Exercises to lecture **Tales of Catalan numbers** by A.N.Kirillov.

(1) Denote by s_n the number of all dissections of a convex $(n+2)$-gon. Show that

$$s_{n+1} = 3\ s_n + 2\left(\sum_{k=1}^{n-1} s_k s_{n-k}\right), \quad n \geq 1.$$

Find a bijective proof.

(2) Let $g_n(x) = x - \frac{x^n}{1-x}$, define formal power series $F_n(x)$ such that $F_n^{'}(0) = 1$, and $g_n(F_n(x)) = x$.

Show that the formal power series $F_n(x)$ has positive coefficients, and give theirs combinatorial interpretation(s).

(3) Assign to each vertex of the Stasheff polytope $\mathcal{K}^3$ a triangulation of a convex hexagon in a such way that the all edges correspond to flips.

(4) Let q be a parameter, define
q-number $[n]_q := \frac{1-q^n}{1-q}$, q-factorial $n]_q! := \prod_{j=1}^{n} [j]_q$, $[0]! := 1$, and
q-binomial $\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[m]_q![n-m]_q!}$, if $n \geq m$, and set $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$, if $n < m$.

Show that

$$\sum_{(a,b,c)} \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \begin{bmatrix} a+c-1 \\ c \end{bmatrix}_q \begin{bmatrix} b+c \\ b \end{bmatrix}_q = 1+\frac{1}{(q;q)^3}\left(\sum_{k\geq 2}(-1)^k \binom{k}{2} q^{\binom{k}{2}}\right),$$

where the sum runs over triples of integers $(a,b,c) \in (Z_{\geq 0})^3$, and $a > 0$.

Find combinatorial interpretation of this identity.

(5) Define polynomials $B_n(a_1,\ldots,a_n) =$

$$\sum_{p_1,\ldots,p_n} \frac{1}{n+1}\binom{n+\sum p_i}{n, p_1, \ldots, p_n} a_1^{p_1} a_2^{p_2}\cdots a_n^{p_n},$$

where the sum runs over n-tuples of nonnegative integers $p_1,\ldots,p_n$ such that $\sum_j\ jp_j = n$, and for any sequence of nonnegative integers $n_1,\ldots,n_k$ such that $\sum_j\ n_j = N$, we define the multinomial coefficient $\binom{N}{n_1,\ldots,n_k} := \frac{N!}{n_1!\cdots n_k!}$.

Show that

(**a**) $B_n(a_1,\ldots,a_n)$ is a polynomial in $a_1,\ldots,a_n$ with *integer coefficients.*

(**b**) $B_n(-1,\ldots,-1) = (-1)^n$.

(**c**) $B_n(1,\ldots,1)$ is equal to the number of dissections of a convex $(n+2)$-gon.

(**d**) $(-1)^n n!\ B_n(-1, \frac{-1}{2!},\ldots,\frac{-1}{n!}) = (n+1)^{n-1}$.

Find combinatorial interpretation of polynomial $B_n(t,\ldots,t)$.

(6) Let x and y be such that $x\ y = x + y + 1$.

Simplify the product $x^n\ y^m$,