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タイトル TITLE	**Eigenvalues of elliptic operators and geometric applications**		
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The purpose of this talk is to present a certain method of obtaining upper estimates of eigenvalues of Schrödinger type operators on Riemannian manifolds, which was introduced in the paper

Grigor'yan A., Netrusov Yu., Yau S.-T. *Eigenvalues of elliptic operators and geometric applications*, Surveys in Differential Geometry **IX** (2004), pp.147-218.

The core of the method is the ability to choose a given number of disjoint sets in X such that one can control simultaneously their volumes from below and their capacities from above. The main technical tool for that is the following theorem.

Theorem 1 *Let (X,d) be a metric space satisfying the following covering property: there exists a constant N such that any metric ball of radius r in X can be covered by at most N balls of radii $r/2$. Let all metric balls in X be precompact sets, and let ν be a non-atomic Radon measure on X. Then, for any positive integer k, there exists a sequence $\{A_i\}_{i=1}^k$ of k annuli in X such that the annuli $\{2A_i\}_{i=1}^k$ are disjoint and, for any $i=1,2,...,k$,*

$$\nu\left(A_i\right) \geq c\frac{\nu\left(X\right)}{k},$$

where $c=c\left(N\right)>0$.

The following results are obtained using this theorem.

Theorem 2 *Let X be a complete Riemannian manifold and μ be its Riemannian measure. Assume that, for some constants N and M, the following is true:*

(i) any geodesic ball $B\left(x,r\right)$ in X can be covered by at most N balls of radii $r/2$;

(ii) for all $x\in X$ and $r>0$, we have $\mu\left(B\left(x,r\right)\right)\leq Mr^2$.

Then, for any $q\in C\left(X\right)$, the number of negative eigenvalues of the operator $-\Delta-q$ admits the estimate

$$Neg\left(-\Delta-q\right)\geq c\int_X\left(\delta q_+-q_-\right)d\mu, \tag{1}$$

where $\delta=\delta\left(N\right)\in(0,1)$, and $c=c\left(N,M\right)>0$.

For example, this result applies for Schrödinger operators in $\mathbb{R}^2$. However, in $\mathbb{R}^n$, $n>2$, the estimate (**??**) fails. It is conjectured that (**??**) holds with $\delta=1$ so that $\left(\delta q_+-q_-\right)$ can be replaced by q.

Theorem 3 *Let X be a compact oriented Riemann surface of genus g, equipped with some Riemannian metric. Then, for any $q\in C\left(X\right)$,and, for any $k=1,2,...,$*

$$\lambda_k(-\Delta-q)\leq\frac{C\left(g+1\right)\left(k-1\right)-\int_X\left(\delta q_+-q_-\right)d\mu}{\sqrt{\delta}\mu(X)}.$$

where $C>0$ and $0<\delta<1$ are absolute constants.

Theorem 4 *For any connected complete oriented minimal surface X embedded in $\mathbb{R}^3$, we have*

$$ind(X)\geq c\sqrt{K_{total}\left(X\right)},$$

where $ind(X)$ is the stability index of X, $K_{total}\left(X\right)$ is the absolute value of the total curvature of X, and $c>0$ is an absolute constant.