The Basel Problem and the Riemann Hypothesis

(Answers)

Exercise 1. For $n = 1, 2, 3, \ldots$ denote the nth partial sum by S_n, i.e.

$$S_n = \sum_{k=1}^{n} \frac{1}{2^{k-1} \cdot k^2}.$$

Then, we compute with a calculator ...

$$\begin{aligned} S_1 &= 1 & S_9 &= 1.1644475\ldots \\ S_2 &= 1.125 & S_{10} &= 1.1644670\ldots \\ S_3 &= 1.1527778\ldots & S_{11} &= 1.1644751\ldots \\ S_4 &= 1.1605903\ldots & S_{12} &= 1.1644785\ldots \\ S_5 &= 1.1630903\ldots & S_{13} &= 1.1644799\ldots \\ S_6 &= 1.1639583\ldots & S_{14} &= 1.1644806\ldots \\ S_7 &= 1.1642772\ldots & S_{15} &= 1.1644808\ldots \\ S_8 &= 1.1643993\ldots & S_{16} &= 1.1644810\ldots \end{aligned}$$

So for S_{16} we get the first six digits of S_∞ as Euler computed. We conclude that Euler calculated at least the first 16 terms, or did he have another secret method?

Exercise 2. Comparing the coefficient of x^5 in the Taylor expansion of the sine, which is $1/120$, with the corresponding coefficient in Euler's product expansion gives

$$\frac{1}{\pi^4} \cdot \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n^2 m^2} = \frac{1}{120}.$$

The double sum on the left hand side runs over all positive integers n and m with the restriction $n < m$. If we sum over all pair of positive integers without this restriction instead, we obtain the following formula

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \cdot \left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right) = 2 \cdot \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n^2 m^2} + \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Now, after inserting this result into the previous equation and using Euler's solution to the Basel problem, we obtain the desired formula

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{\pi^2}{6} \right)^2 - \frac{\pi^4}{60} = \frac{\pi^4}{90}.$$