On Solutions of Polynomial Equations

Exercise 1.
Find all real solutions of the equation $x^3 + 3x^2 + 6x + 2 = 0$, using Cardano's Formula.

Exercise 2.
Find all solutions of the equation $y^4 - 5y^2 + \sqrt{30}y - \frac{3}{2} = 0$.

Hint. Try to guess the solutions of the Lagrange resolvent, but better don't try to use Cardano's Formula in this case!

Exercise 3.
(a) The equation $y^3 + 3y - 4 =$ has the obvious solution $y = 1$ and this is the only real solution. However, Cardano's Formula expresses the solution as $y = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$. Show that the first cubic root in this expression is equal to $\frac{1}{2}(1+\sqrt{5})$ while the second is equal to $\frac{1}{2}(1-\sqrt{5})$. Therefore, their sum is indeed equal to 1.

(b) The equation $y^3 + y = 0$ has the three solutions $y = 0$, $y = 1$ and $y = -1$. Without the knowledge of complex numbers, Cardano's Formula seems to make little sense, in this case. But suppose there where an imaginary number i with the strange property $i^2 = -1$. With this number, Cardano's Formula expresses y in the form

$$y = \frac{1}{\sqrt{3}}\left(\sqrt[3]{i} + \frac{1}{\sqrt[3]{i}}\right).$$

Show that the three cubic roots of i can be expressed as $-i$, $\frac{1}{2}(i+\sqrt{3})$ and $\frac{1}{2}(i-\sqrt{3})$. Conclude that Cardano's Formula then calculates all three solutions correctly!

Remark. Examples like Exercise 3(b) led the Italian mathematicians of the 16th century to introduce and study complex numbers.