Self-similar Solutions to Nonlinear Wave Equations

(Colloquium Lecture, December 8, 2004)

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This talk is mainly based on recent joint work with Jun Kato [2,3,4]. We consider the Cauchy problem for nonlinear wave equations of the form

$$\Box u = f(u), \tag{NLW}$$

where u is a complex-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ with $n \geq 2$, $\Box = \partial_t^2 - \Delta$ is the d'Alembertian, $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian in $\mathbb{R}^n, \partial_j = \partial/\partial x_j$, and $f(u)$ is a homogeneous function of degree $p > 1$, e.g., $f(u) = \pm\ |u|^{p-1}u, \pm u^p, \pm\ |u|^p$, etc. There is a large literature on the Cauchy problem and asymptotic behavior of solutions to NLW in function spaces built over the Lebesgue spaces. Self-similar solutions form an important class of solutions to nonlinear evolution equations with scaling structure, such as nonlinear heat equations, Navier-Stokes equations, nonlinear Schrödinger equations, generalized KdV equations, porus medium equation, mean curvature flow equation, etc. Due to propagation of singularity along the light cone and to an oscillatory phenomenon, self-similar solutions to NLW, however, do not fit into any framework of the available theories above. Recently, two methods were proposed by Pecher [7,8] in three space dimensions. One is based on the $L^{r'} \to L^r$ estimates on the free wavefunction. The other is based on the norm due to F. John taking space-time behavior of the free wavefunction into account. The former is naturally generalized to higher dimensions by Ribaud and Youssfi [9] for p greater than Mochizuki-Motai's critical exponent $p_1(n)$ and less than Sobolev's critical exponent $1 + 4/(n-2)$. The latter is naturally generalized to two and three dimensions by Hidano [1] for p greater than John-Strauss' critical exponent $p_0(n)$ and less than the conformal exponent $1 + 4/(n-1)$. Here we note that $p_0(n) < p_1(n) < 1 + 4/(n-1) < 1 + 4/(n-2)$.

In this talk, we present the third method to study self-similar solutions to NLW. We introduce Strichartz estimates on weighted Lorentz spaces over space-time $\mathbb{R} \times \mathbb{R}^n$. The main result is the existence and uniqueness of self-similar solutions to NLW with $p_0(n) < p < 1 + 4/(n-1)$ for small and radial Cauchy data.

Finally, we present an improvement to non-radial Cauchy data up to five space dimensions [5] based on the expansion by spherical harmonics, inspired by [6].

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