International Workshop

P-positivity in Matroid Theory and related Topics, October 3–8 (2021), RIMS,Kyoto, Japan

Organizing Committee

A.N. Kirillov (RIMS), T. Maeno(Nagoya), Y. Namikawa (RIMS), N. Kishimoto (RIMS)

Overview of Purpose and Subject of Workshop

(**A**) Arrange $3-6$ introductory lectures to the Matroid Theory; On the unimodal and log-concave polynomials; On the Lefschetz Theory (HL) and the Hodge–Riemann Relations(HRR); On Lorentzian polynomials, with orientation on young and senior researches; present and discuss some new Conjectures in Matroid Theory.

The main objective of this Workshop is to attract attention of young researches as well as the leading specialists in Matroid Theory and related areas of Mathematics to some Conjectures concerning the Tutte polynomial of matroids.

(**B**) Introductory lecture to the $\mathcal{P}$-positivity property of polynomials; Unimodality and $\mathcal{P}$-positivity Conjectures in Matroid Theory.

Overview

The study of *matroids* is an investigation of an abstract theory of dependence. It was initiated by H. Whitney in founding paper [11]. Since that time, matroid theory attracted considerable interest among mathematicians and have been actively studied in various directions, see, e.g., [10]. One of the major invariants of a matroid $\mathcal{M}$ is is the so-called Tutte polynomial $Tutte_{\mathcal{M}}(x,y)$ associated with $\mathcal{M}$. The specialization $y=0$ is called by the characteristic polynomial of a matroid $\mathcal{M}$ and denoted by $ch_{\mathcal{M}}(x)$, One of the main open problems in the matroid theory was to prove that the absolute values of the characteristic polynomial coefficients form a Log-concave sequence for any (finite) matroid, the so-called log-concavity conjecture for characteristic polynomial of a matroid. Some special cases of this conjecture had been proved by several authors, see [10], but in the full generality this conjecture has been proved recently by J. Huh, see [4] for details and references.
My interest to this problem was arose from the study of unimodality of certain polynomials which naturally appear in Representation Theory and Combinatorics, [7]. During my study,I came to (far) generalization of the classical log-concavity conjectures in the Matroid Theory, the so-called Mason's Conjectures. As I mentioned, the Mason Conjectures has been proved by J.Huh.The main purpose of this Workshop, as a first step, is to prove my conjectures for graphic matroids and Grothendieck beta-polynomials introduced in [2]. Basic technique I suppose to use, will be based on that develops by J.Huh, E.Katz,.., and that in [].

Main Conjectures

To state the main target of this Workshop, we need a bit of notation and definitions concerning log-cavity and $\mathcal{P}-positivity$ [1] of a polynomial $p(x)$.

Log-concavity and unimodality

(a) (Unompdality) A polynomial $p(x)=a_0+a_1x\ldots+a_nx^n$, is called *unimodal* if there exists $k,\ 0\le k\le n$ such that $a_0\le a_1\le\ldots\le a_k\ge a_{k+1}\ge\ldots\ge a_n$.

(b) (Log-concavity) A polynomial $p(x)$ is called *log-concave* if $a_k^2\ge a_{k-1}a_{k+1}$ for all $k, 1\le k<n$.

(c) (Ultra log-concavity) A polynomial $p(x)$ is called *ultra log-concave*, if

$$\left(\frac{a_k}{\binom{n}{k}}\right)^2\ge\frac{a_{k-1}}{\binom{n}{k-1}}\Big)\frac{a_{k+1}}{\binom{n}{k+1}}\Big),\quad 1\le k<n.$$

(d) A polynomial $p(x)$ is called *strictly unimodal* (resp. *strictly log-concave*; resp. *strictly ultra log-concave*) if the all inequalities in item (a) (resp. item (b); resp. item (c)) are **strict**.

(e) (n-log-concavity, cf [9]) A polynomial $p(x)=a_0+a_1x\ldots+a_nx^n$, is called *$n$-log-concave*, if $\mathcal{L}^i(a_k)\ge 0$ for $0\le k\le n$ and $1\le i\le n$. Here $\mathcal{L}(a_k)=a_k^2-a_{k-1}a_{k+1}$, and we set $a_{-1}=a_{n+1}=0$; $\mathcal{L}^i=\underbrace{\mathcal{L}\cdots\mathcal{L}}_{i}$.

P-positive polynomials

Let $p(x)\in\mathbb{R}[x]$ be a polynomial with positive real coefficients $\{a_o,a_1,\ldots,a_n\}$. Consider matrices $M(p(x))=(m_{ij})$ and $M^{ultra}(p(x))=(\tilde{m}_{ij})$ of the same size $[\frac{n+3}{2}]\times[\frac{n+2}{2}]$ with matrix elements

$$m_{ij}=a_{i+j-2},\quad \tilde{m}_{ij}=\frac{a_{i+j-2}}{\binom{n}{i+j-2}}, 1\le i\le[\frac{n+3}{2}],\quad 1\le j\le[\frac{n+2}{2}],$$

Definition 0.1 *A polynomial $p(x)\in\mathbb{R}_{>0}$ of degree n is said to be $\mathcal{P}$-positive, (resp. strictly $\mathcal{P}$-positive), if the all (non-zero) minors of size $k\times k$ of the matrix $M(p(x))=(m_{ij})$ have the same sign $(-1)^{\binom{k}{2}}$ for all $1\le k\le[\frac{n+2}{2}]$. A polynomial $p(x)$ is called strictly $\mathcal{P}-$ positive if it is P-positive, and the all minors of the matrix $M(p(x))$ are not equal to zero.*

Parabolic Kostka polynomials

Let λ be a partition and $R=\{(\mu_a^{\eta_a})\}_a$ (dominant sequence of rectangular shape partitions, $|\lambda|=\sum_a\mu_a\eta_a$. The parabolic Kostka polynomial $K_{\lambda,R}(q)$ is defined via decomposition

$$\prod_a s_{\mu_a^{\eta_a}}(X)=\sum_\lambda K_{\lambda,R}(q)\ P_\lambda(X;q),$$

where $P_\lambda(X;q)$ stands for the Hall-Littlewood polynomial corresponding to partition λ and (dominant) set of rectangular R, see [8]. Let us write $K_{\lambda,R}(q)=q^{a(\lambda,R)}(b(\lambda,R)+h.d.terms)$, $b(\lambda,R)>0$. For brevity we consider the case $\eta_a=1$ for all a, $\mu=(\mu_1,\mu_2,\ldots)$. Also let us set $\epsilon_{ij}(\lambda,\mu)=2a(\lambda,\mu)--a(\lambda,\mu+\epsilon_{ij})-a(\lambda,\mu-\epsilon_{ij})$. Note that $\epsilon_{ij}(\lambda,\mu)\ge 0$. - -

[1] This notion is different from the well known concept of *Pólya frequency*.

Conjectures, (by A.N. Kirillov)

Conjecture 0.2 *Let $\mathcal{M}$ be a connected, loopless matroid and $ch_M(x) := x\mathrm{Tutte}_{\mathcal{M}}(x+1,0)$ (resp. $\overline{ch}_M(x) := ch(x)/(x+1)$) be the characteristic polynomial (resp. reduced one) of $\mathcal{M}$. Then the polynomials $ch_M(x)$, $ch_M(x)/x$, and $\overline{ch}_M(x)$ are $\mathcal{P}$-positive, in particular, they are Log-concave and unimodal.*

Let $\mathcal{M}$ be a matroid, and $\mathrm{Tutte}_{\mathcal{M}}(x,y) := \sum_{i,j} t_{ij} x^i y^j$ be the Tutte polynomial corresponding to matroid $\mathcal{M}$. Define vertical polynomials to be $v_{\mathcal{M}}^{(i)}(y) := \sum_j t_{ij}\ y^j$ and horizontal one to be $h_{\mathcal{M}}^{(i)}(x) := \sum_i t_{ij}\ x^i$. Similarly, write $\mathrm{Tutte}_{\mathcal{M}}(x+1,y) := \sum_{i,j} b_{ij} x^i\ y^j$, and define big vertical polynomials to be $V_{\mathcal{M}}^{(i)}(y) := \sum_j b_{ij}\ y^j$, and big horizontal polynomials to be $H_{\mathcal{M}}^{(i)}(x) := \sum_i b_{ij}\ x^i$.

Conjecture 0.3 *Let $\mathcal{M}$ be a (finite) simple matroid, that is one without parallel elements. Then the both polynomials $v_{\mathcal{M}}^{(i)}(y)$ and $h_{\mathcal{M}}^{(i)}(x)$ are all unimodal ($\mathcal{P}$-positive ?). However, it is well-known, that except the case $i = 0$, some horizontal polynomials $H_{\mathcal{M}}^{(i)}(x)$ may be not unimodal. i*

Conjecture 0.4 *Let $\mathcal{M}$ be a (finite) loopless matroid. Then the big vertical polynomials $V_{\mathcal{M}}^{(i)}(y)$ and $h_{\mathcal{M}}^{(i)}(x)$ are all unimodal, but not strictly unimodal in general.*

Question Let $\mathcal{M}$ be a simple matroid. Is it true that the Tutte polynomial $\mathrm{Tutte}_{\mathcal{M}}(1+x,0)$ is m-log-concave for $m \geq 2$?

Remark 0.5 *For a polynomial $p(x) = a_0 + \cdots + a_n x^n \in \mathbb{R}_{\geq 0}$, let us set $Np(x) = \sum_{k=0}^{d} a_k \frac{x^k}{k!}$. We expect that for polynomials $Nv_{\mathcal{M}}^{(i)}(y)$, $NV_{\mathcal{M}}^{(i)}(y)$, $Nh_{\mathcal{M}}^{(0)}(y)$, $NH_{\mathcal{M}}^{(0)}(y)$, the set of 2×2 non-positive minors of these polynomials are wider when the set of set of a form $\begin{pmatrix} a_{k-1}, & a_k \\ a_k, & a_{k+1} \end{pmatrix}$, $1 \leq k \leq [(n-1)/2]$. Precise statement of my Conjecture will be discussed during Workshop.*

Conjecture 0.6 *Let p be a prime number, λ be a partition of size p, i.e., $|\lambda| = p$. Let $\{R\} = (\mu_a^{\eta_a})$, $a = 1,\ldots,n$, be a dominant sequence of rectangular shape partitions, $\sum_a\ \mu_a \eta_a = |\lambda|$. Then the parabolic Kostka polynomial $K_{\lambda,\{R\}}(q)$ is a unimodal polynomial.*

Conjecture 0.7 *(cf [5]) Let λ and μ be partitions of the same size, and set $\epsilon_{ij} = (0,\ldots,0,\underbrace{1}_{i},0,\ldots,0,\underbrace{-1}_{j},0,\ldots,0)$, where $1 \leq i < j \leq \ell(\mu)$. Next, define polynomials*

$$q^* A_{\lambda,\mu}(q) := (K_{\lambda,\mu}(q))^2 - q^{\epsilon_{ij}(\lambda,\mu)}\ (K_{\lambda,\mu+\epsilon_{ij}})(q))(K_{\lambda,\mu-\epsilon_{ij}})(q)),\ A_{\lambda,\mu}(0) = 1.$$

Then the all 2×2 minors of the matrix $\mathcal{M}(A_{\lambda,\mu}(q))$ are non-positive. We (AK) expect that the all minors of matrix $\mathcal{M}^{ultra}(A_{\lambda,\mu}(q))$ are non-negative.

List of Participants

Richard Stanley,(University of Miami (USA))
June Huh, KIAS (S.Korea) and IAS, Princeton, (USA)
Petter Rrändén, KTH, Stockholm (Sweden),
Tom Braden, University of Massachusetts Amherst.(USA),
Avery St. Dizier, ISU, Illinois (USA),
Christopher Eur, Harvard U.,(USA),
Nicholas Proudfoot,UOregon, (USA),
Karim Alexander Adiprasito, Hebrew U. Jerusalem (Israel),
Ken Sumi, Kyoto University (Japan),
Anatol Kirillov, RIMS, Kyoto (Japan),
Yoshinori Namikawa, RIMS, Kyoto (Japan),
Tsuyoshi Miezali, Waseda U., Tokyo,(Japan),
Norifumi Kishimoto, RIMS, Kyoto (Japan),
Soichi Okada, Nagoya U., Nagoya (Japan),
Hirishi Naruse, U. of Yamanashi, Yamanashi (Japan),
Suijie Wang, Human U., Beijing (China),
Chenghong Zhao, CSU, Beijing (China),
Satoshi Naito, Tokyo Institute of Technology(TIT), Tokyo (Japan),
So Okada,National Institute of Technology(NIT), Oyama College, Oyama(Japan),
Ye Liu, Xian Jiaotong-Liverpool University (China),
Akishi Kato, University of Tokyo (Japan),
Alice L.L. Gao, Northwestern Polytechnical University (China),
Hideya Watanabe, Osaka City University (Japan),
Weili Guo,Beijing University of Chemical Technology (China)

We will have talks
Cristopher Eur, (Harvard, (USA)
Title: Hodge-Riemann relations in matroid theory

Abstract: We give an overview of developing and using Hodge-Riemann relations in matroid theory. In the first part, we survey the Hodge theory of Chow rings of matroids, discuss its uses, and present a simplified proof of the Hodge-Riemann relations (in degree 1) using the theory of Lorentzian polynomials. In the second part, we discuss tropical Hodge theory and introduce "tautological classes of matroids," and then, by combining these two notions, we obtain a log-concavity result for the Tutte polynomial of a matroid that generalizes previously established log-concavity properties for matroids. We conclude by noting some remaining or new questions concerning positivity in matroid theory. Joint works with Spencer Backman, Andrew Berget, Connor Simpson, Hunter Spink, and Dennis Tseng.

Tropicalizations of Lorentzian polynomials (June Huh (Princeton, USA)

Abstract:

I will give an overview of the theory of Lorentzian polynomials over the field of real Puiseux series. In this setting, one can tropicalize Lorentzian polynomials, and the class of tropicalized Lorentzian polynomials coincides with the class of M-convex functions in the sense of discrete convex analysis. Using the tropical connection, one can produce Lorentzian polynomials from M-convex functions. (Based on joint work with Petter Brändén).

Lorentzian polynomials I and II (Petter Brändén, KTH, Stockholm, Sweden)

Top-heaviness for vector configurations and matroids

(Tom Braden, University of Massachusetts Amherst.(USA),

Abstract:

A 1948 theorem of de Bruijn and Erdós says that if n points in a projective plane do not lie all on a line, then they determine at least n lines. More generally, Dowling and Wilson conjectured in 1974 that for any finite set of vectors spanning a d-dimensional vector space, the number of k-dimensional spaces that they span is at most the number of (d-k)-dimensional spaces they span, for all k smaller than d/2. In fact, Dowling and Wilson conjectured this inequality holds more generally for matroids, a combinatorial abstraction of incidence geometries which do not have to arise from actual vector configurations. The conjecture for vector configurations was proved in 2017 by Huh and Wang, using the hard Lefschetz Theorem for intersection cohomology applied to a singular algebraic variety associated to the vector configuration. I will discuss their proof, and also more recent joint work with Huh, Matherne, Proudfoot and Wang which proves the conjecture for all matroids, using a combinatorial replacement for intersection cohomology which makes sense even for non-realizable matroids.

Kazhdan-Lusztig polynomials of Marroids, I,II,

Nicholas Prodfoot, Oregon U.,]USA).

First Talk

Title: Kazhdan-Lusztig theory of matroids

Abstract:

I will give an introduction to Kazhdan-Lusztig polynomials of matroids and $\mathbb{Z}$-polynomials of matroids, with a focus on examples. I will also say a few words about the proof that these polynomials have non-negative coefficients.

Second Talk

Title: Equivariant log concavity, equivariant real rootedness, and equivariant interlacing

Abstract:

This talk will be about polynomials whose coefficients are virtual representations of a finite group. For example, given a matroid with symmetry group W, one can define the "equivariant characteristic polynomial" be taking the coefficients (up to sign) to be the graded pieces of the Orlik-Solomon algebra, regarded as representations of W. We will define what it means for such a polynomial to be log concave or real rooted, or for two such polynomials to interlace. I'll give lots of conjectures and partial results based on computer calculations.

Log-Concavity of Littlewood-Richardson Coefficients (Avery St. Dizier, ISU, Illinois , USA).

Combinatorial Weak and Hard Lefschetz theorems and their applications. Karim A. Adiprasito, Hebrew U. Jerusalem (Israel)
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A generalization of the Tutte polynomials Tsuyoshi Miezali, Waseda U., Tokyo, (Japan), Merged-log-concavity of rational functions and semi-strongly unimodal sequences. So Okada, National Institute of Technology(NIT), Oyama College, Oyama(Japan),
as well as Tom Braden (Amherst, MA, USA) ,(TBA), Anatol Kirillov (RIMS) , (TBA), Yoshinori Namikawa, RIMS, Kyoto (Japan), Tsuyoshi Miezali, Waseda U., Tokyo,Japan,, Norifumi Kishimoto, RIMS, Kyoto (Japan), Soichi Okada, Nagoya U., Nagoya (Japan), Hirishi Naruse, U. of Yamanashi, Yamanashi (Japan), Suijie Wang, Human U., Beijing (China), Chenghong Zhao, CSU, Beijing (China), ...
If you want participate in the work on our workshop, please fill the registration Form ((please,see web page below).
If you want to give a talk in the subject stream of our Workshop, please send me title and abstractness of of talk you want give, as well as fill Registration Form.
Short Overview of our Workshop, as well as Registration Form, and more information, one can one web page listed below
URL: http://www.kurims.kyoto-u.ac.jp/ kirillov
If you have any questions, suggestions concerning our Workshop, please, don't hesitate to ask me by email.
With best regards and wishes,
Anatol Kirillov

(Some) Titles and Abstracts

- June Huh,

Title: Tropicalizations of Lorentzian polynomials
Abstract
I will give an overview of the theory of Lorentzian polynomials over the field of real Puiseux series. In this setting, one can tropicalize Lorentzian polynomials, and the class of tropicalized Lorentzian polynomials coincides with the class of M-convex functions in the sense of discrete convex analysis. Using the tropical connection, one can produce Lorentzian polynomials from M-convex functions. (Based on joint work with Petter Bränd{]n).

- Petter Rrändén,

Title: Lorentzian polynomials I and II
Abstract:
The class of Lorentzian polynomials contains volume polynomials of convex bodies and projective varieties, as well as homogeneous stable polynomials. I will describe the topology and the combinatorics of the space of Lorentzian polynomials, and provide a useful toolbox of linear operators that preserve the Lorentzian property. In the second part I will show how the theory of Lorentzian polynomials can be applied to problems in combinatorics.

The talks are based on joint work with June Huh and Jonathan Leak

- Christopher Eur,

Title: Hodge-Riemann relations in matroids

Abstract:

Several log-concavity conjectures in matroid theory have been recently resolved via methods inspired by the Hodge-Riemann relations. These include the Hodge theory of matroid for the Chow ring of a matroid, tropical Hodge theory, and the theory of Lorentzian polynomials. We give an overview of these developments, and note some remaining or new questions concerning positivity in matroid theory.

- Avery St. Dizier,

Title: Log-Concavity of Littlewood-Richardson Coefficients

Abstract:

Schur polynomials and Littlewood-Richardson numbers are classical objects arising in symmetric function theory, representation theory, and the cohomology of the Grassmannian. First, I will give a quick introduction and history. Next, I will motivate and describe new log-concavity properties of the Littlewood-Richardson numbers and Schur polynomials. I will finish by explaining the mechanism behind the log-concavity properties, Brändén and Huh's theory of Lorentzian polynomials.

- Karim A. Adiprasito,

Title: Combinatorial Weak and Hard Lefschetz theorems and their applications.

Abstract:

I will explain a second proof of the Hard Lefschetz property beyond positivity, based on residue formulas for deranged toric varieties, and indicate implications towards problems in combinatorial topology and lattice sheaves on polytopes.

- Tsuyoshi Miezaki.

Title: A generalization of the Tutte polynomials

Abstract:

In this talk, we introduce the concept of the Tutte polynomials of genus g and discuss some of its properties. We note that the Tutte polynomials of genus one are well-known Tutte polynomials. The main result of this talk declares that the Tutte polynomials of genus g are complete matroid invariants. We will also discuss connections between matroids, codes, and lattices and give some open problems.

Contact person

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