^j &= \# E_{i,\mathrm{aged}}^{j-1} \end{cases}$$

and

$$\begin{cases} \mathbb{I}_i^j & = \mathrm{Bd}\left(\mathrm{Comb}(\mathbb{I}_i^{j-1})\right) \\ \textbf{with} & \mathrm{Comb}(\mathbb{I}_i^{j-1}) \\ & = \begin{cases} \mathrm{Cbc}(\mathbb{I}_i^{j-1}) \\ \quad \text{if } (\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) < (\widetilde{\mu}_{i-1}^j, \widetilde{\nu}_{i-1}^j) \\ \mathfrak{D}_{E_{i,\mathrm{young}}}^{j-1}\left(\pi^\sharp(\mathrm{Comb}(\mathbb{I}_{i-1}^{j-1}))\right) \\ \quad \text{if } (\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) = (\widetilde{\mu}_{i-1}^j, \widetilde{\nu}_{i-1}^j) \end{cases} \\ E_i^j & = E_i^{j-1} \setminus E_{i,\mathrm{aged}}^{j-1}. \end{cases}$$

2 Detail of the inductive weaving and construction

Case: $\mathrm{inv}^{\leq j-1}(P_i) < \mathrm{inv}^{\leq j-1}(P_{i-1})$

$(\sigma_i^j, \widetilde{\mu}_i^j, \widetilde{\nu}_i^j, s_i^j)$

Start with $\mathbb{I}_i^{j-1}$.

Take the $\mathfrak{D}$-saturation $\mathfrak{D}(\mathbb{I}_i^{j-1})$.

Set

$$\begin{cases} \sigma_i^j &= \sigma\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right), \mathbb{H}; \text{ an LGS of } \mathfrak{D}(\mathbb{I}_i^{j-1}) \\ \widetilde{\mu}_i^j &= \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right) \\ \widetilde{\nu}_i^j &= \nu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right). \end{cases}$$

Lemma

$\mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)$ & $\nu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)$ are independent of the choice of $\mathbb{H}$. Therefore, $\widetilde{\mu}_i^j$ & $\widetilde{\nu}_i^j$ are well-defined.

Also set

$$\begin{cases} s_i^j &= \#\text{ of irred. comp. in } E_{i,\mathrm{aged}}^{j-1} \\ & \quad (\text{passing through } P_i) \\ & = \#\text{ of irred. comp. in } E_i^{j-1} \text{ in this case.} \end{cases}$$

$(\mathbb{I}_i^j, E_i^j)$

∘ **Combined Modification**

$$\mathrm{Comb}(\mathbb{I}_i^{j-1}) := \mathrm{Cbc}(\mathbb{I}_i^{j-1})$$

Construction of "Cbc" and idea behind it

Take

$$\begin{cases} \mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^{l} \text{ LGS} \\ \quad \text{with } h_\alpha = x_\alpha^{p^{e_\alpha}} \bmod \mathfrak{m}_{P_i}^{p^{e_\alpha}+1} \\ \{p^{e_\alpha}\}_{\alpha=1}^{l} = \{p^{e_1} < \cdots < p^{e_m}\} = \{p^{e_\beta}\}_{\beta=1}^{m} \\ \text{and its associated reg. sys. of parameters} \\ (x_1, \cdots, x_l, x_{l+1}, \cdots, x_d) \end{cases}$$

Power Series Expansion: Given $f \in \widehat{\mathcal{O}_{W_i,P_i}}$,

$\exists!\ \ f = \sum c_B(f) H^B$ with $H^B = h_1^{b_1} \cdots h_l^{b_l}$

where

$$\deg_{x_\alpha} c_B(f) \leq p^{e_\alpha} - 1 \text{ for } \alpha = 1, \cdots, l,$$

i.e.,

$$c_B(f) = \sum_{0 \leq n_\alpha \leq p^{e_\alpha}-1} c_{n_1 \cdots n_l} x_1^{n_1} \cdots x_l^{n_l}$$

$$\text{with } c_{n_1 \cdots n_l} \in k[[x_{l+1}, \cdots, x_d]].$$

Observe

$$\begin{aligned}\widetilde{\mu}_i^j &= \mu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)\\ &= \inf\{\mathrm{ord}\left(c_{\mathbb{O}}(f)\right)/a; (f,a)\in\mathfrak{D}(\mathbb{I}_i^{j-1}), a\in\mathbb{Z}_{>0}\}\\ \widetilde{\nu}_i^j &= \nu_{\mathbb{H}}\left(\mathfrak{D}(\mathbb{I}_i^{j-1})\right)\\ &= \inf\{\mathrm{ord}\left(c_{\mathbb{O}}(f)\right)/(p^{e_\beta}-t);\\ &\qquad (f, p^{e_\beta}-t)\in D^t\left(\mathfrak{D}(\mathbb{I}_i^{j-1})_{p^{e_\beta}}\right),\\ &\qquad t\in\mathbb{Z}_{>0}, p^{e_\beta}-t>0\}.\end{aligned}$$

Want to add

$$\{(c_{\mathbb{O}}(f), \widetilde{\mu}_i^j\cdot a); (f,a)\in\mathfrak{D}(\mathbb{I}_i^{j-1}), a\in\mathbb{Z}_{>0}\}.$$

and

$$\{(c_{\mathbb{O}}(f), \widetilde{\nu}_i^j\cdot(p^{e_\beta}-t)); (f,a)\in D^t\left(\mathfrak{D}(\mathbb{I}_i^{j-1})_{p^{e_\beta}}\right), \quad t\in\mathbb{Z}_{>0}, p^{e_\beta}-t>0\}.$$

MAIN MECHANISM OF INDUCTION

Mechanism to guarantee

$$\widetilde{\mu}_i^j \neq \infty \text{ or } 0 \Longrightarrow \sigma_i^j > \sigma_i^{j+1}.$$

Take

$$(f, a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}$$

with

$$\mathrm{ord}\,(c_{\mathbb{O}}(f))\,/a \underset{\text{exactly}}{=} \widetilde{\mu}_i^j.$$

Then

$$\begin{cases} (c_{\mathbb{O}}(f), \widetilde{\mu}_i^j \cdot a) = (c_{\mathbb{O}}(f), \mathrm{ord}\,(c_{\mathbb{O}}(f))) \\ c_{\mathbb{O}}(f) = \sum_{0 \leq n_\alpha \leq p^{e_\alpha}-1} c_{n_1 \cdots n_l} x_1^{n_1} \cdots x_l^{n_l} \\ \quad \text{with } c_{n_1 \cdots n_l} \in k[[x_{l+1}, \cdots, x_d]] \end{cases}$$

At the next $(j+1)$-th stage, we have

δ: an appropriate diff. operator of degree $t < \mathrm{ord}\,(c_{\mathbb{O}}(f))$

such that

$$\begin{cases} (\delta(c_{\mathbb{O}}(f)), \mathrm{ord}\,(c_{\mathbb{O}}(f)) - t) \in \mathbb{H}_i^{j+1} \subset \mathfrak{D}(\mathbb{I}_i^{j+1}) \\ (\delta(c_{\mathbb{O}}(f)), \mathrm{ord}\,(c_{\mathbb{O}}(f)) - t) \notin \mathbb{H}_i^j = \mathbb{H}. \end{cases}$$

$$\Longrightarrow \qquad \sigma_i^j > \sigma_i^{j+1}.$$

Mechanism to guarantee

$$\widetilde{\nu}_i^j \neq \infty \text{ or } 0 \Longrightarrow \sigma_i^j > \sigma_i^{j+1}.$$

is identical.

Naive candidate for "Cbc"

$$\mathrm{NaiveCbc}(\mathbb{I}_i^{j-1}) = G\left(\begin{array}{l} \mathfrak{D}(\mathbb{I}_i^{j-1}) \\ \cup \left\{ \begin{array}{l} (c_{\mathbb{O}}(f), \widetilde{\mu}_i^j \cdot a); \\ (f,a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0} \end{array} \right\} \\ \cup \left\{ \begin{array}{l} (c_{\mathbb{O}}(f), \widetilde{\nu}_i^j \cdot (p^{e_\beta} - t)); \\ (f, p^{e_\beta} - t) \in D^t\left(\mathfrak{D}(\mathbb{I}_i^{j-1})_{p^{e_\beta}}\right), \\ t \in \mathbb{Z}_{>0}, p^{e_\beta} - t > 0 \end{array} \right\} \end{array}\right)$$

Technical Requirements

○ "Cbc" should be independent of the choice of $\mathbb{H}$ and reg. sys. of parameters $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$.

○ "Cbc" should be an idealistic filtration of i.f.g. type.

Real Construction for "Cbc"

$$\mathrm{Cpc}(\mathbb{I}_i^{j-1}) = G\left[IL\left\{\mathfrak{D}\left(\mathrm{NaiveCpc}(\mathbb{I}_i^{j-1})\right)\right\}\right] \text{ where}$$

IL: the operator of taking the elements

at the Integral Level

Note: Description above is at the analytic level.

At the algebraic level ? See Lecture 5 by Kawanoue.

Lemma $\mathrm{Cbc}(\mathbb{I}_i^{j-1})$ **is independent of the choice of** $\mathbb{H}$ **and** $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$**.**

◦ **Boundary Modification**

$$\mathrm{Bd}\left(\mathrm{Comb}(\mathbb{I}_i^{j-1})\right)$$
$$= G\left(\mathrm{Comb}(\mathbb{I}_i^{j-1}) \cup \left\{\begin{array}{l} (f_\lambda, 1); \\ F_\lambda \subset E_{i,\mathrm{aged}}^{j-1} \end{array}\right\}\right)$$

Case: $\mathrm{inv}^{\leq j-1}(P_i) = \mathrm{inv}^{\leq j-1}(P_{i-1})$

MAIN POINTS

- Use of "History"
- Use of "Logarithmic Differentiation"
- Adjustment of the notion of LGS

Go back in "history" to year i_{aged}

when the value $\mathrm{inv}^{\leq j-1}(P_i)$ first started;

$$\begin{aligned}\mathrm{inv}^{\leq j-1}(P_i) &= \mathrm{inv}^{\leq j-1}(P_{i-1}) \\ &\cdots \\ &= \mathrm{inv}^{\leq j-1}(P_{i_{\mathrm{aged}}}) \\ &< \mathrm{inv}^{\leq j-1}(P_{i_{\mathrm{aged}}-1})\end{aligned}$$

Decomposition of the boundary

$$E_i^{j-1} = E_{i,\mathrm{young}}^{j-1} \sqcup E_{i,\mathrm{aged}}^{j-1}$$

where

$$\begin{cases} E_{i,\mathrm{young}}^{j-1} = \text{the collection of the exceptional divisors created after year } i_{\mathrm{aged}} \\ E_{i,\mathrm{aged}}^{j-1} = E_i^{j-1} \setminus E_{i,\mathrm{young}}^{j-1}. \end{cases}$$

Notion of LGS adjusted

$$\begin{cases} V & := \cap_{F_\lambda \subset E^{j-1}_{i,\text{young}}} F_\lambda \\ \mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i) & = \mathfrak{D}_{E^{j-1}_{i,\text{young}}}\text{-saturation of } \mathbb{I}^{j-1}_i \\ \left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V & = \text{ its restriction to } V \end{cases}$$

Lemma $\left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V$ is $\mathfrak{D}$-saturated.

$$\begin{array}{ccc} \mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i) & \overset{\text{surjection}}{\longrightarrow} & \left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V \\ & & \mathfrak{D}\text{-saturated} \\ \cup & & \cup \\ \mathbb{H} & \overset{\text{lift}}{\longrightarrow} & \mathbb{H}_V;\text{ an LGS} \end{array}$$

Definition $\mathbb{H}$ is an LGS of $\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)$.

$$\begin{cases} \sigma_V & := \sigma\left(\left\{\mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i)\right\}|_V\right) \\ c & := \operatorname{codim}_{W_i} V \\ \sigma^j_{i,\log} & := \sigma_V + c^{\mathbb{Z}_{\geq 0}} = (\sigma_{V,e} + c)_{e \in \mathbb{Z}_{\geq 0}} \end{cases}$$

Lemma (Yet to be checked)

$$\sigma^j_{i,\log} = \sigma^j_{i-1,\log} = \sigma^j_{i-1}.$$

$$\boxed{(\sigma_i^j, \widetilde{\mu}_i^j, \widetilde{\nu}_i^j, s_i^j)}$$

Set

$$\begin{cases} \sigma_i^j &= \sigma_{i,\log}^j = \sigma_{i-1,\log}^j = \sigma_{i-1}^j \\ \widetilde{\mu}_i^j &= \mu_{\mathbb{H}, E_{i,\text{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ &= \mu_{\mathbb{H}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) - \sum_{F_\lambda \subset E_{i,\text{young}}^{j-1}} \mu_\lambda \\ \widetilde{\nu}_i^j &= \nu_{\mathbb{H}, E_{i,\text{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ &= \nu_{\mathbb{H}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) - \sum_{F_\lambda \subset E_{i,\text{young}}^{j-1}} \nu_\lambda \\ s_i^j &= \#\text{ of irred. comp. in } E_{i,\text{aged}}^{j-1} \\ & \quad (\text{passing through } P_i) \end{cases}$$

where

$$\begin{cases} \mu_{\mathbb{H}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right) \\ \quad = \inf\left\{\operatorname{ord}(c_{\mathbb{O}}(f))/a; (f,a) \in \mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}),\right. \\ \qquad\qquad \left. a \in \mathbb{Z}_{>0}, f = \sum c_B(f) H^B\right\} \\ \mu_\lambda = \inf\left\{n/a; c_{\mathbb{O}}(f) \text{ divisible by } f_\lambda^n,\right. \\ \qquad\qquad (f,a) \in \mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}, \\ \qquad\qquad \left. f = \sum c_B(f) H^B\right\} \end{cases}$$

and where

$$\begin{cases} \nu_{\mathbb{H}}\left(\mathfrak{D}_{E^{j-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{j-1})\right) \\ \quad = \inf\{\mathrm{ord}\left(c_{\mathbb{O}}(f)\right)/(p^{e_\beta}-t); \\ \qquad (f, p^{e_\beta}-t) \in D^t_{E^{j-1}_{i,\mathrm{young}}}\left(\mathfrak{D}_{E^{j-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{j-1})_{p^{e_\beta}}\right), \\ \qquad t \in \mathbb{Z}_{>0}, p^{e_\beta}-t>0, f=\sum c_B(f)H^B\} \\ \nu_\lambda = \inf\{n/(p^{e_\beta}-t); c_{\mathbb{O}}(f) \text{ divisible by } f_\lambda^n, \\ \qquad (f, p^{e_\beta}-t) \in D^t_{E^{j-1}_{i,\mathrm{young}}}\left(\mathfrak{D}_{E^{j-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{j-1})\right), \\ \qquad t \in \mathbb{Z}_{>0}, p^{e_\beta}-t>0, f=\sum c_B(f)H^B\} \end{cases}$$

Lemma

$\mu_{\mathbb{H},E^{j-1}_{i,\mathrm{young}}}\left(\mathfrak{D}_{E^{j-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{j-1})\right)$ & $\nu_{\mathbb{H},E^{j-1}_{i,\mathrm{young}}}\left(\mathfrak{D}_{E^{j-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{j-1})\right)$ **are independent of the choice of $\mathbb{H}$ (or $\mathbb{H}_V$).**

Therefore, $\widetilde{\mu}_i^j$ & $\widetilde{\nu}_i^j$ are well-defined.

$$\boxed{(\mathbb{I}_i^j, \boldsymbol{E}_i^j)}$$

$$\begin{cases} \mathbb{I}_i^j & = \mathbf{Bd}\left(\mathbf{Comb}(\mathbb{I}_i^{j-1})\right) \\ \textsf{with} & \mathbf{Comb}(\mathbb{I}_i^{j-1}) \\ & = \begin{cases} \mathbf{Cbc}(\mathbb{I}_i^{j-1}) \\ \quad \textsf{if } (\widetilde{\boldsymbol{\mu}}_i^j, \widetilde{\boldsymbol{\nu}}_i^j) < (\widetilde{\boldsymbol{\mu}}_{i-1}^j, \widetilde{\boldsymbol{\nu}}_{i-1}^j) \\ \mathfrak{D}_{E_{i,\mathrm{young}}}^{j-1}\left(\boldsymbol{\pi}^\sharp(\mathbf{Comb}(\mathbb{I}_{i-1}^{j-1}))\right) \\ \quad \textsf{if } (\widetilde{\boldsymbol{\mu}}_i^j, \widetilde{\boldsymbol{\nu}}_i^j) = (\widetilde{\boldsymbol{\mu}}_{i-1}^j, \widetilde{\boldsymbol{\nu}}_{i-1}^j) \end{cases} \\ \boldsymbol{E}_i^j & = \boldsymbol{E}_i^{j-1} \setminus \boldsymbol{E}_{i,\mathrm{aged}}^{j-1}. \end{cases}$$

Description of "Cbc" in case $(\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) < (\widetilde{\mu}_{i-1}^j, \widetilde{\nu}_{i-1}^j)$

Consider

BlackBox $= \mathfrak{D}_{E_{i,\text{young}}^{j-1}}$ **-saturation of**

$$G\left(\begin{array}{l}
\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}) \cup \\
\Big\{(c_{\mathbb{O}}(f) \otimes_k \{(\prod f_\lambda^{\mu_\lambda})^a\}^{-1}, \widetilde{\mu}_i^j \cdot a); \\
\qquad (f, a) \in \mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}, \\
\qquad f = \sum c_B(f) H^B \Big\} \cup \\
\Big\{(c_{\mathbb{O}}(f) \otimes_k \{(\prod f_\lambda^{\nu_\lambda})^{(p^{e_\beta} - t)}\}^{-1}, \\
\qquad\qquad\qquad \widetilde{\nu}_i^j \cdot (p^{e_\beta} - t)); \\
(f, p^{e_\beta} - t) \in D^t_{E_{i,\text{young}}^{j-1}}\left(\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1})\right), \\
\qquad t \in \mathbb{Z}_{>0}, p^{e_\beta} - t > 0, \\
\qquad f = \sum c_B(f) H^B \Big\} \cup \\
\Big\{(f_\lambda \otimes (f_\lambda^{\frac{q}{r_\lambda}})^{-1}, 0); \\
\qquad q = 1, \cdots, r_\lambda - 1, \\
\qquad r_\lambda; \text{ the common denominator of } \mu_\lambda \ \& \ \nu_\lambda, \\
\qquad F_\lambda \subset E_{i,\text{young}}^{j-1} \Big\}
\end{array}\right)$$

$IPIL$; **the operator to take the elements being at the integral levels as well as having only integral powers in $\otimes_k$__.**

$IPIL(\boxed{\textsf{BlackBox}})$

$\downarrow$ **Eliminate $\otimes_k$ by turning it into the real multiplication**

Image

$$\mathrm{Cbc}(\mathbb{I}_i^{j-1}) := \mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathrm{Image}).$$

Lemma $\mathrm{Cbc}(\mathbb{I}_i^{j-1})$ **is independent of the choice of $\mathbb{H}$ (or $\mathbb{H}_V$) and $(x_1, \cdots, x_l, x_{l+1}, \cdots, x_d)$.**

MAIN MECHANISM OF INDUCTION

Mechanism to guarantee

$$\widetilde{\mu}_i^j \neq \infty \text{ or } 0 \Longrightarrow \sigma_i^j > \sigma_i^{j+1}.$$

Take

$$(f, a) \in \mathfrak{D}(\mathbb{I}_i^{j-1}), a \in \mathbb{Z}_{>0}$$

with

$$\operatorname{ord}\left(c_{\mathbb{O}}(f)\right)/a \underset{\text{exactly}}{=} \widetilde{\mu}_i^j.$$

Case: $a \cdot \mu_\lambda \in \mathbb{Z}_{\geq 0}\ \forall \lambda$.

In this case, we have

$$c_{\mathbb{O}}(f) \cdot \left\{\left(\prod f_\lambda^{\mu_\lambda}\right)^a\right\}^{-1} \in \widehat{\mathcal{O}_{W_i,P_i}}.$$

and

$$\left(c_{\mathbb{O}}(f) \cdot \left\{\left(\prod f_\lambda^{\mu_\lambda}\right)^a\right\}^{-1}, \widetilde{\mu}_i^j \cdot a\right) =$$
$$\left(c_{\mathbb{O}}(f) \cdot \left\{\left(\prod f_\lambda^{\mu_\lambda}\right)^a\right\}^{-1}, \operatorname{ord}\left(c_{\mathbb{O}}(f) \cdot \left\{\left(\prod f_\lambda^{\mu_\lambda}\right)^a\right\}^{-1}\right)\right)$$

Observe

$$c_{\mathbb{O}}(f) \cdot \left\{\left(\prod f_\lambda^{\mu_\lambda}\right)^a\right\}^{-1} = \sum_{o \leq n_\alpha \leq p^{e_\alpha}-1} b_{n_1 \cdots n_l} x_1^{n_1} \cdots x_l^{n_l}$$

with $b_{n_1 \cdots n_l} \in k[[x_{l+1}, \cdots, x_d]]$.

Note: We take $(x_{l+1}, \cdots, x_d)$ to contain

$$\{f_\lambda; F_\lambda \subset E_{i,\text{young}}^{j-1}\}.$$

At the next $(j+1)$-th stage, taking the $\mathfrak{D}$-saturation $\mathfrak{D}(\mathbb{I}_i^j)$, we create a new element in LGS.

$$\Longrightarrow \qquad \sigma_i^j > \sigma_i^{j+1}.$$

Case: $\exists\lambda$ (say, λ_o) s.t. $a \cdot \mu_\lambda \notin \mathbb{Z}_{\geq 0}$.

In this case, $\exists n \in \mathbb{Z}_{>0}$ s.t.

$$\left[c_{\mathbb{O}}(f) \cdot \{(\prod f_\lambda^{\mu_\lambda})^a\}^{-1}\right]^n \in \widehat{\mathcal{O}_{W_i,P_i}}$$

and

divisible by $f_{\lambda_o}^{n_o}$ for some $n_o \in \mathbb{Z}_{>0}$,

and that

$$(\left[c_{\mathbb{O}}(f) \cdot \{(\prod f_\lambda^{\mu_\lambda})^a\}^{-1}\right]^n, n \cdot \widetilde{\mu}_i^j \cdot a) =$$

$$(\underbrace{\left[c_{\mathbb{O}}(f) \cdot \left\{(\prod f_\lambda^{\mu_\lambda})^a\right\}^{-1}\right]^n}_{\text{divisible by } f_{\lambda_o}^{n_o}} = *, \mathrm{ord}\,(*))$$

At the next $(j+1)$-th stage, taking the $\mathfrak{D}$-saturation $\mathfrak{D}(\mathbb{I}_i^j)$, we create a new element of the form $(f_{\lambda_o}^{m_o}, m_o)$ in LGS.

$$\Longrightarrow \qquad \sigma_i^j > \sigma_i^{j+1}.$$

Mechanism to guarantee $\widetilde{\nu}_i^j \neq \infty$ **or** $0 \Longrightarrow \sigma_i^j > \sigma_i^{j+1}$.

is identical.

3 Termination in the horizontal direction (revisited)

$$(\sigma_i^1, t_i^0) > (\sigma_i^2, t_i^1) > \cdots$$

In fact, we have

$$(\sigma_i^j, t_i^{j-1}) > (\sigma_i^{j+1}, t_i^j),$$

since

$$\begin{cases} (\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) \neq (\infty, \infty) \text{ or } (0,0) \\ \qquad \rightarrow \sigma_i^j > \sigma_i^{j+1} \\ (\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) = (\infty, \infty) \text{ or } (0,0) \ \& \ s_i^j \neq 0 \\ \qquad \rightarrow \sigma_i^j \geq \sigma_i^{j+1} \ \& \ t_i^{j-1} > t_i^j \\ (\widetilde{\mu}_i^j, \widetilde{\nu}_i^j) = (\infty, \infty) \text{ or } (0,0) \ \& \ s_i^j = 0 \\ \qquad \rightarrow \text{End of weaving} \\ \qquad\qquad \text{with } (\sigma_i^j, \infty, \infty, 0) \text{ or} \\ \qquad\qquad\qquad (\sigma_i^j, 0, 0, 0, \Gamma) \\ \qquad \text{except when} \ \text{ with } (\sigma_i^j, 0, 0, 0, \odot) \\ \qquad \rightarrow \text{Continue weaving } \sigma_i^j > \sigma_i^{j+1} \end{cases}$$

$\{(\sigma, t)\}$ **satisfies the descending chain condition.**

$\Longrightarrow$

In a fixed year i**, weaving of the strand "**inv**" ends after finitely many years.**

Main mechanism of induction on σ **(and** t**)**

4 Choice of the center (revisited); how to end weaving of the strand

Choose $C_i = \mathrm{Supp}(\mathbb{I}_i^{m_i})$.

Case: $(\mathbb{I}_i^{m_i-1}, E_i^{m_i-1}) \qquad (\mathbb{I}_i^{m_i}, E_i^{m_i})$

$$(\sigma_i^{m_i}, \infty, \infty, 0)$$

Note: When $\widetilde{\mu}_i^j = \infty$**, the invariant** $\widetilde{\nu}_i^j$ **is necessarily equal to** ∞**, i.e.,** $\widetilde{\nu}_i^j = \infty$**.**

(i) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ **nonsingular.**

$\because$

$$\left\{\begin{array}{l} \mathbb{I}_i^{m_i} = \mathfrak{D}(\mathbb{I}_i^{m_i-1}) \\ \quad ;\ \mathfrak{D}\text{-saturated} \\ \mu_{\mathbb{H}} = \infty \end{array}\right. \quad \underset{\text{Nonsingularity Principle}}{\Longrightarrow} \quad \mathrm{Supp}(\mathbb{I}_i^{m_i}) \text{ nonsingular.}$$

Note: Even when

$$\begin{array}{ll} \mathrm{inv}^{\leq m_i-1}(P_{i_{\mathrm{aged}}}) & (\sigma^{m_i}_{i_{\mathrm{aged}}}, 0, 0, s^{m_i}_{i_{\mathrm{aged}}} \neq 0) \\ \quad \| & \quad \cdots \\ \quad \cdots & \quad \cdots \\ \quad \| & \quad \cdots \\ \mathrm{inv}^{\leq m_i-1}(P_i) & (\sigma^{m_i}_{i_{\mathrm{aged}}}, 0, 0, s^{m_i}_{i} = 0) \end{array}$$

and hence when $\mathbb{I}_i^{m_i} = \mathfrak{D}_{E^{m_i-1}_{i,\mathrm{young}}}(\mathbb{I}_i^{m_i-1})$ **is only** $\mathfrak{D}_{E^{m_i-1}_{i,\mathrm{young}}}$**-saturated a priori, we see from the construction that** $\mathbb{I}_i^{m_i}$ **is** $\mathfrak{D}$**-saturated with** $\mu = \infty$**.**

(ii) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ **transversal to** E_i**.**

$\because$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp E_i^{m_i} = E_i^{m_i-1} \setminus E^{m_i-1}_{i,\mathrm{aged}} = E^{m_i-1}_{i,\mathrm{young}}$$

by construction

and

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \subset \cap_{F_\lambda \subset E^0_{i,\mathrm{aged}} \cup \cdots \cup E^{m_i-1}_{i,\mathrm{aged}}} F_\lambda$$

by construction of "Bd"

$\therefore$

$$\mathrm{Supp}(\mathbb{I}_i^{m_i}) \perp \underbrace{\left(E^0_{i,\mathrm{aged}} \cup \cdots \cup E^{m_i-1}_{i,\mathrm{aged}}\right) \cup E^{m_i-1}_{i,\mathrm{young}}}_{\substack{\| \\ E_i}}$$

Case: $(\mathbb{I}_i^{j-1}, E_i^{j-1}) \quad (\mathbb{I}_i^j, E_i^j)$

$(\sigma_i^j, 0, 0, 0)$

This should always be the MONOMIAL CASE.

However, we have the following example.

Example

$\boxed{\mathrm{char}(k) = 5}$

$\mathbb{I}_i^{j-1}$ $\quad (x^5 + f^4 y, 5)$

$(f^4, 4)$

$$\mathfrak{D}_{E_{i,\text{young}}^{j-1}}(\mathbb{I}_i^{j-1}) \quad \frac{\partial}{\partial y}(x^5 + f^4 y, 5) = (f^4, 4)$$

$$E_{i,\text{young}}^{j-1} = \{F\}, F = \{f = 0\}$$

$$\mathbb{H} = \{(x^5 + f^4 y, 5)\}$$

$$\begin{cases} \mu_F = 4 \text{ divisible mod } \mathbb{H} \text{ by } f^4 \text{ per level} \\ \widetilde{\mu}_i^j = 0 \\ \nu_F = 4 \text{ divisible mod } \mathbb{H} \text{ by } f^4 \text{ per level} \\ \widetilde{\nu}_i^j = 0 \end{cases}$$

Say $E_{i,\text{aged}}^{j-1} = \emptyset$.

Then we should be in the MONOMIAL CASE,

since we have $(\sigma_i^j, \widetilde{\mu}_i^j, \widetilde{\nu}_i^j, s_i^j) = (\sigma_i^j, 0, 0, 0)$.

However, we can NOT take the expected center

$$\mathrm{Supp}(\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1})|_F) \not\subset \mathrm{Supp}(\mathfrak{D}_{E_{i,\mathrm{young}}^{j-1}}(\mathbb{I}_i^{j-1}))$$

defined by	**defined by**
(x, f)	(x, f, y)

Observation: When $\sum \mu_\lambda = 1$ & $\sum \nu_\lambda = 1$, the expected center may NOT be included in the support of the idealistic filtration.

This observation leads us to the classification into the following two subcases.

Subcase: $\sum_{F_\lambda \subset E^{j-1}_{i,\text{young}}} \mu_\lambda = 1$

In this subcase,

weaving of the strand does NOT end

at the j-th stage.

We replace

$(\sigma^j_i, 0, 0, 0)$ **the original j-th unit**

with

$(\sigma^j_i, 0, 0, 0, \odot)$ **the new j-th unit.**

We leave

$(\mathbb{I}^j_i, E^j_i)$ **the original j-th modification**

as it is.

From the subcase assumption,

$$\exists(\prod f_\lambda^{n_\lambda}, \sum n_\lambda) \in \mathfrak{D}_{E^{j-1}_{i,\text{young}}}(\mathbb{I}^{j-1}_i) \subset \mathbb{I}^j_i.$$

$\longrightarrow$

At the next $(j+1)$-th stage,

taking the $\mathfrak{D}$-saturation (when $\mathrm{inv}^{\leq j}(P_i) < \mathrm{inv}^{\leq j}(P_{i-1})$),

we create a new element in LGS

$\Longrightarrow \qquad \sigma^j_i > \sigma^{j+1}_i$.

Note: When $\mathrm{inv}^{\leq j}(P_i) = \mathrm{inv}^{\leq j}(P_{i-1})$, we go back in history to the time when the value $\mathrm{inv}^{\leq j}(P_i)$ first started (i.e., year i_{aged}). The new element created at that time survives to year i.

$\Longrightarrow \qquad \sigma_i^j > \sigma_i^{j+1}$.

Subcase: $\sum_{F_\lambda \subset E_{i,\text{young}}^{j-1}} \mu_\lambda > 1$

This subcase is **GENUINE MONOMIAL CASE**

via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method (and we set $j = m_i$).

We introduce the invariant $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$

where

$$\begin{cases} \Gamma_1 = -\min\{n; \exists(\lambda_1, \cdots, \lambda_n) \\ \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i - 1} \\ \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, \\ \qquad \nu_{\lambda_1} + \cdots + \nu_{\lambda_n} > 1, \\ \quad P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \end{cases}$$

Note: From the subcase assumption

$\sum_{F_\lambda \subset E_{i,\text{young}}^{j-1}} \mu_\lambda > 1$ and from the general fact

$\nu_\lambda \geq \mu_\lambda \; \forall \lambda$, it follows that there exists at least

one $(\lambda_1, \cdots, \lambda_n)$ satisfying the conditions mentioned

in the definition of Γ_1.

$$
\begin{cases}
\Gamma_2 & = \max\{\mu_{\lambda_1} + \cdots + \mu_{\lambda_n}; \exists(\lambda_1, \cdots, \lambda_n) \\
& \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\
& \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, \\
& \qquad \nu_{\lambda_1} + \cdots + \nu_{\lambda_n} > 1, \\
& \quad P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \\
& \quad -n = \Gamma_1\} \\
\Gamma_3 & = \max\{\nu_{\lambda_1} + \cdots + \nu_{\lambda_n}; \exists(\lambda_1, \cdots, \lambda_n) \\
& \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\
& \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, \\
& \qquad \nu_{\lambda_1} + \cdots + \nu_{\lambda_n} > 1, \\
& \quad P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \\
& \quad -n = \Gamma_1, \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} = \Gamma_2\} \\
\Gamma_4 & = \max\{(\lambda_1, \cdots, \lambda_n); \\
& \quad \text{with } F_{\lambda_1}, \cdots, F_{\lambda_n} \subset E_{i,\text{young}}^{m_i-1} \\
& \quad \text{s.t. } \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} \geq 1, \\
& \qquad \nu_{\lambda_1} + \cdots + \nu_{\lambda_n} > 1, \\
& \quad P_i \in F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}\} \\
& \quad -n = \Gamma_1, \mu_{\lambda_1} + \cdots + \mu_{\lambda_n} = \Gamma_2 \\
& \quad \nu_{\lambda_1} + \cdots + \nu_{\lambda_n} = \Gamma_3\}
\end{cases}
$$

We replace

$(\sigma_i^{m_i}, 0, 0, 0)$ **the original** m_i**-th unit**

with

$(\sigma_i^{m_i}, 0, 0, 0, \Gamma)$ **the new** m_i**-th unit.**

We also replace

$$\mathbb{I}_i^{m_i} = \mathrm{Bd}(\mathrm{Comb}(\mathbb{I}_i^{m_i})) = \mathfrak{D}_{E_{i,\text{young}}^{m_i-1}}(\mathbb{I}_i^{m_i})$$

the original m_i**-th modification**

with

$$\mathbb{I}_i^{m_i} = G\begin{pmatrix} \mathfrak{D}_{E_{i,\text{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1}) \cup \\ \{(f_{\lambda_1}, 1), \cdots, (f_{\lambda_n}, 1); (\lambda_1, \cdots, \lambda_n) = \Gamma_4\} \end{pmatrix}$$

the new m_i**-th modification.**

(i) $\mathrm{Supp}(\mathbb{I}_i^{m_i})$ **nonsingular.**

$\because$

$$\mathrm{Supp}\left(\mathfrak{D}_{E_{i,\text{young}}^{m_i-1}}(\mathbb{I}_i^{m_i-1})|_Z)\right) \underset{\text{from the definition of } \Gamma}{\overset{\text{obvious}}{\underset{\subset}{\overset{\supset}{=}}}} \mathrm{Supp}(\mathbb{I}_i^{m_i})$$

where

$Z = F_{\lambda_1} \cap \cdots \cap F_{\lambda_n}$ **with** $(\lambda_1, \cdots, \lambda_n) = \Gamma_4$.

$$\mathfrak{D}_{E^{m_i-1}_{i,\text{young}}}(\mathbb{I}^{m_i-1}_i)|_Z$$

;$\mathfrak{D}$-saturated & $\mu = \infty$

$\underset{\text{nonsingularity proinciple}}{\Longrightarrow}$

$$\text{Supp}(\mathbb{I}^{m_i}_i) = \text{Supp}\left(\mathfrak{D}_{E^{m_i-1}_{i,\text{young}}}(\mathbb{I}^{m_i-1}_i)|_Z\right)$$

;nonsingular.

(ii) $\text{Supp}(\mathbb{I}^{m_i}_i)$ transversal to E_i.

$\because$

$$\text{Supp}(\mathbb{I}^{m_i}_i) \perp E^{m_i}_i = E^{m_i-1}_i \setminus E^{m_i-1}_{i,\text{aged}} = E^{m_i-1}_{i,\text{young}}$$

by construction

and

$$\text{Supp}(\mathbb{I}^{m_i}_i) \subset \cap_{F_\lambda \subset E^0_{i,\text{aged}} \cup \cdots \cup E^{m_i-1}_{i,\text{aged}}} F_\lambda$$

by construction of "Bd"

$\therefore$

$$\text{Supp}(\mathbb{I}^{m_i}_i) \perp \underbrace{\left(E^0_{i,\text{aged}} \cup \cdots \cup E^{m_i-1}_{i,\text{aged}}\right) \cup E^{m_i-1}_{i,\text{young}}}_{\substack{\| \\ E_i}}$$

5 Termination in the vertical direction (revisited)

Crucial Claim **The strand of invariants "inv" never increases after blowup, i.e.,**

$$\text{inv}(P_i) \leq \text{inv}(P_{i-1}).$$

We have yet to check this **Crucial Claim** **for the algorithm via $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-method !**

Claim **The strand of invariants "inv" actually strictly decreases after blowup, i.e.,**

$$\mathrm{inv}(P_i) < \mathrm{inv}(P_{i-1}).$$

Proof of the claim using Crucial Claim

Observe

(i) $P_i \in \mathrm{Supp}(\mathbb{I}_i^j)\ \forall j$

(ii) $\mathrm{inv}^{\leq j}(P_i) = \mathrm{inv}^{\leq j}(P_{i-1})$

$$\Longrightarrow \mathfrak{D}_{E^j_{i,\mathrm{young}}}(\mathbb{I}_i^j) = \mathfrak{D}_{E^j_{i,\mathrm{young}}}(\pi^\sharp(\mathbb{I}_{i-1}^j)).$$

Suppose

$$\begin{array}{ccc} \mathrm{inv}(P_i) & = & \mathrm{inv}(P_{i-1}) \\ \| & & \| \\ \mathrm{inv}^{\leq m_i}(P_i) & & \mathrm{inv}^{\leq m_{i-1}}(P_{i-1}) \end{array} \quad \text{with } m_i = m_{i-1}.$$

Then by (ii) with $j = m_i$, we have

$$\mathfrak{D}_{E^{m_i}_{i,\mathrm{young}}}(\mathbb{I}_i^{m_i}) = \mathfrak{D}_{E^{m_i}_{i,\mathrm{young}}}(\pi^\sharp(\mathbb{I}_{i-1}^{m_i})).$$

On the other hand,

$$\mathrm{Supp}\left(\mathfrak{D}_{E^{m_i}_{i,\mathrm{young}}}(\pi^\sharp(\mathbb{I}^{m_i}_{i-1}))\right) = \mathrm{Supp}\left(\pi^\sharp(\mathbb{I}^{m_i}_{i-1})\right) = \emptyset,$$

since

the last (m_i-th) modification has

the distinguished feature that

its transformation after blowup has NO support.

But then by (i)

$$P_i \in \mathrm{Supp}\left(\mathbb{I}^{m_i}_{i}\right) = \mathrm{Supp}\left(\mathfrak{D}_{E^{m_i}_{i,\mathrm{young}}}(\mathbb{I}^{m_i}_{i})\right) = \emptyset,$$

a contradiction !

Last Claim The strictly decreasing sequence

$$\mathrm{inv}(P_0) > \mathrm{inv}(P_1) > \cdots$$
$$\cdots \quad > \mathrm{inv}(P_{i-1}) > \mathrm{inv}(P_i) > \cdots$$

stops after finitely many years.

Caution No descending chain condition

for the value set of "inv", since

denominators of $\widetilde{\mu}$ and Γ_2, Γ_3 in $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$

are NOT a priori bounded.

Proof of the last claim

Suppose inductively "$\mathrm{inv}^{\leq j}$" stabilizes, i.e.,

$\exists i_j$ s.t. $\mathrm{inv}^{\leq j}(P_i) = \mathrm{inv}^{\leq j}(P_{i_j}) \ \forall i \geq i_j$.

Then

$$\begin{aligned}
&\mathfrak{D}_{E^j_{i,\mathrm{young}}}(\mathbb{I}^j_i)\\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp(\mathbb{I}^j_{i-1})\right)\\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\left(\mathfrak{D}_{E^j_{i-1,\mathrm{young}}}(\mathbb{I}^j_{i-1})\right)\right)\\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\left(\mathfrak{D}_{E^j_{i-1,\mathrm{young}}}\left(\pi^\sharp(\mathbb{I}^j_{i-2})\right)\right)\right)\\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\pi^\sharp(\mathbb{I}^j_{i-2})\right)\cdots\\
&= \mathfrak{D}_{E^j_{i,\mathrm{young}}}\left(\pi^\sharp\pi^\sharp\cdots\pi^\sharp(\mathbb{I}^j_{i_j})\right)
\end{aligned}$$

$\Longrightarrow$

Denominators of $\widetilde{\mu}_i^j$ are uniformly bounded by the number determined by the levels of the generators of $\mathbb{I}_{i_j}^j$.

(Similarly denominators of Γ_2 & Γ_3 are uniformly bounded.)

$\Longrightarrow$

"$\mathrm{inv}^{\leq j+1}$" stabilizes after finitely many years.

$\Longrightarrow$

"inv" stabilizes after finitely many years. Q.E.D.

Note: We can NOT extend "inv" infinitely in the horizontal direction (i.e., can NOT increase "j" infinitely), since the set $\{(\sigma, t)\}$ satisfies the descending chain condition !