Invitation to Enumerative Combinatorics

Goal:

To count certain combinatorial objects
and
Construct bijection(s) <u>between</u> equinumerable sets

<u>Literature</u>

G. Andrews, Theory of partitions, Cambridge Univ. Press, 1984

I. Macdonald, Symmetric functions and Hall polynomials, Oxford Science Publ., 1995

R. Stanley, Enumerative Combinatorics vol 1, vol 2, Cambridge Univ. Press, 2001.

R. Stanley, Catalan addendum

R. Stanley, Bijective proof problems 2007~

Tales of Catalan numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...

Definition. The n-th Catalan number

$$C_n = \frac{(2n)!}{n!\,(n+1)!}, \quad n = 0, 1, 2, \dots .$$

This is an integer, since

$$\boxed{C_n = \binom{2n}{n} - \binom{2n}{n-1}}$$

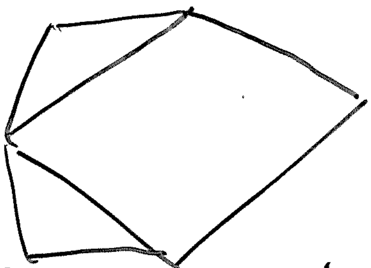
The Catalan numbers C_n are known from the ancient times for small n.
It was L. Euler who had proved a theorem that the number of triangulations of a convex $(n+2)$-gon is equal to C_n, (1758).
Eugen Catalan (1814–1894) had published several papers about triangulations of $(n+2)$-gon and its connections with parenthesations. Related ~~topics~~ were studied by A. Cayley, H. Rothe, D. Andre, N. Fuss and many other mathematicians of the 19-th century.

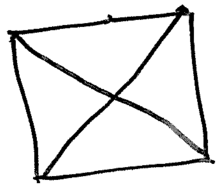
Nowadays, there are more then 200 different combinatorial interpretations of Catalan numbers (see 198 in Catalan Addendum, by R. Stanley)

§1 Polygon dissections

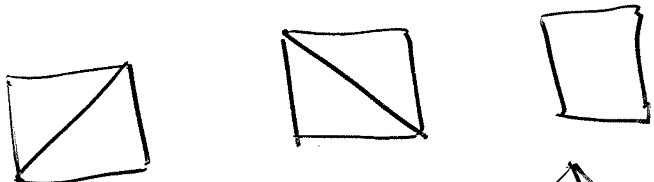
Let P be a convex polygon.

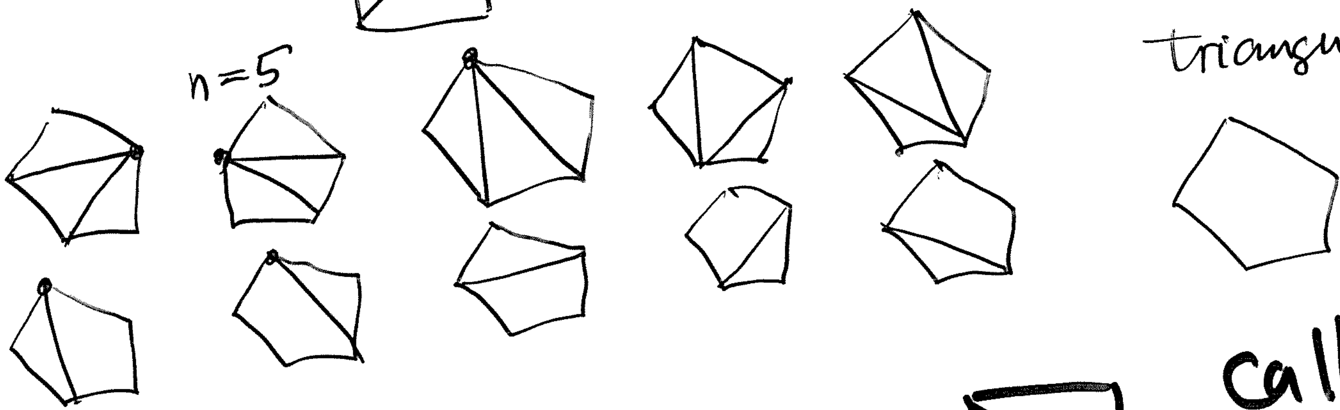
Definition. A **dissection** of P is obtained by drawing some diagonals that don't intersect in their interior.

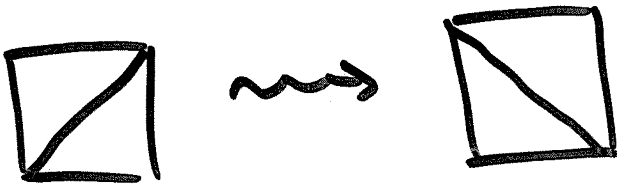
(a)  OK

(b)  forbidden

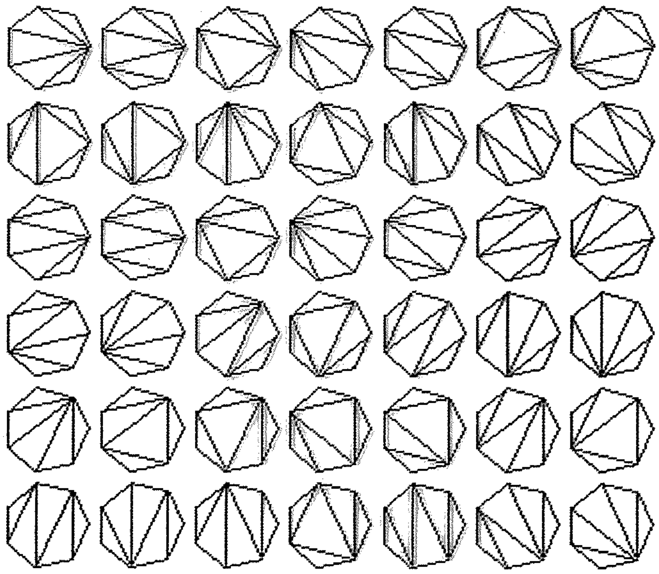
Thus, P is divided up into regions that are themselves convex polygons. In the case when all regions are triangles, dissection is called **triangulation**

Ex. $n=4$ 

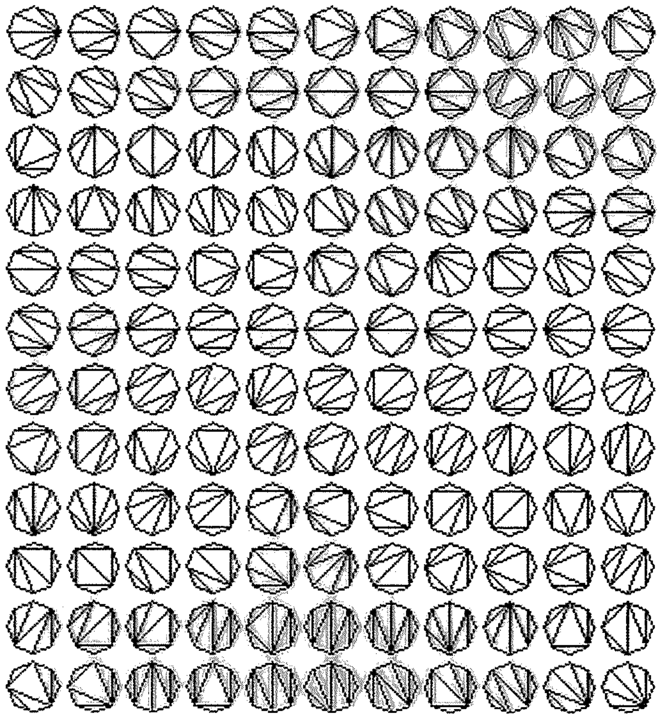
$n=5$  triangulations

Operation  called **flip**

Theorem Any two triangulation of a convex n-gon can be connected by a sequence of flips.



8 sides, 132 ways:



9 sides, 429 ways:
(Hidden in file catalan9.gif; around 29K.)

Multiplication diagrams

3 numbers:

```
(1 (2 3))    ((1 2) 3)
```

4 numbers:

```
(1 (2 (3 4)))     (1 ((2 3) 4))
((1 2) (3 4))     ((1 (2 3)) 4)
(((1 2) 3) 4)
```

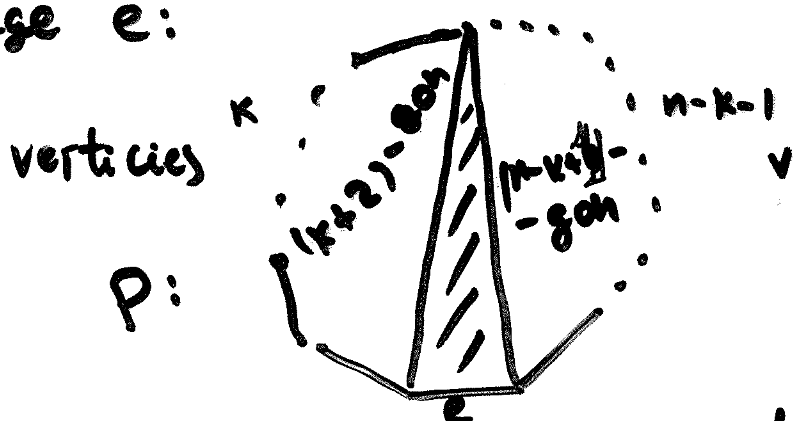
5 numbers:

```
(1 (2 (3 (4 5))))    (1 (2 ((3 4) 5)))
(1 ((2 3) (4 5)))    (1 ((2 (3 4)) 5))
(1 (((2 3) 4) 5))    ((1 2) (3 (4 5)))
((1 2) ((3 4) 5))    ((1 (2 3)) (4 5))
((1 (2 (3 4))) 5)    ((1 ((2 3) 4)) 5)
```

Q: Find the number T_n of triangulations of a convex $(n+2)$-gon. ②

It's clear that $T_0=1, T_1=1, T_2=2, T_3=5$;

Now take a $(n+2)$-gon, and choose an edge e:



Therefore, there exist $T_k \cdot T_{n-k-1}$ triangulations of P with fixed triangle

. Clear that k can be $0,\dots,n-1$.

Thus, $\boxed{T_n = \sum_{k=0}^{n-1} T_k T_{n-k-1}}$.

Now, let $T(x) = \sum_{n=0}^{\infty} T_n x^n = \sum_{n=0}^{\infty} x \cdot \sum_{k=0}^{n-1} T_k x^k \cdot T_{n-k-1} x^{n-k-1}$

$= 1 + xT^2(x)$. So we find that

$$xT^2(x) - T(x) + 1 = 0,$$ so that

Taylor series

$$T(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x}\left(1 - \sum_{n\geq 0} \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)\cdot(-4x)^n\right)$$

$$= \sum_{n\geq 1} \frac{1\cdot 3\cdots(2n-3)}{n!}\cdot 2^{n-1} x^{n-1} = \sum_{n\geq 0} \frac{1\cdot 3\cdots(2n-3)\cdot(n-1)!\cdot 2^{n-1}}{n!\,(n-1)!} x^{n-1}$$

$$= \sum_{n\geq 1} \frac{(2n-2)!}{n!\,(n-1)!} x^{n-1} \Longrightarrow T_n = \frac{(2n)!}{n!\,(n+1)!} = C_n.$$

Generalizations.

Let n and $0 \le d \le n-1$ be fixed, and s_n be the number of all dissections of convex $(n+2)$-gon, and $s_n(d)$ be the number of dissections by $(n-1-d)$ diagonals.

It's clear that $s_n(0) = C_n$, $s_n(n-1) = 1$.

Theorem (T. Kirkman (1857), A. Cayley (1890))

- $s_n(d) = \frac{1}{n+1}\binom{2n-d}{n}\binom{n-1}{d}$, $n \geq 1$.

•• Generating function $S(x) = \sum_{n \geq 1} s_n x^n =$

$$= \frac{1}{4}\left(1 + x - \sqrt{1 - 6x + x^2}\right).$$

Proof of (••) is based on observation that if we put $y = S(x)$, then

$$y = x + y^2 + y^3 + \dots = x + \frac{y^2}{1-y},$$

and therefore,

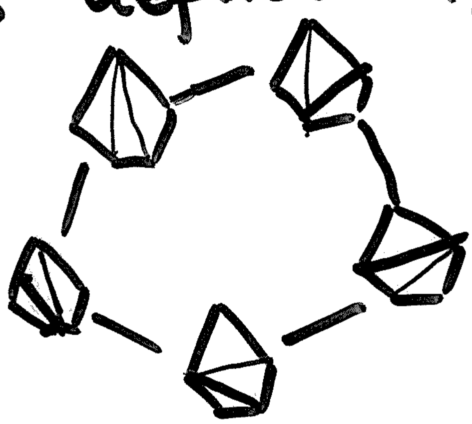
$$2y^2 - (1+x)y + x = 0.$$

$\boxed{\text{see Exercises}}$

The numbers s_n is called $\boxed{\textbf{little Schröder numbers}}$

Now consider graph with vertices parametrized by triangulations of a convexe $(n+2)$-gon, and edges defined by flips.

Ex. $n=3$



We see that set of triangulation of pentagon FORMS a convex polytope (in our case just pentagon).

Important remark.

Let $f(x) = x - x^2$. Then

$$f(xT(x)) = xT(x) - x^2 T^2(x) = x.$$

Therefore, the generating function for Catalan numbers is the **composition inverse series** for $f(x) = x - x^2$:

$$f(\underline{xT(x)}) = x.$$

Similarly, if we put $g(x) = x - \frac{x^2}{1-x}$, then $g(S(x)) = x$, and the generating function for the number of dissections of a convex $(n+2)$-gon, is ~~the~~ **composition inverse series** of $g(x) = x - \frac{x^2}{1-x}$.

such that $f'(0) = 1$

If series $f(x)$ is given, the composition inverse series $g(x)$, s.t. $f(g(x)) = x$, can be found by means of **Lagrange Inversion Formula** (≈ 1770)

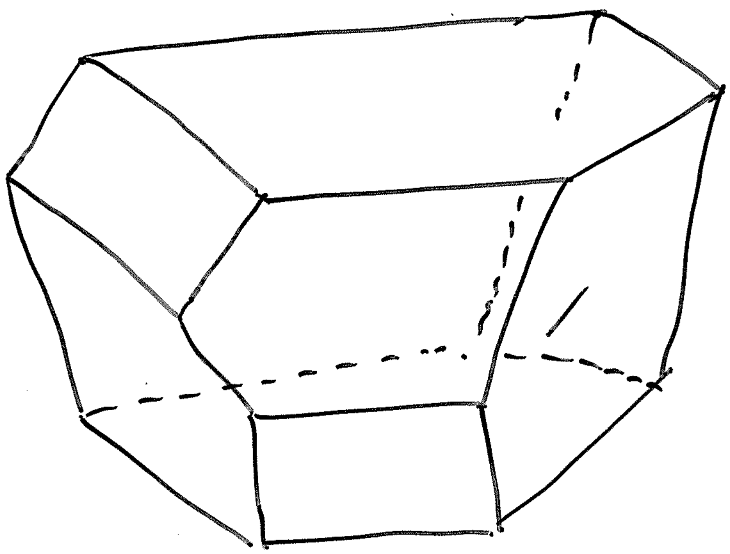
Exercise. Let $g_n(x) = x - \frac{x^n}{1-x}$, define $T_n(x)$ such that $g_n(xT_n(x)) = x$.

Show ~~that~~ $T_n(x)$ has positive coefficients, and find their Combinatorial interpretation (for $n = 3, 4, 5$) polygon dissections.

Theorem (J. Stasheff (1963), M. Haiman (1981))

The set of all triangulations of a convex (n+2)-gon together with edges difined by flips, FORMS a 1-skeleton of a convex (integral) polytope of dimension (n-1), ~~the~~ the so-called associahedron, or Stasheff polytope, K^{n-1}

For $n=4$ it looks like



K^3

$$(\underset{f_3}{1}, \underset{f_2}{9}, \underset{f_1}{21}, \underset{f_0}{14})$$

Exercese.

To assign to each vertex ~~the~~ corresponding triangulation of a convex hexagon, such ~~that the~~ edges correspond to flips.

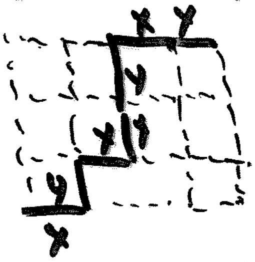
Theorem. The number of cells (of dimension $(n-k)$) in K^{n-1} isomorphic to $(K^0)^{n_1}\times\dots\times(K^{k-1})^{n_k}$ is equal to

$$\frac{1}{n+1}\binom{n+\sum_{i=1}^{k} n_k}{n, n_1, \dots, n_k}$$, where $\sum_{i=1}^{k} i\, n_i = n$.

f_k correspond to [redacted] the number of all $(n-k)$-dimension cells.

(§2) Lattice paths

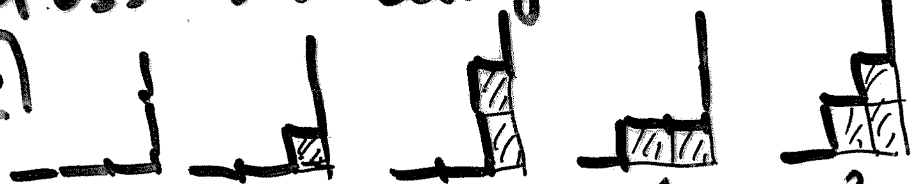
Definition A lattice path is a path starting from point $(0,0)$ with steps $(1,0)$ or $(0,1)$.



$\rightsquigarrow$ xyxyyxy

Monotonic, or Dyck, path is a lattice path from point $(0,0)$ to that (n,n), which never cross the diagonal.

$\boxed{\text{Take } n=3}$



# of picks:	0	1	1	1	2	x
volume:	0	1	2	2	3	y

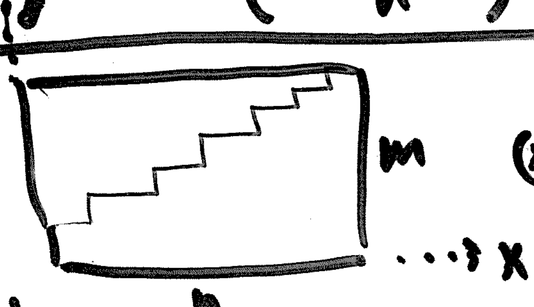
Generating function:

$$1+xy+2xy^2+x^2y^3$$

Theorem. The number of lattice paths from point $(0,0)$ to that (n,m) is equal to the binomial coefficient

$$\binom{n+m}{n} = \frac{(n+m)!}{n!\, m!}$$

Proof:



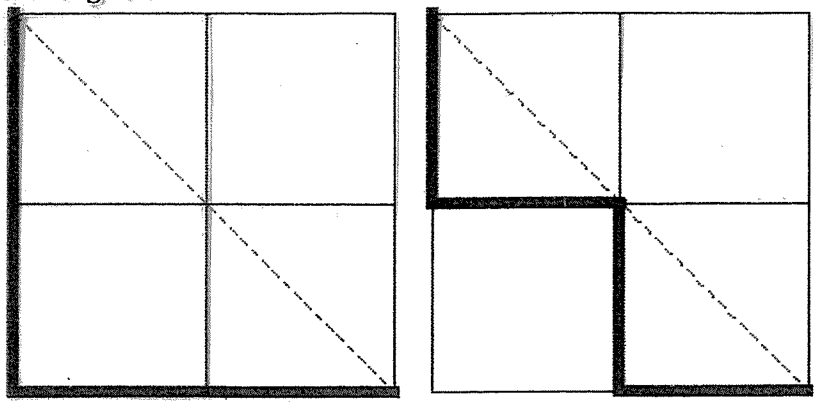
Let (i, a_i) be the (x, y)th coordinate of a point on the path.

Clearly, $0 \le a_0 \le \ldots \le a_{n-1} = a_n = m$
$\quad\quad +1 \quad +2 \ldots \quad +n \quad\quad +n$

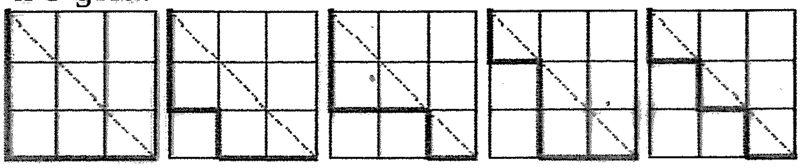
We obtaine the sequence $1 \le b_1 < b_2 < \ldots < b_n \le n+m$

Path diagrams

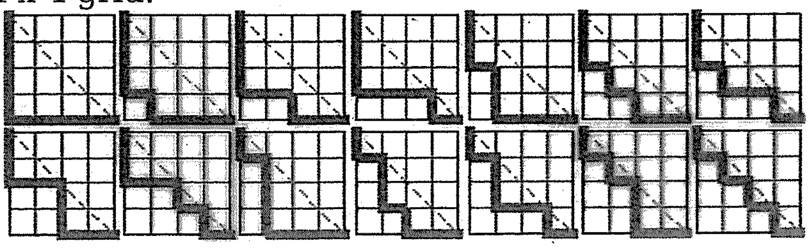
2 x 2 grid:



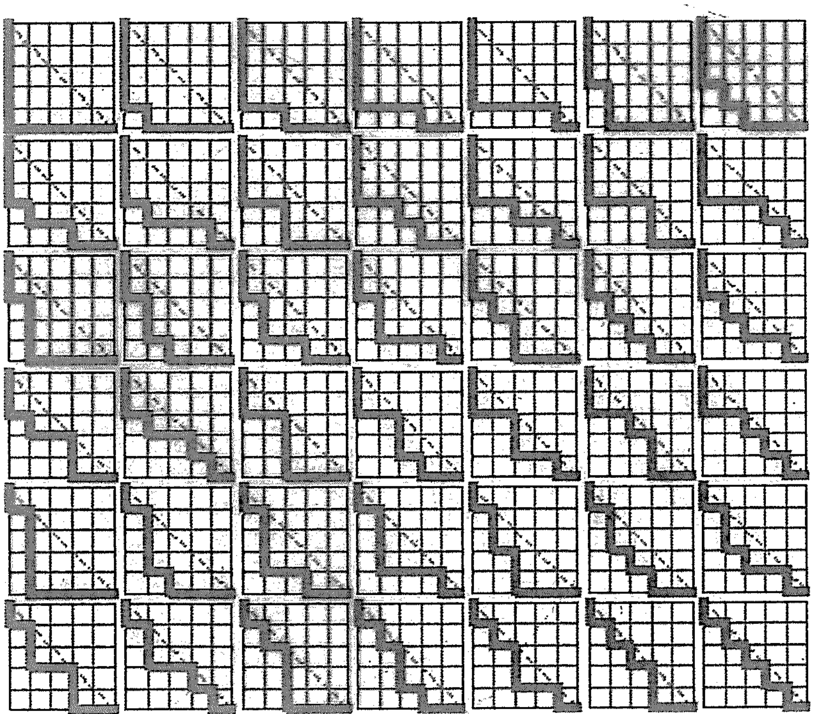
3 x 3 grid:



4 x 4 grid:



5 x 5 grid:



Correspondence ⑥

$$\text{path} \longrightarrow \{ b_1 < b_2 < \ldots < b_n \}$$

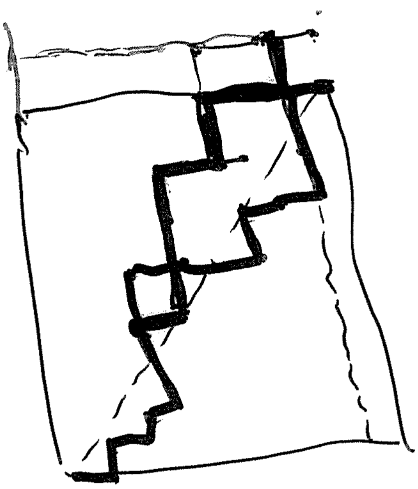
clearly is a bijection between the lattice paths in rectangular $m \times n$, and the set of subsets of cardinality n of the set consisting of $(n+m)$ elements. By definition, this number is $\binom{n+m}{n}$.

Q: How to compute the number of monotonic paths?

<u>Theorem</u> The number of monotonic paths between points $(0,0)$ and (n,n) is equal to the Catalan number C_n.

Proof (D. André, 1871).

$$\#\left|\begin{matrix}\text{Monotonic}\\ \text{paths}\end{matrix}\right| = \underbrace{\#|\text{all paths}|}_{\binom{2n}{n}} - \#\underbrace{\left|\begin{matrix}\text{paths}\\ \text{crossing}\\ \text{diagonal}\end{matrix}\right|}_{\substack{\text{paths from}\\ (0,0) \text{ to } (n-1,n+1)\\ \| \\ \binom{2n}{n-1}}}$$



Therefore,

$$\#\left|\begin{matrix}\text{Monotonic}\\ \text{paths}\end{matrix}\right| = \binom{2n}{n} - \binom{2n}{n-1} = C_n$$

§3) Parenthesis or bracketing.

Definition. Recursively define a bracketing as follows:

First, x itself is considered as a bracketing. Now define a bracketing to be a sequence $(B_1, \dots, B_k)$, where $k \geq$ and each B_i is bracketing.

Binary bracketing is one without ~~fragments of~~ (a ~~form~~ $(\dots \boxed{x}\boxed{x}\boxed{x} \dots)$ In other words, k

Example There are exactly eleven bracketin of four letters:
$xxxx$, $(xx)xx$, $x(xx)x$, $xx(xx)$, $(xxx)x$, $x(xxx)$
$(((xx)x)x)$, $((x(xx))x)$, $((xx)(xx))$, $(x((xx)x))$, $(x(x(xx)))$.

Note ~~that the~~ last five of ~~these~~ are binary bracketings.

Another interpretation: multiplication diagrams:

associativity rule

How many ways to multiply ~~the~~ numbers $1, 2, \dots, n$ (without reoderings).

$n=3$: $((1\,2)\,3)$, $(1\,(2\,3))$

$n=4$: $(1\,(2\,(34)))$, $(1\,(23)\,4)$, $((12)\,(34))$

$((1(23))4)$, $(((12)3)4)$

Clearly, this is a bijection with binary bracketings.

A Bijection between paths and brackettings. (7')

(Take a brackettings of $n+1$ letters)

<u>Rules:</u>

- $(ab) \longmapsto xy$
- $a(\ldots\ldots) \longmapsto x(\ldots\ldots)Y$
- $(\ldots\ldots)b \longmapsto (\ldots\ldots)XY$
- $(\ldots\ldots)(\ldots\ldots) \longmapsto (\ldots\ldots)X(\ldots\ldots)Y.$

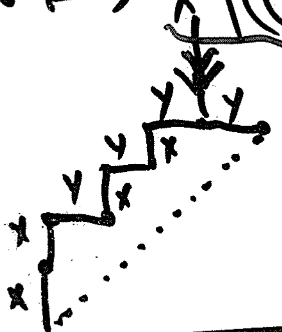
As a result, we will have a sequence $a_1 a_2 \ldots\ldots a_{2n}$ such that $a_i \in \{X, Y\}$, which satisfies ~~the~~ Yamanuchi condition: in any subsequence $\underbrace{a_1 \ldots\ldots a_p}$, $\#|X| \geq |Y|$.

Construction of a path:

- replace X by $\uparrow$, and Y by $\rightarrow$.

<u>Example</u> Take $(((bc)d)e) := w.$

$w \longmapsto X(((bc)d)e)Y \longrightarrow X((bc)d)XY\cdot Y \rightarrow X\cdot xy\ xy\ xy\ y$



$(ab)\ (((cd)e) \rightarrow (ab)\ X\ ((cd)e)Y$

$\rightarrow x\,y\,x\ xy\ xyy$

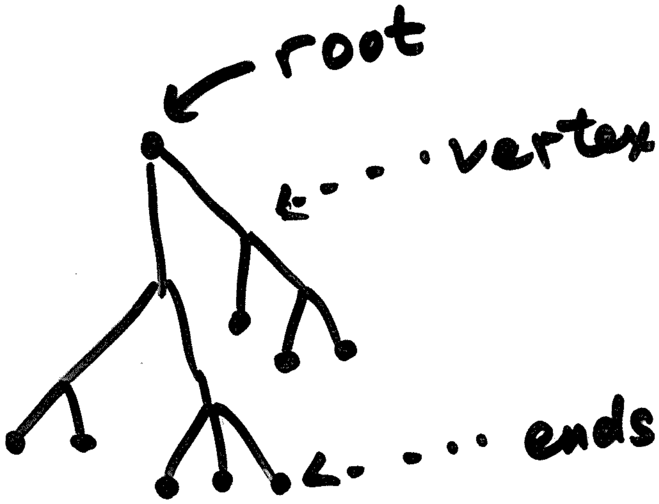
<u>Note</u> $(xy)(x(xy)(xy)y) \rightsquigarrow ()(()())$

$(x(xy)(xy)(xy)y) \rightsquigarrow (()()()).$

In other words, C_n is equal to the number of expressions containing n pairs of parentheses which are correctly matched.

§4) Trees

A plane tree is



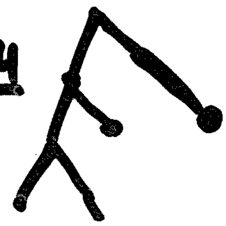
A plane tree for which every non-endpoint vertex has exactly two successors, is called <u>plane binary tree</u> with n ends.

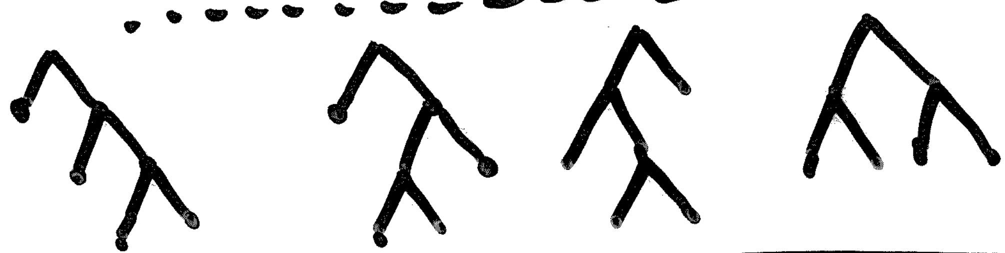
Examples: Binary trees

$\underline{n=2}$

$\underline{n=3}$

$\underline{n=4}$





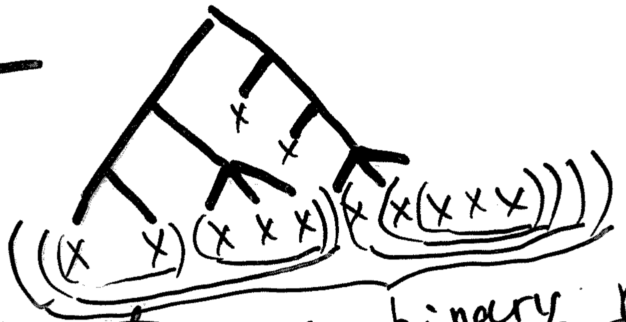
<u>Theorem</u> $\#\left|\begin{array}{c}\text{Bracketing of } n \text{ letters} \\ \text{with } k \text{ blocks}\end{array}\right|$

$$= \#\left|\begin{array}{c}\text{planar trees with } k \\ \text{vertex and } n \text{ ends}\end{array}\right|$$

<u>Proof</u>: let B be a bracketings, define the plane tree $\tau(B)$ recursively. If B consists of a single letter, then $\tau(B)$ is a single root vertex. If $B = (B_1, \dots, B_k)$, then

$\tau(B)$ consists of a root vertex drawn at the top, with subtrees $\tau(B_1), \dots, \tau(B_k)$, drawn in that order from the left to right.
Clearly, this construction can be inverted.

Example



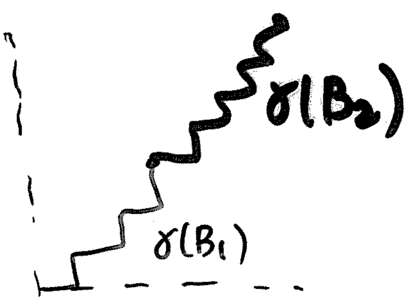
By construction, a binary plane tree correspon to a binary bracketing, i.e.

$$(!)\ \left\{\begin{array}{c}\text{Binary trees}\\ \text{with } n \text{ ends}\end{array}\right\} \xleftrightarrow[1:1]{} \left\{\begin{array}{c}\text{Binary bracketings}\\ \text{of } n \text{ letters}\end{array}\right\}$$

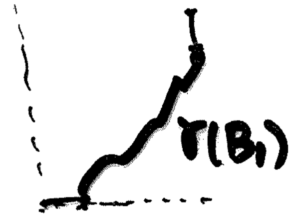
Now let us construct a bijection between binary bracketings and monotonic lattice paths.
Eeach bracketings is either a letter x or an ordered pair of binary bracketings $B=(B_1,B_2)$. If $B_i \neq x$, $i=1,2$, define a monotonic lattice path $\gamma(B)$ to be the concatenation of those $\gamma(B_1)$ and $\gamma(B_2)$:

$$\gamma(B) = \gamma(B_1) * \gamma(B_2).$$

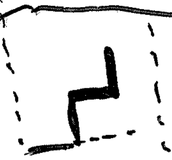
Define $\gamma(x, B_2) =$



Define $\gamma(B_1, x)$



Examples

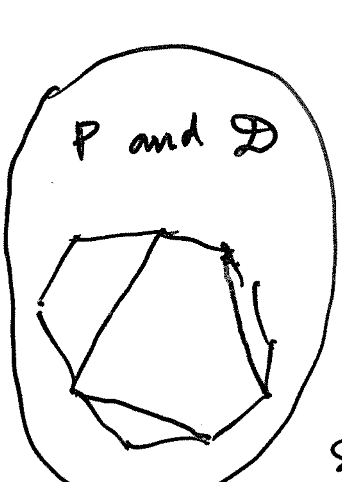


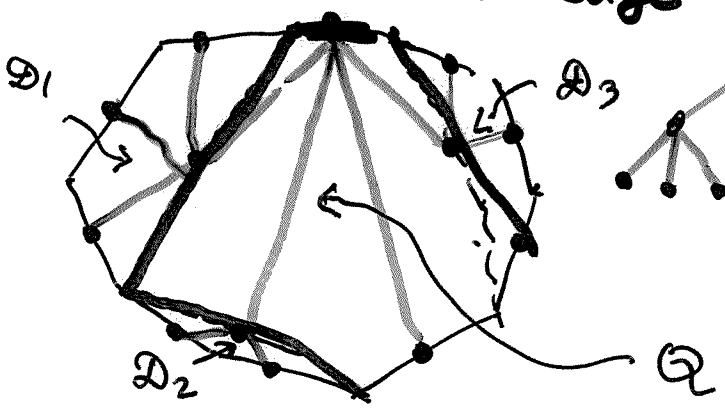
$(x\,(x,x))$

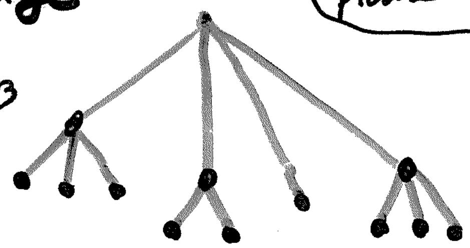
$((x,x)\,x)$

Now let us define a bijection between polygon dissections of a convex n-gon, by d diagonals and the {plane trees with $(n-1)$ ends and d vertex}

To start, fix once and for all an edge e of the polygon P, called the root edge. In a given dissection $\mathcal{D}$, the edge e is contained in a unique polygon Q which is a region of $\mathcal{D}$. let $(k+1)$ is the number of edges of Q. If we move the edge e and the interior of Q from $\mathcal{D}$, we are left with dissections $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k$ of k polygons, reading counterclockwise from e along the boundary of Q,





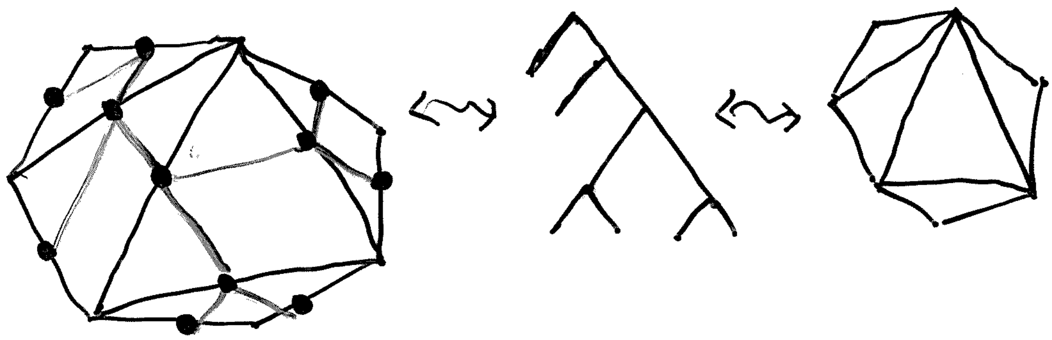


such that $\mathcal{D}_i$ and $\mathcal{D}_j$ inersect in a single point, $1 \le i \le k-1$. The tree $\tau(\mathcal{D})$ is the plane tree whose subrees of the root are $\tau(\mathcal{D}_1), \ldots, \tau(\mathcal{D}_k)$

It is clear from our construction that

$$\left\{\begin{array}{c}\text{triangulations}\\ \text{of a convex}\\ (n+2)\text{-gon}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{binary plane trees}\\ \text{with } (n+1) \text{ ends}\end{array}\right\}$$

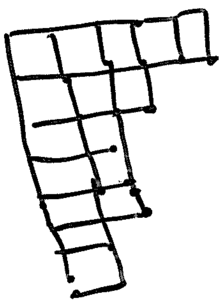
Example



---

§5 Young diagrams and tableaux.

~~Def~~. Young diagram



$\lambda := (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$, where λ_k is the length of k-th row.

let $n = \sum_{j=1}^{k} \lambda_j := |\lambda|$.

A Young tableau of shape λ is a filling of boxes of the diagram λ by the numbers $1, 2, \dots, n$ $(= |\lambda|)$ in a such way that they are strictly increasing along the rows and columns.

Example

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 6 & & \end{array}$$

. Here $\lambda = (3, 3, 1)$, $n = 7$.

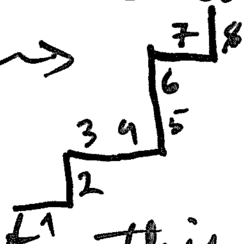
Now let us consider the case $\lambda = (n, n)$

Example $n=1$ $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ $n=2$ $\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & 4\\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 4\\ \hline\end{array}$

$n=3$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline 4 & 5 & 6\\ \hline\end{array}$ $\begin{array}{ccc} 1 & 2 & 4\\ 3 & 5 & 6\end{array}$ $\begin{array}{ccc} 1 & 2 & 5\\ 3 & 4 & 6\end{array}$ $\begin{array}{ccc} 1 & 3 & 4\\ 2 & 5 & 6\end{array}$ $\begin{array}{ccc} 1 & 3 & 5\\ 2 & 4 & 6\end{array}$

Exactly 5 Young tableaux! Why?

we constructed a bijection between the set (18) monotonic paths from $(0,0)$ to (n,n), and the set of Young tableaux of shape (n,n).
Namely, the numbers in the first row correspond to step $(1,0)$, and those in the second ones, to step $(0,1)$. $\boxed{\text{Example}}$ $\begin{matrix} 1\,3\,4\,7 \\ 2\,5\,6\,8 \end{matrix} \rightsquigarrow$



It is clearly seen that this is a bijection.

Noncrossing partitions

Noncrossing partition is a partition of the set $[n] = \{1, 2, \dots, n\}$ into blocks $(B_1, \dots, B_k)$ such that

$\bigcup_i B_i = [n]$, $B_i \cap B_j = \emptyset$, if $i \neq j$, <u>and</u>

if $a < b < c < d$, and $a, c \in B_i$, $b, d \in B_j \Rightarrow i = j$.

Denote by NC_n the set of noncrossing partitions of $[n]$.

<u>Example</u> $12|34$, $14|23$, but $13|24 \notin NC_4$.

Now let's list the set NC_3:

123, $1|23$, $12|3$, $13|2$, $1|2|3$, so that

$$|NC_3| = 5 = C_3 \quad !!!$$

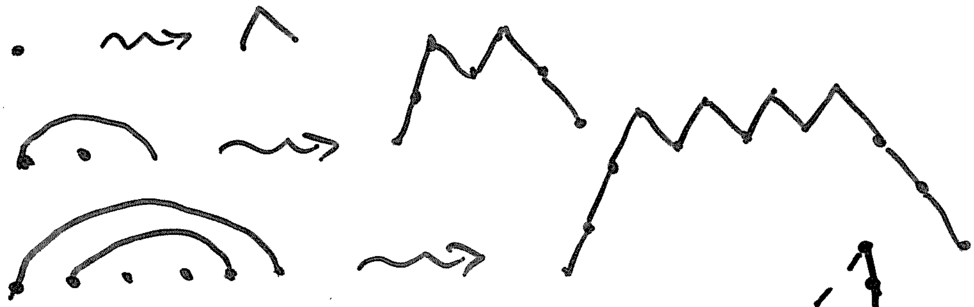
Theorem $|NC_n| = C_n$ the n-th Catalan number.

Noncrossing partitions can be displayed as monotonic paths.
First of all lets present a noncrossing partition by the following picture:

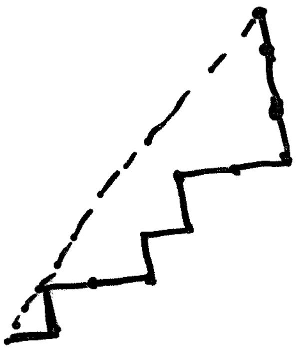
1|24|3|567 ⇝ 

13|2|6|7|4958 ⇝ 

Bijection $NC_n \longrightarrow$ { monotonic paths } looks like:

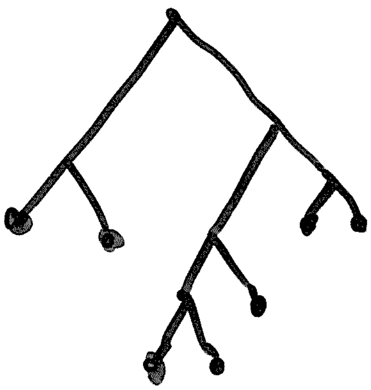


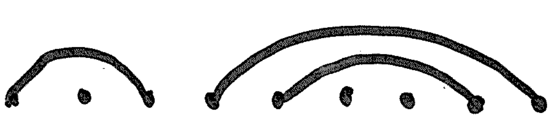
For example

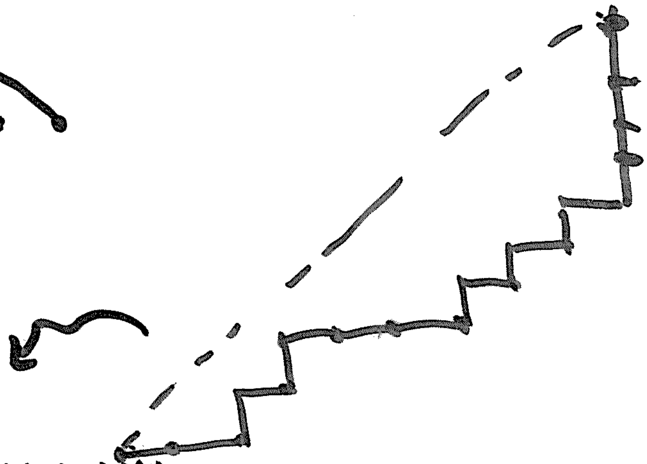
 ⇝ 

⇝ xyxxyxyxxyyy

(ab)((cd)e)(fg))







xxyx~~y~~xxx yxyxyxyyyy

§7 Jacobi game.

Consider an alphabet $\{x_{ij} \mid 1 \le i < j \le n\}$ together with the following rules:

① $x_{ij} \cdot x_{kl} = x_{kl} x_{ij}$, if i, j, k, l are distinct;

② $x_{ij} x_{jk} = x_{ik} x_{ij} + x_{jk} x_{ik} + x_{ik}$, if $1 \le i < j < k \le n$.

Examples $n=3$, $\{x_{12}, x_{13}, x_{23}\} \,\|\, x_{12}x_{23} = x_{13}x_{12} + x_{23}x_{13} + x_{1}$

$n=4$ $\{x_{12}, x_{13}, x_{23}, x_{14}, x_{24}, x_{34}\} \,\|\, x_{12}x_{34} = x_{34}x_{12}$, $x_{13}x_{24} = x_{24}x_{13}$, $x_{14}x_{23} = x_{23}x_{14}$;

$$\left.\begin{aligned} x_{12}x_{23} &= x_{13}x_{12} + x_{23}x_{13} + x_{13} \\ x_{12}x_{24} &= x_{14}x_{12} + x_{24}x_{14} + x_{14} \\ x_{13}x_{34} &= x_{14}x_{13} + x_{34}\cdot x_{14} + x_{14} \\ x_{23}\cdot x_{34} &= x_{24}\cdot x_{23} + x_{34}\cdot x_{24} + x_{24} \end{aligned}\right\}$$

Jacobi game:

Consider word $w = x_{12} \cdot x_{23} \cdot x_{34} \cdots x_{n-1\,n}$, and let us consecutively apply the above rules to the element w *in any order* until unable to do so. Denote the resulting polynomial by $P(x_{ij})$. In principal, the polynomial itself can depend on the order in which the rules ① & ② are applied.

Examples $n=3$, $x_{12}x_{23} = \underbrace{x_{13}x_{12} + x_{23}x_{13}}_{2} + \underbrace{x_{13}}_{1}$;

$n=4$ $x_{12}x_{23}x_{34} = (x_{13}x_{12} + x_{23}x_{13} + x_{13}) \cdot x_{34} =$

$= (x_{13}x_{34})x_{12} + x_{23}(x_{13}x_{34}) + (x_{13}x_{34}) = x_{14}x_{13}x_{12} + x_{34}x_{14}x_{12} + x_{13}x_{12} +$

$+ (x_{23}x_{34})x_{14} + x_{23}x_{14}x_{13} + x_{23}x_{14} = \underbrace{x_{14}x_{13}x_{12} + x_{34}x_{14}x_{12} + x_{23}x_{14}x_{13} + x_{34}x_{24}x_{14} + x_{24}x_{23}x_{14}}_{5} + \underbrace{x_{13}x_{12} \;\; x_{34}x_{14} \;\; x_{14}x_{13} \;\; x_{23}x_{1} \;\; x_{24}x_{1}}_{5} \;\; \underbrace{x_{14}}_{1}$

Show that the number of terms in the polynomial $P(x_{ij})$ is equal to the number of dissection of a convex $(n+1)$-gon by some diagonals that don't intersect in their interior. **Give a bijective proof!**

38) Inversion of power series.

Let $f(x) = x + a_1 x^2 + a_2 x^3 + \dots + a_n x^{n+1} + \dots$

$g(x) = x + B_1 x + B_2 x^2 + B_3 x^3 + \dots + B_n x^n + \dots$,

be two formal series, such that

$$f(g(x)) = x.$$

In this case ~~the~~ series $g(x)$ is called <u>inverse</u> series of $f(x)$.

<u>Example</u> $f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$g(x) = e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$.

Clearly, $f(g(x)) = \log(e^x) = x$.

For example, $x = f(g(x)) = g(x) + a_1 g(x)^2 + a_2 g(x)^3 + \dots$

$= x + b_1 x^2 + b_2 x^3 + \dots + a_1 x^2 + 2a_1 b_1 x^3 + a_2 x^3 + \dots$

$\Rightarrow a_1 + b_1 = 0, \quad b_2 + a_2 + 2a_1 b_1 = 0 \Rightarrow b_1 = -a_1, \quad b_2 = 2a_1 b_1 - a_2$

It is clear that B_n is a polynomial in $a_1, \dots, a_n$ with integer coefficients.

Theorem $B_n(a_1, \dots, a_n) =$

$$= \sum_{\substack{p_1, \dots, p_n \geq 0 \\ \sum i p_i = n}} (-1)^{\sum p_i} \frac{1}{n+1} \binom{n + \sum p_i}{n, p_1, \dots, p_n} a_1^{p_1} a_2^{p_2} \dots a_n^{p_n},$$

where $\binom{N}{k_1, \dots, k_n} = \frac{N!}{k_1! \dots k_n!}$, if $k_1 + \dots + k_n = N$.

Examples

Take $n=3$. Possibilities $p_1=3$; $\begin{cases} p_1=1 \\ p_2=1 \end{cases}$; $p_3=1$

and $B_3(a_1,a_2,a_3) =$

$$= -5a_1^3 + 5a_1a_2 - a_3.$$

Now take $p_1=n$, $p_2=\dots=0$, then

$$B_n(a_1,\dots,a_n) = \frac{1}{n+1}\binom{2n}{n}a_1^n = C_n a_1^n.$$

Theorem.

① $B_n(-t,\dots,-t) = \sum_{d=0}^{n-1} s_n(d)\, t^{n-d}$,

therefore $B_n(-1,\dots,-1) = \#\{\text{Dissections of a convex } (n+2)\text{-gon}\}$

② $B_n\left(1, \frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{n!}\right) = (n+1)^{n-1}$.

$(-1)^n \cdot n!$

Welcome
to
CATALANIA!