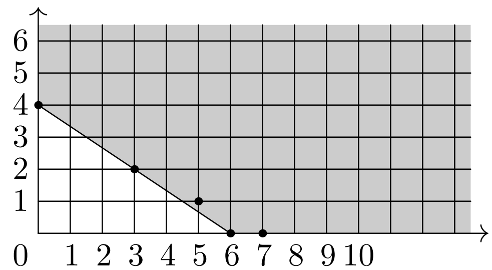
On the Newton-Puiseux Series

(Answers)

Exercise 1.

(a) The Newton polygon of $F(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$ has only one edge:



(b) The slope of the Newton polygon is $-2/3$ and the corresponding initial form is $H(x,y) := y^4 - 2x^3y^2 + x^6$. So we put $\mu = 3/2$ and insert $y = tx^\mu$ into $H(x,y)$,

$$H(x, tx^{3/2}) \;=\; x^6(t^4 - 2t^2 + 1) \;=\; x^6(t^2-1)^2$$

with the solutions $t = 1$ and $t = -1$. We choose the first $t = 1$. Thus $y(x) = x^{3/2}$ is a solution to our equation $H(x,y) = 0$.

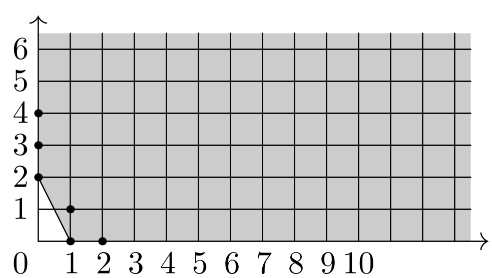
We use this as our first approximation for a solution of $F(x,y) = 0$. To improve this approximation, we make the substitution

$$x \;=\; x_1^2 \quad \text{and} \quad y \;=\; x_1^3(1+y_1)$$

and insert this into the original polynomial

$$F\big(x_1^2, x_1^3(1+y_1)\big) \;=\; x_1^{12}F_1(x_1,y_1),$$

where $F_1(x_1,y_1) = y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1y_1 - 4x_1 - x_1^2$. The Newton polygon of this new polynomial $F_1(x_1,y_1)$ again has only one edge:



The slope is -2 and the initial form is $H_1(x_1,y_1) = 4y_1^1 - 4x_1$. Obviously, the equation $H_1(x_1,y_1) = 0$ can be solved by $y_1(x_1) = x_1^{1/2}$. Coincidentally, this also solves the equation $F_1(x_1,y_1) = 0$ and hence, $y(x) = x_1^3(1 + x_1^{1/2}) = x^{3/2} + x^{7/4}$ solves $F(x,y) = 0$. The other Newton-Puiseux series for $F(x,y)$ are $x^{3/2} - x^{7/4}$, $-x^{3/2} - ix^{7/4}$ and $-x^{3/2} + ix^{7/4}$, where $i = \sqrt{-1}$.