P-ADIC COHOMOLOGY THEORIES WITH A VIEW TOWARD ℓ-ADIC APPLICATIONS

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1. BIG PICTURE

Let k be a field of characteristic $p \geq 0$ with algebraic closure $\overline{k}$. Fix a smooth, geometrically connected, separated curve of finite type X over k and denote with $X_{\overline{k}}$ the base change of X to $\overline{k}$. Let $|X|$ be the set of closed points. For $x \in |X|$, write $k(x)$ for its residue field. For any integer $d \geq 1$, let $X(\leq d)$ be set of all $x \in |X|$ such that $[k(x) : k] \leq d$. Fix a smooth proper morphism $Y \to X$ of smooth k-variety. For $x \in X$ denote with Y_x the fibre of f in x and with $Y_{\overline{x}}$ the base change of Y_x to the algebraic closure of $k(x)$. Write η for the generic point of X and fix $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. By smooth-proper base change theorem in étale cohomology we have, for every $\ell \neq p$, a representation

$$\rho_\ell : \pi_1(X) \to GL(H^i(Y_{\overline{\eta}}, \mathbb{Z}_\ell(j)))$$

of the étale fundamental group of X. This is obtained by the natural representation

$$\pi_1(k(\eta)) \to GL_r(H^i(Y_{\overline{\eta}}, \mathbb{Z}_\ell(j))$$

that factors trough the canonical surjection

$$\pi_1(k(\eta)) \to \pi_1(X)$$

For every $x \in |X|$, by the functoriality of the étale fundamental group, we have a map

$$\pi_1(x) := \pi_1(Spec(k(x))) \to \pi_1(X)$$

and hence a representation

$$\rho_{\ell,x} : \pi_1(x) \to \pi_1(X) \to GL_r(\mathbb{Z}_l)$$

By smooth proper base change, for every, $\rho_{\ell,x}$ is isomorphic to the natural Galois representation

$$\pi_1(Spec(k(x))) \to GL(H^i(Y_{\overline{x}}, \mathbb{Z}_\ell(j)))$$

Write:

$$\Pi_\ell = \rho_\ell(\pi_1(X)) \quad \Pi_{\ell,x} = \rho_{\ell,x}(\pi_1(x))$$

and consider the inclusion:

$$\Pi_{\ell,x} \subseteq \Pi_\ell$$

We have the following:

Fact 1. *Assume that k is infinite finitely generated. Then:*

1. *There exists a $d \geq 1$ such that $\Pi_{\ell,x} = \Pi_\ell$ for infinitely many $x \in X(\leq d)$*
2. *Assume X is a curve, $p = 0$ and $d \geq 1$. Then $\Pi_{\ell,x} \subseteq \Pi_\ell$ is open for all but finitely many $x \in X(\leq d)$ and of index bounded independently of $x \in X(\leq d)$*
3. *Assume X is a curve and $p > 0$. Then $\Pi_{\ell,x} \subseteq \Pi_\ell$ is open for all but finitely many $x \in X(\leq 1)$ and of index bounded independently of $x \in X(\leq 1)$*

The proof of these facts relies heavily on the fact that Π_ℓ is an ℓ-adic Lie group. If one want to extend this theorem to more general or geometric situation one as to find some transfer principle. For example, we can consider the adelic representation:

$$\rho_\infty : \pi_1(S) \to \prod_{\ell \neq p} GL(H^i(Y_{\overline{\eta}}, \mathbb{Z}_\ell(j)))$$

and its specialization

$$\rho_{\infty,x} : \pi_1(k(x)) \to \prod_{\ell \neq p} GL(H^i(Y_{\overline{\eta}}, \mathbb{Z}_\ell(j)))$$

Write

$$\Pi_\infty = \rho_\infty(\pi_1(X)) \quad \Pi_{\infty,x} = \rho_{\infty,x}(\pi_1(x))$$

Fact 2. *Assume k finitely generated and that either $p > 0$ or either $Y \to X$ is an abelian scheme.*

(1) Assume that $Y \to X$ is an abelian scheme. Then $\Pi_{\infty,x}$ is open in Π_∞ if and only if $\Pi_{\ell,x}$ is open in Π_ℓ

(2) Assume $p > 0$. Then $\Pi_{\infty,x}$ is open in Π_∞ if and only if $\Pi_{\ell,x}$ is open in Π_ℓ

So one can transfer fact 1 from ℓ-adic representation to adelic representation.
The aim of this course is to understand the theory that allows us to obtain other two highly related transfer principles in positive characteristic. In particular we study the specialization theory of Néron-Severi groups and p-adic monodromy groups.

2. NERON-SEVERI GROUPS

For every smooth proper variety V over an algebraically closed field denote with $NS(V)$ its Néron-Severi group. If we have a smooth proper morphism $f : Y \to X$, we can consider the variation of $NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ with $x \in X$. There is an injective cycle class map

$$NS(Y_{\overline{x}}) \otimes \mathbb{Q} \to H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

and an injective specialization map:

$$sp_{\eta,x} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to NS(Y_{\overline{x}}) \otimes \mathbb{Q}$$

and we prove the following:

Theorem 3. *Assume $char(k) = 0$ or that $Y \to X$ is projective. If $\Pi_{\ell,x}$ is open in Π_ℓ then $sp_{\eta,x}$ is an isomorphism.*

When $char(k) = 0$ this is a consequence of the so called variational Tate conjecture, i.e. a combination of:

(1) Variation Hodge conjecture
(2) Comparison isomorphism between Betti and ℓ-adic cohomology

To extend this result to positive characteristic we have to find replacements for these ingredients. A general philosophy is that the analogue of Hodge theory in positive characteristic is some form of p-adic cohomology theory (crystalline cohomology or rigid cohomology). In our situation this philosophy is verified. For example, a Variational Tate conjecture in Crystalline cohomology has been recently proved by M.Morrow (2014). The first part of the course is devoted to explain the main ingredients in its proof and to introduce some basic tools in crystalline cohomology (crystalline site, F-isocrystals, De Rham-Witt complex) used in it. Actually crystalline cohomology and the variational Tate conjecture work well only over perfect fields, while our fields are in general not perfect (the main example is $\mathbb{F}_p(T)$). So some extra work is necessary to apply it in our situation.

3. P-ADIC MONODROMY GROUPS

So we have a replacement for (1). The problem now is that we don't known how to compare directly crystalline cohomology with ℓ-adic cohomology. The idea is to replace this comparison with the comparison of monodromy groups. More precisely we start replacing Π_ℓ with its Zariski closure G_ℓ in $GL((H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$. This is an algebraic group with the property that the category of its $\mathbb{Q}_\ell$ representations is equivalent to the Tannaka category generated by the Π_ℓ representation $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))$. Moreover we have that $\Pi_{\ell,x}$ is open in Π_ℓ if and only if $G_{\ell,x}$ and G_ℓ have the same connected component of the identity, $G^0_{\ell,x}$ and G^0_ℓ.
The category of F-isocrystals in the crystalline site is also a Tannaka category and we can associate to

f an element of this category $R^2 f_* \mathcal{O}_{Y/K}$. Via the Tannaka formalism we obtain an algebraic group G_p^{conv}. This group is quite different from the ℓ-adic one. For example, if $Y \to X$ is a non isotrivial family of elliptic curves without supersingular fibres, there is a filtration of $R^2 f_* \mathcal{O}_{Y/K}$ in two one dimensional pieces, coming from the decomposition in the étale and connected part of the p-divisible group associated to $Y_{\overline{\eta}}$, that does not exists in the ℓ-adic setting. This leads to consider the smaller and better behaved category of F-overconvergent isocrystals $\mathbf{F\text{-}Isoc}^{\dagger}(X)$. Recent works of T.Abe, D.Caro, C.Lazda, H.Esnault, M. D'addezio, A.Pàl show that this category looks like the category of ℓ-adic lisse sheaves. Again this is a Tannaka category and $R^2 f_* \mathcal{O}_{Y/K}$ leaves inside it. So we can associate to it a monodromy group G_p. Using an easy variation of independence techniques developed by M.Larsen and R.Pink we prove the following:

Theorem 4. *$G^0_{\ell,x} = G^0_{\ell}$ if and only if $G^0_{p,x} = G^0_p$*

Again, the category of overconvergent isocrystals is well behaved when k is perfect, so a little of work is needed to extend the definitions and the results to our setting.

4. Companion conjectures

Theorems 4 and 1 gives us "lots" of points with the same p-adic monodromy group of the generic one for overconvergent isocrystals arising from geometry. Recent works of T.Abe and H.Esnault give us the possibility to extend this result for a big class of overconvergent isocrystals. Indeed they prove that we can associated to a "pure and p-plain" overconvergent isocrystals a "pure and p-plain" ℓ-adic representation of $\pi_1(X)$ for some $\ell \neq p$. Theorems 1 and 4 can be extended to these objects to get "lots" of points with the same p-adic monodromy group of the generic one for this class of overconvergent isocrystals.
The second part of the course is aimed to explain the definition and the proprieties of overconvergent isocrystals, the independence results that we need and finally how these can be used to obtain results over finitely generated fields.

5. Plan of the lectures

5.1. **Part I:Variational Tate conjecture in crystalline cohomology.**

5.1.1. *Lecture 1:Motivation and crystalline site.* We give some motivations to the study of p-adic cohomology theory. We define the classical crystalline site and state some properties of it.

5.1.2. *Lecture 2:Pro De Rham-Witt complex and slopes filtration.* We define and study the Pro De Rham-Witt complex and we show the classical comparison theorem between its continuous cohomology and crystalline cohomology.

5.1.3. *Lecture 3:Variational Tate conjecture in crystalline cohomology.* We state and explain the significance of the Variational Tate conjecture in crystalline cohomology. We show how the topic of the previous two lectures play a role in its proof.

5.2. **Part II:Overconvergent isocrystals, independence and applications.**

5.2.1. *Lecture 4:Overconvergent isocrystals.* We define overconvergent isocrystals and we explain how they are related to the F-isocrystals defined in the crystalline site. We state some theorems that should convince the audience that they are the right analogue of ℓ-adic lisse sheaves.

5.2.2. *Lecture 5:Compatible systems of coefficient objects and their monodromy groups.* We survey on ℓ (and p) independence of compatible systems over finite fields. We define monodromy groups of overconvergent isocrystals and lisse sheaves and we state the main theorems about them.

5.2.3. *Lecture 6:Application to finitely generated fields.* We explain how to extend the previous definitions to infinite finitely generated fields of positive characteristic and how to use the previous machinery to get theorems 4 and 3