Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps

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Abstract

We investigate the relationships between the parabolic Harnack inequality, heat kernel estimates, some geometric conditions, and some analytic conditions for random walks with long range jumps. Unlike the case of diffusion processes, the parabolic Harnack inequality does not, in general, imply the corresponding heat kernel estimates.

1 Introduction

This paper investigates the relationships between the parabolic Harnack inequality, heat kernel estimates, some geometric conditions, and some analytic conditions for random walks with long range jumps. By random walks with long range jumps, also known as random walks with unbounded range, we mean random walks for which there does **not** exist a positive integer K such that the probability of a jump larger in size than K is zero.

Our investigation combines two lines of research that have received much attention. For the past few decades there has been a great deal of interest in extending the results of DeGiorgi, Nash, Moser, and others on the regularity of solutions to the heat equation on $\mathbb{R}^d$ with respect to elliptic operators in divergence form to much more general state spaces. Among the state spaces considered are manifolds, graphs, and fractals. A typical result for the case of diffusions on manifolds or nearest neighbor random walks on graphs is along the lines of the following. (For a precise statement of the results, see [Gr, SC1, Del].)

Theorem 1.1 *The following are equivalent:*
(a) Gaussian upper and lower bounds on the heat kernel;
(b) the parabolic Harnack inequality;
(c) volume doubling and a family of Poincaré inequalities.

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The other line of research leading to this paper is the study of Harnack inequalities and heat kernel estimates for processes with jumps on $\mathbb{R}^d$, $\mathbb{Z}^d$, or state spaces with similar structure. These results are more recent; among the early papers here are [BL, CK1]. The motivation is that researchers in mathematical physics, mathematical finance, and other areas want to allow their models to have jumps, but many of the basic properties of jump processes are still unknown. A typical result (see the cited references for exact statements) is the following.

Theorem 1.2 *If there exists $\alpha \in (0,2)$ such that the probability of a jump from x to y is comparable to $|x-y|^{-d-\alpha}$, then the following hold:*
(a) polynomial type upper and lower bounds on the heat kernel;
(b) the parabolic Harnack inequality.

It is therefore quite natural to study heat kernel estimates and the parabolic Harnack inequality for random walks on more general graphs where there is the possibility of arbitrarily large jumps. Besides being interesting in its own right, we believe this study sheds additional light on pure jump processes of all types. It should also be mentioned that there are significant differences between the results for the diffusion or nearest neighbor case (Theorem 1.1) and the results we obtain here for the long range case (Theorems 1.5 and 1.6)

In this paper we investigate these connections in the framework of continuous time random walks on graphs. We believe that our results should extend with only minor changes to jump processes on metric measure spaces. However, in that context some issues of regularity would have to be treated.

Let $\Gamma = (G, E)$ be an infinite connected graph, where G is the set of vertices and E the set of edges. We write $d(x, y)$ for the graph distance, and we assume that Γ is locally finite. We let $B(x, r) = \{y : d(x, y) \le r\}$ denote balls in the graph metric; we allow $r \in [0, \infty)$. The notation $x \sim y$ means that $d(x, y) = 1$.

Let $J(x, y) = J(y, x)$ be a symmetric non-negative function on $G \times G$ with $J(x, x) = 0$ for all x. We write

$$J(x, A) = \sum_{y \in A} J(x, y), \tag{1.1}$$

and assume there exists $C_J \in [1, \infty)$ such that

$$C_J^{-1} \le J(x, G) \le C_J, \quad x \in G. \tag{1.2}$$

Let μ be a measure on G such that $\mu_x = \mu(\{x\})$ satisfies for some constant $C_M \in [1, \infty)$

$$C_M^{-1} \le \mu_x \le C_M, \quad x \in G. \tag{1.3}$$

We write

$$V(x, r) = \mu(B(x, r)). \tag{1.4}$$

For each $p > 0$, let $L^p(G, \mu) = \{f \in \mathbb{R}^G : \sum_{x \in G} f(x)^p \mu_x < \infty\}$, and let $\|f\|_p$ be the L^p norm of f with respect to μ. We define the operator

$$\mathcal{L} f(x) = \frac{1}{\mu_x} \sum_y (f(y) - f(x)) J(x, y). \tag{1.5}$$

and the quadratic form

$$\mathcal{E}(f,f) = \tfrac{1}{2} \sum_x \sum_y (f(x) - f(y))^2 J(x,y), \quad f \in L^2(G,\mu). \tag{1.6}$$

An easy application of Cauchy-Schwarz shows that $\mathcal{E}(f,f) \le 2C_J C_M ||f||_2^2$. We consider the continuous time Markov process $X = (X_t, t \ge 0, \mathbb{P}^x, x \in G)$ with jump rates from x to y of $\mu_x^{-1} J(x,y)$. This is the Markov process associated with the generator $\mathcal{L}$ and the Dirichlet form $(\mathcal{E}, L^2(G,\mu))$. Note that since $||\mathcal{L}f||_2 \le 2C_J C_M ||f||_2$, $\mathcal{L}$ is defined on $L^2(G,\mu)$. We write $p_t(x,y)$ for the heat kernel on Γ; this is the transition density of the process X with respect to μ:

$$p_t(x,y) = \frac{\mathbb{P}^x(X_t = y)}{\mu_y}. \tag{1.7}$$

Since the total jump rate out of x is $\mu_x^{-1} J(x,G) \le C_J C_M$, the process X is conservative and $\sum_y p_t(x,y)\mu_y = 1$ for all x, t.

We now consider various conditions which could be imposed on Γ, J, X and p.

1. Volume growth. G satisfies volume doubling VD if there exists a constant C_V such that

$$V(x,2r) \le C_V V(x,r) \quad \text{for all } x \in G, r > 0. \tag{VD}$$

It is easy to check that VD implies that there exists $\alpha_1 > 0$ such that if $x, y \in G$ and $0 < r < R$ then

$$\frac{V(x,R)}{V(y,r)} \le c_1 \Big(\frac{d(x,y) + R}{r} \Big)^{\alpha_1}. \tag{1.8}$$

Thus $V(x, d(x,y)) \asymp V(y, d(x,y))$, where $\asymp$ means that the ratio of the two sides is bounded above and below by two positive constants not depending on x or y.

VD also implies that

$$\frac{V(x,2r)}{V(x,r)} \ge 1 + C_V^{-4}, \quad \text{provided } r \ge 1. \tag{1.9}$$

To see this choose $y \in B(x,2r) - B(x,r)$, such that $B(y,r/4) \subset B(x,2r)$ and $B(y,r/4) \cap B(x,r) = \emptyset$. So, $V(x,2r) \ge V(x,r) + V(y,r/4)$, while by VD, $C_V^4 V(y,r/4) \ge V(y,4r) \ge V(x,r)$. Combining these gives (1.9).

A more restrictive condition is that $V(x,r)$ grows like r^d (where $d \in [1,\infty)$):

$$C_1 r^d \le V(x,r) \le C_2 r^d, \quad r \ge 1. \tag{$V(d)$}$$

2. Transition density estimates. Next we introduce various conditions on the heat kernel $p_t(x,y)$. X satisfies UHKP(α) if

$$p_t(x,y) \le c_1 \Big(\frac{1}{V(x,t^{1/\alpha})} \wedge \frac{t}{V(x,R)R^\alpha} \Big), \quad t > 0, \text{ where } R = d(x,y), \tag{$UHKP(\alpha)$}$$

and LHKP(α) if

$$p_t(x,y) \ge c_2 \Big(\frac{1}{V(x,t^{1/\alpha})} \wedge \frac{t}{V(x,R)R^\alpha} \Big), \quad t > 0, \text{ where } R = d(x,y). \tag{$LHKP(\alpha)$}$$

If both UHKP(α) and LHKP(α) hold, we say HKP(α) holds. The 'P' here stands for 'polynomial' – this kind of decay arises frequently for processes with long range jumps, instead of the Gaussian type behavior associated with continuous processes. This decay also occurs for Markov chains obtained by subordination of nearest neighbor random walks. Note that the first term in UHKP(α) (and in LHKP(α)) is smaller than the second term if and only if $t > R^\alpha$.

If we just have the upper bound for $x = y$, then we say UHD(α) holds:

$$p_t(x,x) \le \frac{c}{V(x,t^{1/\alpha})}. \qquad (UHD(\alpha))$$

If $V(d)$ holds, then HKP(α) takes the form:

$$p_t(x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{R^{d+\alpha}}, \quad t \ge 0, \text{ where } R = d(x,y).$$

Let $p_t^{B(x,r)}(\cdot,\cdot)$ be the heat kernel of the process X killed on exiting $B(x,r)$. (Γ, J) satisfies NDLB(α) if there exist constants c_1, c_2, c_3 such that

$$p_t^{B(x,r)}(x',y') \ge \frac{c_1}{V(x,r)}, \qquad x',y' \in B(x,r/2), \quad c_2 r^\alpha \le t \le c_3 r^\alpha. \qquad (NDLB(\alpha))$$

This lower bound plays a key role in the proof of the parabolic Harnack inequality.

3. Harnack inequalities. Let I be an open subset of $\mathbb{R}$, $A \subset G$, and $Q = I \times A$. Let $u(t,x)$ be defined on $I \times G$. We say that u is a *caloric on Q* if

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x), \quad t \in I, x \in A. \qquad (1.10)$$

We can interpret (1.10) in the weak sense in t, that is, for any $\psi \in C_c^\infty(I)$

$$-\int_I \psi'(t)u(t,x)dt = \int_I \mathcal{L}u(t,x)\psi(t)dt, \quad x \in A. \qquad (1.11)$$

Here $C_c^\infty(I)$ is the set of C^∞ functions with compact support contained in I.

Let $Q = Q(x_0,T,R) = (0,T) \times B(x_0,R)$, and set

$$\begin{aligned} Q_- &= Q_-(x_0,T,R) = [T/4, T/2] \times B(x_0,R/2), \\ Q_+ &= Q_+(x_0,T,R) = [3T/4, T] \times B(x_0,R/2). \end{aligned}$$

For Q as above we write $s + Q = (s, s+T) \times B(x_0,R)$. We say the parabolic Harnack inequality PHI(α) holds, if for $\lambda \in (0,1]$ there exist constants $C_P(\lambda)$, depending only on λ, such that whenever $u = u(t,x) \ge 0$ is caloric in $Q(x_0, \lambda R^\alpha, R)$ and continuous at time T, then

$$\sup_{Q_-} u \le C_P(\lambda) \inf_{Q_+} u. \qquad (PHI(\alpha))$$

The continuity of u at time T is assumed since we often use PHI(α) at time T. Alternatively, one may define PHI(α) as "$\cdots$ whenever $u = u(t,x) \ge 0$ is caloric in $Q(x_0, \lambda R^\alpha + \varepsilon, R)$ for some $\varepsilon > 0$, then $\cdots$ ".

We say a function h defined on G is *harmonic* on a subset A if

$$\mathcal{L}h(x) = 0, \qquad x \in A.$$

The elliptic Harnack inequality holds if there exists a constant c not depending on x_0 or R such that if h is non-negative on G and harmonic in $B(x_0, 2R)$, $x_0 \in G$, $R > 1$, then

$$h(x) \leq ch(y), \qquad x, y \in B(x_0, R). \tag{EHI}$$

Remark 1.3 (a) The classical parabolic Harnack inequality for diffusions on manifolds has $\alpha = 2$. Parabolic Harnack inequalities with anomalous scaling $\alpha > 2$ are given in [HSC, BB]. The case $\alpha < 2$ is discussed in [BL, CK1, CK2].

(b) If $R < 1$, then $B(x, R)$ and $B(x, \frac{1}{2}R)$ are both just the single point $\{x\}$. Nevertheless the parabolic Harnack inequality as stated above still makes sense, and in fact it is easy to check that this local parabolic Harnack inequality, (i.e., the parabolic Harnack inequality with $R < 1$), will always hold under the condition that $C_J < \infty$.

(c) If $\alpha > 1$ then the introduction of λ is not necessary, since the parabolic Harnack inequality for $\lambda = 1$ implies the parabolic Harnack inequality for any $\lambda \in (0, 1]$. To see why we need to introduce λ in the case when $\alpha \leq 1$, let $x_0, x_1 \in G$ with $d(x_0, x_1) = R$, let $0 < T < R^\alpha$ and suppose we wish to find a chain of n space-time boxes $Q_i = s_i + Q(x_i, r^\alpha, r)$ linking $(x_0, 0)$ with (x_1, T). We need $nr \geq R$, and $nr^\alpha \leq T$, which implies that $n^{\alpha-1} \geq R^\alpha/T$. Since $n \geq 1$, chaining of this type is only possible when $\alpha > 1$.

(d) Since a harmonic function is caloric, PHI(α) implies EHI.

4. Analytic estimates. Let P_t^B be the semigroup for X_t killed on exiting B. For an operator T on functions, $\|T\|_{p\to q} = \sup\{\|Tf\|_q : \|f\|_p \leq 1\}$. We consider the following semigroup bound: there exist $c_1, c_2, c_3 > 0$ such that

$$\|P_t^{B(x_0,r)}\|_{1\to\infty} \leq c_1 V(x_0, r)^{-1}, \quad \text{for all } c_2 r^\alpha \leq t \leq c_3 r^\alpha, r \geq 0, x_0 \in G. \tag{SB(α)}$$

5. Jump kernel. We also consider bounds on J:

$$J(x, y) \leq \frac{c\mu_x\mu_y}{d(x, y)^\alpha V(x, d(x, y))}, \tag{UJ(α)}$$

$$J(x, y) \geq \frac{c\mu_x\mu_y}{d(x, y)^\alpha V(x, d(x, y))}. \tag{LJ(α)}$$

If J satisfies both UJ(α) and LJ(α) we say it satisfies J(α). (The μ_x, μ_y are superfluous in view of (1.3). We put them here and elsewhere for the sake of possible future generalizations.)

We introduce the following hypotheses concerning the smoothness of the jump kernel J. As we see below these play an important role in the characterization of HKP(α) and PHI(α). We say UJS holds if

$$J(x, y) \leq \frac{c\mu_x}{V(x, r)} \sum_{x' \in B(x,r)} J(x', y) \quad \text{whenever } r \leq \tfrac{1}{2}d(x, y), \tag{UJS}$$

and we say that LJS holds if

$$J(x,y) \geq \frac{c\mu_x}{V(x,r)} \sum_{y \in B(x,r)} J(x',y) \quad \text{whenever } r \leq \tfrac{1}{2} d(x,y), \quad \text{and} \tag{1.12}$$

$$J(x,y) \geq c_0 > 0 \quad \text{whenever } x \sim y. \tag{1.13}$$

We recall that $x \sim y$ means $d(x,y) = 1$. We say JS holds if the local non-degeneracy condition (1.13) holds and in addition

$$J(x_1,y) \leq cJ(x_0,y) \quad \text{if } d(x_0,x_1) \leq \tfrac{1}{2} d(x_0,y); \tag{1.14}$$

cf. [Fo], where a similar condition is used. Given VD, then combining UJS and LJS gives JS by a straightforward argument (see Lemma 5.5).

6. Exit times. For $A \subset G$ we write

$$\tau_A = \min\{t \geq 0 : X_t \notin A\}.$$

(Γ, J) satisfies E_α if for all $x \in G$, $r \geq 1$,

$$c_1 r^\alpha \leq \mathbb{E}^x \tau_{B(x,r)} \leq c_2 r^\alpha. \tag{E_α}$$

In [GT] it is proved that for simple random walks the condition VD+EHI + E_2 is also equivalent to conditions (a)–(c) of Theorem 1.1. (The case $\alpha > 2$ is also treated there.)

7. Poincaré inequality. (Γ, J) satisfies PI(α) if there exists a constant C_Q such that for any ball $B = B(x,R) \subset G$ with $R \geq 1$ and $f : B \to \mathbb{R}$,

$$\sum_{x \in B} (f(x) - \overline{f}_B)^2 \mu_x \leq C_Q R^\alpha \sum_{x,y \in B} (f(x) - f(y))^2 J(x,y), \tag{$PI(\alpha)$}$$

where $\overline{f}_B = \mu(B)^{-1} \sum_{x \in B} f(x) \mu_x$.

The main results of this paper are as follows. First, we see that some of the implications in Theorem 1.1 do hold in the long range jump case.

Theorem 1.4 *Let $\alpha > 0$.*
(a) HKP(α) implies PHI(α).
(b) PHI(α) implies VD+ EHI + E_α.

A counterexample in Section 6 shows that the converse of Theorem 1.4(a) does not hold. We also sketch an example in that section which shows that the converse of (b) fails as well. It is easy to see that VD plus PI(α) is not enough to prove PHI(α) – see the example at the start of Section 6.

Given the gap between HKP(α) and PHI(α), one wishes to find good conditions equivalent to each of these.

Theorem 1.5 *Assume $V(d)$, and $\alpha \in (0,2)$. The following are equivalent:*
(a) J(α),
(b) HKP(α),
(c) PHI(α) and LJS.

Theorem 1.6 *Assume $V(d)$ and $\alpha \in (0,2)$. The following are equivalent:*
(a) PHI(α)
(b) UJ(α)+ $PI(\alpha)$ + UJS.

Remark 1.7 1. Theorems 1.5 and 1.6 are enough to prove 'stability' of HKP(α) and PHI(α) in the following sense. We say a property P is stable if whenever J and J' are comparable, i.e., $J(x,y) \asymp J'(x,y)$ for $x, y \in G$, and P holds for (Γ, J), then P also holds for (Γ, J').
2. Our results are actually slightly stronger than those stated, in that the constants which arise in the conclusions only depend on those in the various hypotheses. So, for example, a more careful statement of Theorem 1.4(a) would be "Suppose the graph Γ, jump kernel J and measure μ satisfy (1.2) and (1.3) with constants C_J and C_M, and that (Γ, J) satisfies HKP(α) with constants C_1 and C_2. Then (Γ, J) satisfies PHI(α) with a constant C_P, where C_P depends only on the constants C_1, C_2, C_J and C_M."

Section 2 shows that HKP(α) implies a lower bound on the heat kernel of the killed process in a ball. Section 3 proves the parabolic Harnack inequality, using the 'balayage' argument introduced in [BBCK]. Section 4 looks at consequences of the parabolic Harnack inequality – see Proposition 4.11 for a summary of these. Section 5 looks at consequences of the condition SB(α), and combining these with the results of Sections 2–4 completes the proofs of Theorems 1.4 – 1.6. In Section 6 we give some counterexamples, which show that the converses of (a) and (b) in Theorem 1.4 do not hold.

Note. By Theorem 1.4 each of PHI(α) and HKP(α) implies VD. Some of the implications in the Theorems 1.5 and 1.6 do not need $V(d)$; in addition some do not need the condition $\alpha < 2$. We summarize this in the following table.

Statement	*Volume condition*	*Range of α*
Theorem 1.5(a)⇒(b)	$V(d)$	$0 < \alpha < 2$
Theorem 1.5(b)⇒(c)	None	$0 < \alpha < \infty$
Theorem 1.5(c)⇒(a)	None	$0 < \alpha < 2$
Theorem 1.6(a)⇒(b)	None	$0 < \alpha < \infty$
Theorem 1.6(b)⇒(a)	$V(d)$	$0 < \alpha < 2$

For the convenience of the reader, we list the abbreviations we have used and in which subsection of Section 1 they can be found.

Abbreviation	*Meaning*	*Subsection*
E_α	Exit time	1.6
EHI	Elliptic Harnack inequality	1.3
HKP(α), LHKP(α), UHKP(α)	Upper and lower heat kernel	1.2
J(α), LJ(α), UJ(α)	Jump kernel bounds	1.5
JS, LJS, UJS	Jump smoothness	1.5
NDLB(α)	Near diagonal lower heat kernel	1.2

PHI(α)	parabolic Harnack inequality	1.3
PI(α)	Poincaré inequality	1.7
SB(α)	Semigroup bounds	1.4
UHD(α)	On diagonal upper heat kernel	1.2
VD	Volume doubling	1.1
V(d)	Volume growth	1.1

Throughout the paper, we use c, c' to denote strictly positive finite constants whose values are not significant and may change from line to line. We write c_i for positive constants whose values are fixed within theorems and lemmas. We adopt the convention that if we cite elsewhere the constant c_1 in Lemma 2.2 (for example), we denote it as $c_{2.2.1}$.

2 The heat kernel killed outside a ball

We begin by proving that HKP(α) gives a near-diagonal lower bound on the heat kernel killed outside a ball. We assume HKP(α) holds with constants C_1 and C_2, so that, writing $r = d(x,y)$,

$$\frac{C_1}{V(x,t^{1/\alpha})} \wedge \frac{C_1 t}{V(x,r)r^\alpha} \le p_t(x,y) \le \frac{C_2}{V(x,t^{1/\alpha})} \wedge \frac{C_2 t}{V(x,r)r^\alpha}.$$

Lemma 2.1 *HKP(α) implies VD.*

Proof. Let $t^\alpha = r$. Then

$$\frac{C_2}{V(x,(2t)^{1/\alpha})} \ge p_{2t}(x,x) \ge \sum_{y\in B(x,r)} p_t(x,y)^2 \mu_y \ge V(x,r)\frac{C_1^2}{V(x,r)^2}.$$

Rearranging gives $V(x, 2^{1/\alpha}r) \le cV(x,r)$, which implies VD. □

Remark 2.2 Note that HKP(α) is equivalent to the following:

$$C_1\Big(\frac{1}{V(x,C_2t^{1/\alpha})} \wedge \frac{t}{V(x,r)r^\alpha}\Big) \le p_t(x,y) \le C_3\Big(\frac{1}{V(x,C_4t^{1/\alpha})} \wedge \frac{t}{V(x,r)r^\alpha}\Big), \tag{2.1}$$

where $r = d(x,y)$. Indeed, one can prove VD from (2.1) similarly to the proof of Lemma 2.1, so (2.1) implies HKP(α).

Lemma 2.3 *Assume HKP(α). Let $B = B(x_0,R) \subset G$. For each $\lambda \in (0,1)$ there exists a constant $c_1(\lambda)$ such that*

$$p^B_{\lambda R^\alpha}(x,y) \ge \frac{c_1(\lambda)}{V(x_0,R)}, \quad x,y \in B' = B(x_0,R/2). \tag{2.2}$$

Proof. Let $\tau = \tau_B$. By the strong Markov property of X, for $x, y \in B' = B(x_0, R/2)$ and for any $t > 0$

$$p_t(x,y) = p_t^B(x,y) + \mathbb{E}^x 1_{(\tau<t)} p_{t-\tau}(X_\tau, y) \le p_t^B(x,y) + \sup_{0\le s\le t} \sup_{z\in B^c} p_s(z,y). \tag{2.3}$$

For $z \in B^c$, $y \in B'$, and $s \le (R/2)^\alpha$ the upper bound in HKP(α) gives

$$p_s(z,y) \le \frac{C_2 s}{V(x, d(z,y)) d(z,y)^\alpha} \le \frac{c_3 s}{V(x,R)R^\alpha},$$

where we used VD to obtain the final expression.

Now choose δ such that $2\delta^\alpha = 2^{-\alpha} \wedge (C_1/c_3)$. Let $\kappa \in (0,1]$, and let $s = \kappa(\delta R)^\alpha$. If $x, y \in B'' = B(x_0, \frac{1}{2}\delta R)$, then $d(x,y) \le \delta R$. So

$$p_s(x,y) \ge \frac{C_1}{V(x, s^{1/\alpha})} \wedge \frac{C_1 s}{d(x,y)^\alpha V(x, d(x,y))} \ge \frac{C_1 s}{(\delta R)^\alpha V(x, \delta R)} \ge \frac{C_1 \kappa}{V(x_0, R)}.$$

So,

$$p_s^B(x,y) \ge \frac{C_1\kappa}{V(x_0,R)} - \frac{c_3 s}{V(x_0,R)R^\alpha} = \frac{\kappa}{V(x_0,R)}(C_1 - c_3\delta^\alpha) \ge \frac{C_1\kappa}{2V(x_0,R)}. \tag{2.4}$$

Now let $x_1, y_1 \in B'$. Choose $n = 1 + \lfloor 4/\delta \rfloor$, and let $x_1 = z_0, z_1 \dots, z_n = y_1$ be a sequence of points in B' with $d(z_{i-1}, z_i) = \frac{1}{4}\delta R$. Let $B_i = B(z_i, \delta R/4)$, and note that (2.4) implies that $p_s^B(x,y) \ge c_4\kappa/V(x_0,R)$ for $x \in B_{i-1}$, $y \in B_i$. A standard chaining argument then gives

$$p_{ns}^B(x_1, y_1) \ge c_5 \kappa^n / V(x_0, R). \tag{2.5}$$

We have

$$ns = (1 + \lfloor 4/\delta \rfloor)\kappa(\delta R)^\alpha \tag{2.6}$$

so choosing κ such that $ns = \lambda R^\alpha$ we obtain (2.2). □

Remark 2.4 As already mentioned in the introduction, the need for λ (and so for κ) only arises when $\alpha \le 1$; when $\alpha > 1$ the usual chaining argument with sufficiently small balls allows one to bound $p^B_{\lambda R^\alpha}(x,y)$ from below once one has the bound (2.4) with $\kappa = 1$.

3 Parabolic Harnack inequality

In this section we show (under the assumption $V(d)$) that PI(α), UJ(α) and UJS together imply PHI(α).

Lemma 3.1 *Let $0 < \alpha < 2$. Suppose $V(d)$, PI(α) and UJ(α) hold. Then the upper bound UHKP(α) holds.*

Proof. It is well known (see for example [SC1]) that PI(α) implies the Nash inequality

$$||f||_2^{2+(2\alpha/d)} \le C_N \mathcal{E}(f,f) ||f||_1^{2\alpha/d}. \tag{3.1}$$

Given (3.1), we have UHKP(α) by the arguments in [BL, CK1]. (See also [BGK, CK2] for a simpler version of the proof.) $\square$

Given PI(α) and VD the argument of [SS] gives a weighted Poincaré inequality. This takes the following form. Let $x_0 \in G$, $R \geq 1$, $B = B(x_0, R)$, and

$$\varphi_R(x) = c_1(R - d(x, x_0))^+,$$

where c_1 is chosen so that $\sum_{x \in B} \varphi_R(x) = 1$. Set

$$\overline{f}_{\varphi_R} = \sum_{x \in B} f(x)\varphi_R(x)\mu_x.$$

Then there exists a constant C not depending on R, f, or x_0 such that

$$\sum_{x \in B} |f(x) - \overline{f}_{\varphi_R}|^2 \mu_x \leq C \sum_{x,y \in B} (f(x) - f(y))^2 \big(\varphi_R(x) \wedge \varphi_R(y)\big) J(x,y). \tag{3.2}$$

Lemma 3.2 *Suppose $V(d)$, PI(α) and UJ(α) hold. Then NDLB(α) holds.*

Proof. This follows from the weighted Poincaré inequality by a standard argument; see, for example, [BBCK], Section 3. So we have

$$p_T^B(x', y') \geq \frac{c}{V(x_0, R/2)}, \quad x', y' \in B(x_0, R/4), \quad T \asymp R^\alpha. \tag{3.3}$$

$\square$

Proposition 3.3 *Suppose VD, UHKP(α), NDLB(α) and UJS hold. Then PHI(α) holds.*

Proof. Let $\lambda \in (0, 1]$, $R \geq 1$, $T = \lambda R^\alpha$, $x_0 \in G$, and write:

$$B_0 = B(x_0, R/2), \quad B' = B(x_0, 3R/4), \quad B = B(x_0, R),$$

and

$$Q = Q(x_0, T, R) = [0, T] \times B, \quad E = (0, T] \times B'.$$

We consider the space time process on $\mathbb{R} \times G$ given by $Z_t = (V_0 - t, X_t)$, for $t \geq 0$.

Let $u(t, x)$ be non-negative and caloric on Q. Define the réduite u_E by

$$u_E(t, x) = \mathbb{E}^x\big(u(t - T_E, X_{T_E}); T_E < \tau_Q\big),$$

where T_E is the hitting time of E by Z, and τ_Q the exit time by Z from Q. Then $u_E = u$ on E, $u_E = 0$ on Q^c, and $u_E \leq u$ on $Q - E$.

The process Z has as a dual the process $\widehat{Z}_t = (V_0 + t, X_t)$; we may therefore apply the results of Chapter VI of [BG]. The balayage formula gives

$$u_E(t, x) = \int_E p_{t-r}^B(x, y)\nu_E(dr, dy), \quad (t, x) \in Q,$$

where ν_E is a measure on $\overline{E}$. We write

$$\nu_E(dr, dy) = \sum_{z \in B'} \nu_E(dr, z)\delta_z(dy)\mu_z.$$

We can divide each of the measures $\nu_E(dr, z)$ into two parts: an atom at 0, and the remainder. Given this we can write

$$u_E(t, x) = \sum_{z \in B'} p_t^B(x, z)u(0, z)\mu_z + \sum_{z \in B'} \int_{(0,t]} p_{t-r}^B(x, z)\mu_z\nu_E(dr, z). \tag{3.4}$$

To identify $\nu_E(dr, z)$ note that if $(t, x) \in E$ then

$$\frac{\partial u_E}{\partial t} = \frac{\partial u}{\partial t} = \mathcal{L}u = \mathcal{L}(u - u_E) + \mathcal{L}u_E. \tag{3.5}$$

Differentiating (3.4) we deduce that each measure $\nu_E(dr, z)$ is absolutely continuous with respect to Lebesgue measure, and that, writing $\nu_E(dr, z) = v(r, z)\, dr$,

$$\frac{\partial u_E}{\partial t}(t, x) = v(t, x) + \mathcal{L}u_E(t, x). \tag{3.6}$$

Using (3.5) this gives

$$v(t, x) = \mathcal{L}(u - u_E)(t, x) = \mu_x^{-1} \sum_{z \in B - B'} J(x, z)(u(t, z) - u_E(t, z)). \tag{3.7}$$

Let

$$w_t(x) = u(t, x) - u_E(t, x), \qquad Jw_r(z) = \sum_{y \in B - B'} J(z, y)w_r(y). \tag{3.8}$$

Then combining (3.4) and (3.7), for $x \in B_0$, $t \in [0, T]$,

$$u(t, x) = \sum_{z \in B'} p_t^B(x, z)u(0, z)\mu_z + \sum_{z \in B'} \int_{(0,t]} p_{t-r}^B(x, z)Jw_r(z)dr. \tag{3.9}$$

Now let $(t_1, x_1) \in Q_-$ and $(t_2, x_2) \in Q_+$. To prove the parabolic Harnack inequality it is enough, using (3.9), to show that:

$$p_{t_1}^B(x_1, z) \le Cp_{t_2}^B(x_2, z) \quad \text{ for } z \in B', \tag{3.10}$$

$$\sum_{z \in B'} p_{t_1 - r}^B(x_1, z)Jw_r(z) \le C \sum_{z \in B'} p_{t_2 - r}^B(x_2, z)Jw_r(z), \quad 0 \le r \le t_1. \tag{3.11}$$

Of these (3.10) is immediate from UHKP(α) and NDLB(α). So we consider (3.11). Since $t_2 - r \ge t_2 - t_1 \ge T/4$, using NDLB($\alpha$), and writing $V = V(x_0, R)$,

$$\sum_{z \in B'} p_{t_2 - r}^B(x, z)Jw_r(z) \ge cV^{-1} \sum_{z \in B'} Jw_r(z), \quad x \in B_0. \tag{3.12}$$

Let $s = t_1 - r \in [0, T/2]$. To complete the proof of (3.11) it is enough to show that

$$\sum_{z \in B'} p_s^B(x,z) J w_r(z) \le c V^{-1} \sum_{z \in B'} J w_r(z). \tag{3.13}$$

If $s \ge T/8$, then using the upper bound on p^B we obtain (3.13). So suppose $s \le T/8$. Let $B_1 = B(x_0, 5R/8)$. Then

$$\sum_{z \in B'} p_s^B(x,z) J w_r(z) = \sum_{z \in B_1} p_s^B(x,z) J w_r(z) + \sum_{z \in B' - B_1} p_s^B(x,z) J w_r(z). \tag{3.14}$$

If $z \in B' - B_1$ then $d(x,z) \ge R/8$ and so by UHKP(α)

$$p_s^B(x,z) \le \frac{cs}{(R/8)^\alpha V(x, R/8)} \le c' V^{-1}.$$

Hence

$$\sum_{z \in B' - B_1} p_s^B(x,z) J w_r(z) \le c V^{-1} \sum_{z \in B' - B_1} J w_r(z) \le c V^{-1} \sum_{z \in B'} J w_r(z). \tag{3.15}$$

If $z \in B_1$ then using UJS

$$\begin{aligned} J w_r(z) &= \sum_{y \in B - B'} J(z,y) w_r(y) \\ &\le \sum_{y \in B - B'} \frac{c\mu_z}{V(z, R/8)} \sum_{z' \in B(z, R/8)} J(z', y) w_r(y) \\ &= \frac{c\mu_z}{V(z, R/8)} \sum_{z' \in B(z, R/8)} J w_r(z') \le c\mu_z V^{-1} \sum_{z' \in B'} J w_r(z'). \end{aligned}$$

So,

$$\sum_{z \in B_1} p_s^B(x,z) J w_r(z) \le c V^{-1} \sum_{z' \in B'} J w_r(z') \sum_{z \in B_1} p_s^B(x,z) \mu_z \le c V^{-1} \sum_{z' \in B'} J w_r(z'). \tag{3.16}$$

Combining (3.15) and (3.16) proves (3.13), and hence (3.11). □

4 Consequences of the parabolic Harnack inequality

Throughout this section we assume PHI(α).

Lemma 4.1 *Let G satisfy PHI(α). Then the on-diagonal upper bound UHD(α) holds:*

$$p_t(x,x) \le \frac{C}{V(x, t^{1/\alpha})}, \quad x \in G,\ t > 0. \tag{4.1}$$

Proof. Let $r = t^{1/\alpha}$ and $\lambda = 1$. Let $u(t, y) = p_t(x, y)$ and apply PHI(α) to u in $Q = (0, 4t) \times B(x, 2r)$; this gives

$$p_t(x,x) \le \sup_{Q_-} u \le c_0 \inf_{Q_+} u \le c_0 p_{3t}(x,y), \quad y \in B(x,r).$$

Integrating over $B = B(x, r)$,

$$\mu(B) p_t(x,x) \le c_0 \sum_{y \in B} p_{3t}(x,y)\mu_y \le c_0.$$

□

Let $B \subset G$, and $p_t^B(x, y)$ be the heat kernel for X killed on exiting B. A key consequence of the parabolic Harnack inequality is a lower bound for $p_t^B(x, y)$. For continuous processes a standard argument (see [SC2], p. 153) is to apply the parabolic Harnack inequality to the function

$$v(s,x) = \begin{cases} \psi(x), & \text{if } s < t/2, \\ P_{s-t/2}\psi(x), & \text{if } t/2 \le s. \end{cases} \tag{4.2}$$

where $\psi = 1$ on a ball B and $\psi = 0$ on $G - B^*$, where B^* is the ball with the same center as B but radius twice as large. However, for v to be caloric one needs $\mathcal{L}\psi = 0$ on B, and this fails if the process can jump from B to $G - B^*$. Instead we use the argument below.

Theorem 4.2 *Let Γ satisfy PHI(α). Then if $x_0 \in G$, $T = R^\alpha$, $B = B(x_0, R)$,*

$$p_T^B(x', y') \ge \frac{c}{V(x_0, R/2)}, \quad x', y' \in B(x_0, R/2). \tag{4.3}$$

Proof. Let $R_0 = R/2$ and $c_0 = 1 - (3/4)^{1/\alpha}$. Set $r_k = c_0 R_0 (3/4)^{k/\alpha}$, and let

$$R_k = R_{k-1} - r_{k-1} = R_0 - \sum_0^{k-1} r_i = R_0 (3/4)^{k/\alpha}.$$

Let $t_k = r_k^\alpha$, and let $B_k = B(x_0, R_k)$ for $0 \le k < \infty$; for large k the ball B_k is just $\{x_0\}$.

Set

$$u_n(t,x) = \mathbb{P}^x(\tau_{B_n} > t) = \sum_{y \in B_n} p_t^{B_n}(x,y)\mu_y,$$

$$\theta_n(t) = \sup_{x \in B_n} u_n(t,x).$$

Since u_n is a sum of caloric functions, u_n is caloric in $(0, \infty) \times B_n$. Note that $u_{n+1} \le u_n$ and that $u_n(x, t)$ and $\theta(t)$ are decreasing in t. Also note that

$$u_n(t, x_0) \ge \mathbb{P}^{x_0}(X_s = x_0, 0 \le s \le t) \ge e^{-J(x_0,G)t}. \tag{4.4}$$

For any ball B_j

$$\begin{aligned} u_j(t+s,x) &= \sum_{y \in B_j} p_{t+s}^{B_j}(x,y)\mu_y = \sum_{z \in B_j} \sum_{y \in B_j} p_t^{B_j}(x,z) p_s^{B_j}(z,y)\mu_y\mu_z \\ &= \sum_{z \in B_j} p_t^{B_j}(x,z) u_j(s,z)\mu_z \le u_j(t,x)\theta_j(s). \end{aligned}$$

Therefore $\theta_j(t+s) \le \theta_j(t)\theta_j(s)$.

Now let $n \ge 0$ and let $x \in B_{n+1}$. Then $B(x, r_n) \subset B_n$, so that $u_n(t,x)$ is caloric in $Q = (0, t_n) \times B(x, r_n)$. Applying the parabolic Harnack inequality we obtain

$$u_n(t_n/4, x) \le \sup_{Q_-} u_n \le C_1 \inf_{Q_+} u_n \le C_1 u_n(t_n, x) \le C_1 \theta_n(t_n) \le C_1 \theta_n(t_n/3)^3.$$

Since $t_{n+1}/3 = t_n/4$,

$$\theta_{n+1}(t_{n+1}/3) = \sup_{B_{n+1}} u_{n+1}(t_n/4, x) \le \sup_{B_n} u_n(t_n/4, x) \le C_1 \theta_n(t_n/3)^3. \tag{4.5}$$

Write $a_n = \theta_n(t_n/3)$; we have

$$a_{n+1} \le C_1 a_n^3, \qquad n \ge 0. \tag{4.6}$$

Note that $a_n \le 1$ for all n. Suppose that $a_0 \le (C_1 \vee e)^{-1}$. Then $a_1 \le (C_1 a_0) a_0^2 \le a_0$, and so iterating we deduce that $C_1 a_n \le 1$ for all n. Therefore, using (4.6) again,

$$a_n \le (C_1 a_{n-1}) a_{n-1}^2 \le a_{n-1}^2 \le a_{n-2}^4 \le (a_0)^{2^n} \le e^{-2^n}.$$

Hence $u_n(t_n, x_0) \le \exp(-2^n)$ for all $n \ge 0$, which contradicts (4.4).

So $a_0 \ge (C_1 \vee e)^{-1} = c_2$, and thus $\theta_0(s) \ge c_2$ for $s \in [0, t_0/3]$. Let $s = t_0/3 \wedge (T/8)$. Then there exists $x' \in B_0 = B(x_0, R/2)$ such that

$$u_0(s, x') \ge c_2.$$

Applying the parabolic Harnack inequality to u_0 enough times to compare $u_0(s, x')$ with $u_0(T/4, x')$ it follows that $u_0(T/4, x') \ge c_4$. Thus as

$$u_0(T/4, x') = \sum_{y \in B_0} p_{T/4}^{B_0}(x', y) \mu_y,$$

writing $V_0 = \mu(B_0)$, there exists $y' \in B_0$ with

$$c_4 V_0^{-1} \le p_{T/4}^{B_0}(x', y') \le p_{T/4}^{B}(x', y').$$

Applying the parabolic Harnack inequality to $v(t,y) = p_t^B(x', y)$ in the region $(0,T) \times B(x_0, R)$ we obtain

$$c_4 V_0^{-1} \le p_{T/4}^B(x', y) \le \sup_{Q_-} v \le C_1 \inf_{Q_+} v \le C_1 p_{3T/4}^B(x', y), \text{ for all } y \in B_0.$$

Fix $y \in B_0$; applying the parabolic Harnack inequality again to $w(t,x) = p_{t+T/4}^B(x, y)$ in the region $(0,T) \times B(x_0, R)$ we obtain, for any $x \in B_0$,

$$c_4 (C_1 V_0)^{-1} \le p_{3T/4}^B(x', y) = w(T/2, x') \le \inf_{Q_-} w \le C_1 \inf_{Q_+} w \le C_1 w(3T/4, x) = C_1 p_T^B(x, y),$$

which completes the proof of (4.3). □

Corollary 4.3 *Suppose* (Γ, J) *satisfies PHI*(α). *Then* Γ *satisfies VD.*

Proof. This is immediate given (4.1) and (4.3). Let $R > 0$, $T = R^\alpha$, $x \in G$ and $B = B(x, R)$. Then

$$\frac{c_1}{V(x, R/2)} \le p_T^B(x, x) \le p_T(x, x) \le \frac{c_2}{V(x, R)},$$

giving VD. $\square$

Corollary 4.4 *PHI*(α) *implies SB*(α).

Proof. First, note that SB(α) is equivalent to the following: There exist $c_1, c_2, c_3 > 0$ such that for any $x_0 \in G$, $r > 0$, and writing $B = B(x_0, r)$,

$$\sup_{x,y \in B} p_t^B(x, y) \le c_1 V(x_0, t^{1/\alpha})^{-1}, \qquad \text{for all } c_2 r^\alpha \le t \le c_3 r^\alpha. \tag{4.7}$$

Now $p_t(x, y)^2 \le p_t(x, x) p_t(y, y)$. So, by Lemma 4.1, for $x, y \in B$,

$$p_t^B(x, y)^2 \le p_t(x, y)^2 \le \frac{c}{V(x, t^{1/\alpha}) V(y, t^{1/\alpha})} \le \frac{c_4}{V(x_0, t^{1/\alpha})^2},$$

where we used (1.8) in the last line. $\square$

Lemma 4.5 *Suppose* G *satisfies PHI*(α). *Then*

$$c_1 R^\alpha \le \mathbb{E}^x \tau_{B(x,R)} \le c_2 R^\alpha, \qquad R \ge 1. \tag{4.8}$$

Proof. Let $B = B(x_0, R)$ and $B' = B(x_0, R/2)$; then if $T = (2R)^\alpha$, (4.3) gives

$$p_T(x, y) \ge c V(x_0, R)^{-1}, \quad x, y \in B.$$

Fix y_0 with $d(x_0, y_0) = [3R/4]$; then if $x \in B$

$$\mathbb{P}^x(X_T \notin B') \ge \mathbb{P}^x(X_T \in B(y_0, R/4)) = \sum_{y \in B(y_0, R/4)} p_T(x, y) \mu_y \ge c \frac{V(y_0, R)}{V(x_0, R)} \ge c_3.$$

So we have $\mathbb{P}^x(\tau_{B'} > T) \le 1 - c_3$ for all $x \in B'$. Hence by the Markov property $\mathbb{P}^x(\tau_{B'} > kT) \le (1 - c_3)^k$, and thus $\mathbb{E}^x \tau_{B'} \le c_4 T$. Since $\mathbb{E}^{x_0} \tau_{B(x_0, R/2)} = \mathbb{E}^{x_0} \tau_{B'}$, replacing $R/2$ by R gives the upper bound in (4.8).

The lower bound is easy; Theorem 4.2 gives

$$\mathbb{P}^x(\tau_{B(x,R)} > R^\alpha) = \sum_{y \in B(x,R)} p_{R^\alpha}^B(x, y) \mu_y \ge c_4 > 0,$$

and thus $\mathbb{E}^x \tau_{B(x,R)} \ge c_4 R^\alpha$. $\square$

Remark 4.6 Lemma 4.5 shows that (Γ, J) cannot satisfy PHI(α) for two different values of α.

Proposition 4.7 *Suppose (Γ, J) satisfies PHI(α). Then UJS holds.*

Proof. Let $A \subset G$ and $f(t,x)$, $t \in \mathbb{R}_+$, $x \in G - A$, be a bounded non-negative function. Consider the equations

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x), \qquad x \in A, \tag{4.9}$$

$$u(0,x) = 0, \qquad x \in A, \tag{4.10}$$

$$u(t,x) = f(t,x), \qquad x \in A^c. \tag{4.11}$$

Then u is caloric on $(0,\infty) \times A$ and

$$u(t,x) = \mathbb{E}^x(f(t-\tau_A, X_{\tau_A}); \tau_A \le t). \tag{4.12}$$

Let $x_0, y_0 \in G$ and $R \le d(x_0,y_0)$. Take $A = B(x_0,R)$, let $T = R^\alpha$, $h > 0$ be small and define $f_h(t,x)$ by

$$f_h(t,x) = 1_{(x=y_0)} 1_{(T/2-h,T/2)}(t), \quad x \in G - B.$$

Let $u_h(t,x)$ be the solution of (4.9)–(4.11). Thus

$$u_h(t,x) = \mathbb{P}^x(X_{\tau_B} = y_0, t - T/2 < \tau_B < t + h - T/2).$$

Since $C_J < \infty$ we have

$$\lim_{h \downarrow 0} h^{-1} u_h(T/2,x) = \mu_x^{-1} J(x,y_0). \tag{4.13}$$

Applying the parabolic Harnack inequality to u in $(0,T) \times B(x_0,R)$ we obtain

$$u_h(T/2,x_0) \le C_1 u_h(T,x_0).$$

Now by (4.7)

$$u_h(T,x_0) = \sum_{z \in B} p^B_{T/2}(x_0,z) u_h(T/2,z)\mu_z \le c\mu(B)^{-1} \sum_{z \in B} u_h(T/2,z)\mu_z.$$

Thus

$$u_h(T/2,x_0) \le c\mu(B)^{-1} \sum_{z \in B} u_h(T/2,z)\mu_z,$$

and using (4.13) gives

$$J(x_0,y_0) \le \frac{c\mu_{x_0}}{V(x_0,R)} \sum_{z \in B(x_0,R)} J(z,y), \tag{4.14}$$

proving UJS. $\square$

Lemma 4.8 *Suppose PHI(α) holds. Let $B = B(x_0,R)$, and $B' = B(x_0,R/2)$. Then*

$$\sum_{y \in B'} J(y, G-B) \le \frac{c\mu(B')}{R^\alpha}. \tag{4.15}$$

Proof. Let $\tau = \tau_B$, and consider the martingale

$$M_t = 1_{[\tau,\infty)}(t) - \int_0^t 1_{(\tau>s)} \mu_{X_s}^{-1} J(X_s, G-B) ds.$$

Then $\mathbb{E}^x M_t = 0$, and hence

$$1 \ge \mathbb{E}^x \int_0^t 1_{(\tau>s)} \mu_{X_s}^{-1} J(X_s, G-B) ds = \int_0^t \sum_{y\in B} p_s^B(x,y) J(y, G-B) ds. \tag{4.16}$$

Using the lower bound (4.3), and writing $T = R^\alpha$,

$$1 \ge \sum_{y\in B'} J(y, G-B) \int_{T/2}^T p_s^B(x_0, y) ds \ge cT\mu(B')^{-1} \sum_{y\in B'} J(y, G-B).$$

□

Proposition 4.9 *If PHI(α) holds then UJ(α) holds, i.e.,*

$$J(x,y) \le \frac{c\mu_x \mu_y}{d(x,y)^\alpha V(x, d(x,y))}. \tag{4.17}$$

Proof. Using (1.2) and (1.3), (4.17) holds if $d(x,y) \le 3$. If $d(x,y) > 3$ let $r = \lfloor d(x,y)/3 \rfloor$. Then using Proposition 4.7 twice and (4.15) once,

$$\begin{aligned} J(x,y) &\le \frac{c\mu_x}{V(x,r)} \sum_{x' \in B(x,r)} J(x', y) \\ &\le \frac{c\mu_x}{V(x,r)} \sum_{x' \in B(x,r)} \frac{c\mu_y}{V(y,r)} \sum_{y' \in B(y,r)} J(x', y') \\ &\le \frac{c\mu_x\mu_y}{V(x,r)V(y,r)} \sum_{x' \in B(x,r)} J(x', B(x_0, 2r)^c) \\ &\le \frac{c\mu_x\mu_y}{V(x,r)V(y,r)} \frac{V(x,r)}{r^\alpha} = \frac{c\mu_x\mu_y}{r^\alpha V(y,r)}. \end{aligned}$$

Using (1.8) completes the proof. □

Lemma 4.10 *Suppose PHI(α) holds. Then PI(α) holds.*

Proof. Let B be a ball and let $\overline{X}$ denote the process X 'reflected on the boundary of B.' That is, $\overline{X}$ is the process with jump rates

$$\overline{J}(x,y) = \begin{cases} J(x,y), & \text{if } x, y \in B, \\ 0, & \text{otherwise.} \end{cases} \tag{4.18}$$

Write $\overline{p}_t(x,y)$ for the heat kernel of $\overline{X}$. Then by Theorem 4.2

$$\overline{p}_t(x,y) \ge p_t^B(x,y) \ge \frac{c}{V(x_0, R/2)}, \quad x, y \in B(x_0, R/2). \tag{3.16}$$

This lower bound then gives PI(α) by a standard argument, as in [SC2], p. 159–160. □

We summarize the results of this section in the following Proposition.

Proposition 4.11 *Suppose PHI(α) holds.*
(a) (Γ, J) satisfies VD, UHD(α), NDLB(α), SB(α), E_α, UJS, UJ(α) and PI(α).
(b) Suppose that in addition (Γ, J) satisfies $V(d)$. Then UHKP(α) holds.

5 Consequences of the on-diagonal upper bound

In this section, we assume that (Γ, J) satisfies VD and $SB(\alpha)$. For $A \subset G$ let

$$\mathcal{F}_A = \{u \in L^2(G, \mu) : u|_{G-A} = 0\}. \tag{5.1}$$

Lemma 5.1 *Suppose VD and $SB(\alpha)$ hold. Then there exists $c_1 > 0$ such that for all $x_0 \in G$, $r \geq 1$,*

$$\mathcal{E}(u, u) \geq c_1 \frac{\|u\|_2^2}{r^\alpha}, \quad \textit{for all} \;\; u \in \mathcal{F}_{B(x_0,r)}. \tag{5.2}$$

Proof. Let $\lambda \geq 1$ (to be chosen later) and $B = B(x_0, \lambda r)$. Using the log-convexity of $t \mapsto \|P_t^B u\|_2^2$ (see [Cou] Lemma 3.2 for the proof), we have

$$\frac{(P_t^B u, u)}{\|u\|_2^2} \geq \exp\Big(-\frac{\mathcal{E}(u,u)}{\|u\|_2^2} t\Big), \qquad \text{for all } u \in \mathcal{F}_B, \quad t \geq 0. \tag{5.3}$$

By interpolating $SB(\alpha)$ with $\|P_t^B\|_{1\to 1} \leq 1$, and using VD, we obtain

$$\|P_t^B u\|_2^2 \leq c_1 V(x_0, \lambda r)^{-1} \|u\|_1^2, \qquad c_2(\lambda r)^\alpha \leq t \leq c_3 (\lambda r)^\alpha.$$

Substituting this into (5.3) with $2t$ instead of t,

$$\frac{c_1 \|u\|_1^2}{V(x_0, \lambda r)\|u\|_2^2} \geq \exp\Big(-2\frac{\mathcal{E}(u,u)}{\|u\|_2^2} t\Big), \qquad c_2(\lambda r)^\alpha \leq t \leq c_3(\lambda r)^\alpha, \; u \in \mathcal{F}_B \cap L^1. \tag{5.4}$$

Let $u \in \mathcal{F}_{B(x_0,r)} \cap L^1$, and $t = c_3(\lambda r)^\alpha$; then using the Cauchy-Schwarz inequality,

$$\frac{c_1 \|u\|_1^2}{V(x_0, \lambda r)\|u\|_2^2} \leq \frac{c_1 V(x_0, r)}{V(x_0, \lambda r)}. \tag{5.5}$$

As $r \geq 1$, using (1.9), we can choose λ so that the right hand side of (5.5) is less than e^{-1}. (5.4) with $t = c_3(\lambda r)^\alpha$ and (5.5) now give (5.2). □

Let

$$M_1(x, r) = \sum_{y \in B(x,r)} d(x, y)^2 J(x, y), \qquad M_2(x, r) = \sum_{y \in B(x,r)^c} J(x, y).$$

Lemma 5.2 *Suppose VD and $SB(\alpha)$ hold. Then for all x_0, r*

$$\frac{c_1 V(x_0, r)}{r^\alpha} \leq \sum_{x \in B(x_0,r)} (r^{-2} M_1(x, r) + M_2(x, r)). \tag{5.6}$$

Proof. Let $x_0 \in G$, and let $r > 0$. Consider the function

$$f(y) = (1 - r^{-1} d(x_0, y))_+.$$

Let $A(x) = \{y : d(x_0, y) \geq d(x_0, x)\}$, and

$$\Gamma f(x) = \sum_{y \in A(x)} (f(x) - f(y))^2 J(x, y).$$

Then $f(x) = 0$ and $\Gamma f(x) = 0$ if $x \in B(x_0, r)^c$, so

$$\mathcal{E}(f, f) \leq 2 \sum_{x \in G} \sum_{y \in A(x)} (f(x) - f(y))^2 J(x, y) = \sum_{x \in B(x_0, r)} \Gamma f(x).$$

Since $|f(x) - f(y)| \leq (c_0 r)^{-1} d(x, y)$, and $0 \leq f \leq 1$, for $x \in B(x_0, r)$ we have

$$\Gamma f(x) \leq c \sum_{B(x,r)} (d(x, y)/r)^2 J(x, y) + \sum_{B(x,r)^c} J(x, y) \leq c r^{-2} M_1(x, r) + M_2(x, r).$$

Combining these inequalities

$$\mathcal{E}(f, f) \leq C \sum_{B(x_0, r)} (r^{-2} M_1(x, r) + M_2(x, r)), \tag{5.7}$$

and (5.6) follows by Lemma 5.1.

Proposition 5.3 *Let $0 < \alpha < 2$ and assume VD. Suppose $SB(\alpha)$ and $UJ(\alpha)$ hold. Then there exist $\delta > 0$, $\lambda < \infty$ (depending only on α and on the constants C in $SB(\alpha)$ and $UJ(\alpha)$) so that for all $x_0 \in G$, $r \geq 1$,*

$$\sum_{x \in B(x_0, r)} \Big[\sum_{y \in B(x, \lambda r) - B(x, \delta r)} J(x, y) \Big] \geq c_1 \frac{V(x_0, r)}{r^\alpha}. \tag{5.8}$$

Proof. Note that the term in brackets on the left side of (5.8) is $M_2(x, \delta r) - M_2(x, \lambda r)$.

Using UJ(α) and VD, and the fact that $\alpha < 2$, we have

$$\begin{aligned}
M_1(x, r) &\leq c_1 \sum_{B(x,r)} \frac{d(x, y)^{2-\alpha} \mu_x \mu_y}{V(x, d(x, y))} \\
&\leq c_2 \mu_x \sum_{i=0}^{\infty} \sum_{y \in B(x, 2^{-i} r) - B(x, 2^{-i-1} r)} \frac{d(x, y)^{2-\alpha} \mu_y}{V(x, d(x, y))} \\
&\leq c_2 \mu_x \sum_{i=0}^{\infty} (2^{-i} r)^{2-\alpha} \frac{V(x, r 2^{-i})}{V(x, 2^{-i-1} r)} \\
&\leq c_3 \mu_x \sum_{i=0}^{\infty} (2^{-i} r)^{2-\alpha} = c_4 \mu_x r^{2-\alpha},
\end{aligned}$$

and

$$\begin{aligned}
M_2(x,r) &\le c_5 \sum_{i=0}^{\infty} \sum_{B(x,2^{i+1}r)-B(x,2^i r)} \frac{\mu_x \mu_y}{d(x,y)^\alpha V(x,d(x,y))} \\
&\le c_5 \mu_x \sum_{i=0}^{\infty} (2^i r)^{-\alpha} \frac{V(x,2^{i+1}r)}{V(x,2^i r)} \le c_6 \mu_x r^{-\alpha}.
\end{aligned}$$

So, for $x \in B(x_0,r)$,

$$\begin{aligned}
r^{-2}M_1(x,r) + M_2(x,r) &= \sum_y (r^{-2}d(x,y)^2 \wedge 1) J(x,y) \\
&\le r^{-2}M_1(x,\delta r) + M_2(x,\delta r) \\
&\le \mu_x r^{-\alpha}(c_4 \delta^{2-\alpha} + c_6 \lambda^{-\alpha}) + M_2(x,\delta r) - M_2(x,\lambda r).
\end{aligned}$$

Now choose $\delta > 0$ small enough and $\lambda > 0$ large enough so that

$$c_4 \delta^{2-\alpha} + c_6 \lambda^{-\alpha} \le \tfrac{1}{2} c_{5.2.1};$$

then summing over $x \in B(x_0,r)$ and using (5.6) we deduce (5.8). □

Lemma 5.4 *Suppose VD and LJS hold. Then if $x \sim y$ and $z \ne x,y$,*

$$J(x,z) \ge cJ(y,z).$$

Proof. If $d(x,z) = 1$ then by (1.13) $J(x,z) \ge c_0$, while by (1.2) $J(y,z) \le C_J$. If $d(x,z) \ge 2$ then by (1.12) and VD

$$J(x,z) \ge c \frac{\mu_x}{V(x,1)} \sum_{w \in B(x,1)} J(w,z) \ge c \frac{\mu_x}{V(x,1)} J(y,z) \ge c' J(y,z).$$

□

Lemma 5.5 *Suppose VD, LJS and UJS hold. Then JS holds.*

Proof. We prove (1.14). Let $d(x_0,y) = R$. If $R \le 4$ then we can use Lemma 5.4, so suppose $R \ge 4$. Suppose first that $d(x_1,x_0) \le R/4$. Then writing $s = R/4$, and using UJS and LJS,

$$J(x_1,y) \le \frac{c\mu_{x_1}}{V(x_1,s)} \sum_{z \in B(x_1,s)} J(z,y) \le \frac{c\mu_{x_1}}{V(x_1,s)} \sum_{z \in B(x_0,2s)} J(z,y) \le \frac{c\mu_{x_1} V(x_1,2s)}{\mu_{x_0} V(x_1,s)} J(x_0,y).$$

Using VD and (1.3) then gives $J(x_1,y) \le c_1 J(x_0,y)$, proving (1.14). If $d(x_1,x_0) \ge R/4$ then (1.14) follows by an easy chaining argument. □

We need a general lemma on symmetric functions on $G \times G$ which satisfy conditions similar to JS. See [Ba] for a similar argument.

Lemma 5.6 *Let $g : G \times G \to \mathbb{R}_+$ satisfy $g(x,y) = g(y,x)$ for all x, y, and also the conditions*

$$g(x,y) \geq c_0, \qquad \text{if } x \sim y, \tag{5.9}$$
$$g(x,z) \geq c_0 g(y,z), \qquad \text{if } x \sim y, z \neq x, y. \tag{5.10}$$

Suppose that for some $\kappa \in (0,1)$, $c_1 < \infty$,

$$g(x,y) \leq c_1 g(x,y') \qquad \text{if } d(y,y') \leq \kappa d(x,y). \tag{5.11}$$

Then given $0 < \delta < \lambda < \infty$, there exists a constant C_1, depending only on $c_1, \kappa, \delta, \lambda$, such that the following holds. If $x_0, y_0 \in G$ with $d(x_0,y_0) = r$ then

$$C_1^{-1} g(x_0,y_0) \leq g(x,y) \leq C_1 g(x_0,y_0) \qquad \text{whenever } x, y \in B(x_0, \lambda r),\ d(x,y) \geq \delta r. \tag{5.12}$$

Proof. Let H be the metric space obtained by replacing each edge of the graph G by a line segment of length 1. (In [BB] this is called the 'cable system' of G.) We write d for the metric on H. Extend g to a function h on $H \times H$ by linearity on each cable; then the conditions (5.9) and (5.10) imply that (5.11) also holds for h. So it is now enough to prove the Lemma for h.

We can assume $\delta \leq \frac{1}{4}$ and $\lambda \geq 2$. Also, by an easy chaining argument we can assume $\kappa = \frac{1}{2}$. Note first that (5.11) implies

$$h(x,y) \asymp h(x,y') \qquad \text{whenever } d(y,y') \leq \tfrac{1}{2}(d(x,y) \vee d(x,y')). \tag{5.13}$$

Given $x, y \in H$ let $\gamma(x,y)$ denote a shortest (geodesic) path between x and y. Suppose $x, y \in H$, $d(x,y) = s$, and $z \in \gamma(x,y)$ with $d(x,z) \geq s/2$. Then by (5.13)

$$h(x,y) \asymp h(x,z). \tag{5.14}$$

Using this repeatedly, we can compare h on any geodesic path. More precisely, if $x, y \in H$, $d(x,y) = s$ then we have

$$h(x,y) \asymp h(x',y') \qquad \text{for } x', y' \in \gamma(x,y) \text{ with } d(x',y') \geq \tfrac{1}{2}\delta s. \tag{5.15}$$

Now let $x_0, y_0 \in H$ with $d(x_0,y_0) = r$, and $x_1, y_1 \in B(x_0, \lambda r)$, $d(y_1,x_1) \geq \delta r$. As G is infinite there exists $w \in H$ with $d(x_0,w) = 5\lambda r$. Suppose we can prove:

$$h(x',y') \asymp h(x',w) \qquad \text{for all } x', y' \in B(x_0, \lambda r) \text{ with } d(x',y') \geq \delta r. \tag{5.16}$$

Then we have $h(x_j,y_j) \asymp h(x_j,w)$ for $j = 0, 1$. But since $d(x_0,x_1) \leq \lambda r \leq d(x_0,w)$, using (5.13) we have $h(x_0,w) \asymp h(x_1,w)$, and so we obtain

$$h(x_0,y_0) \asymp h(x_1,y_1). \tag{5.17}$$

It remains to prove (5.16). Suppose first that

$$d(x',z) \geq \delta r/2 \qquad \text{for all } z \in \gamma(y',w). \tag{5.18}$$

Then chaining the relation (5.13) along $\gamma(y',w)$ gives $h(x',y') \asymp h(x',w)$, proving (5.16).

Now suppose that (5.18) fails. Then there exists $z \in \gamma(y',w)$ with $d(x',z) \leq \delta r/2$. By (5.13) $h(y',x') \asymp h(y',z)$. Also, $d(y',z) \geq \delta r/2$, so using (5.15) we obtain $h(y',z) \asymp h(y',w)$. Finally, as $d(x',y') \leq 2\lambda r$ and $d(y',w) \geq 4\lambda r$, (5.13) gives $h(x',w) \asymp h(y',w)$. Combining these inequalities gives (5.16) in this case also. $\square$

Lemma 5.7 *(a) VD, (5.8) and JS imply LJ(α).*
(b) Assume $0 < \alpha < 2$. Then VD, $SB(\alpha)$, UJ(α) and JS imply J(α).

Proof. (a) Let $x_0, y_0 \in H$, and $d(x_0, y_0) = r$. Then by (5.8), JS and Lemma 5.6,

$$\begin{aligned} c\frac{V(x_0,r)}{r^\alpha} &\le \sum_{x\in B(x_0,r)} \Big(\sum_{y\in B(x,\lambda r)-B(x,\delta r)} J(x,y) \Big) \\ &\le c' J(x_0,y_0) \sum_{x\in B(x_0,r)} V(x,\lambda r) \\ &\le c' J(x_0,y_0) V(x_0,r) V(x_0,(1+\lambda)r), \end{aligned}$$

and using VD we obtain LJ(α).
(b) By Proposition 5.3, VD, $SB(\alpha)$ and UJ(α) imply (5.8). (a) now gives LJ(α), and so J(α) holds. □

Proposition 5.8 *Assume $0 < \alpha < 2$.*
(a) PHI(α) implies (5.8).
(b) PHI(α) and LJS imply J(α).

Proof. (a) Assume $PHI(\alpha)$. Then by Corollary 4.3, Corollary 4.4 and Proposition 4.9, VD, $SB(\alpha)$ and UJ(α) hold. Hence, by Proposition 5.3, (5.8) holds.
(b) Since PHI(α) also implies UJS (due to Proposition 4.7) and VD (due to Corollary 4.3), by Lemma 5.5 we obtain JS and hence, by Lemma 5.7(a), J(α) holds. □

Proof of Theorem 1.5. (a) ⇒ (b). This has been proved in the context of Markov processes on $\mathbb{Z}^d$ and on d-sets in [BL, CK1, CK2]. The transfer of these arguments to a graph satisfying $V(d)$ is straightforward. (b) ⇒ (a) is immediate, since we have $J(x,y) = \mu_x\mu_y \lim_{t\to 0} t^{-1} p_t(x,y)$.

Now suppose J(α) and HKP(α) hold. Then UJS holds, and by Lemma 2.3, NDLB(α) holds. Therefore, by Proposition 3.3 PHI(α) holds. Thus ((a) and (b)) together imply (c). Finally, by Proposition 5.8 we have (c) ⇒ (a). □

Proof of Theorem 1.6. That (a) implies (b) is immediate from Proposition 4.11. We remark that this does not need $V(d)$ or $\alpha < 2$.
(b) ⇒ (a). This follows by combining Lemmas 3.1 and 3.2, and Proposition 3.3. □

Proof of Theorem 1.4. (a) is contained in the implication (b) ⇒ (c) in Theorem 1.5. (Note that this part of the argument does not use $V(d)$ or $\alpha < 2$.) (b) is immediate from Proposition 4.11. □

Remark 5.9 One might ask if the three conditions in Theorem 1.6 are independent.

1. If we drop UJ(α) then we have no upper bound on J. If UJS and PI(α) hold, then since PI(α) implies PI(α') for any $\alpha' > \alpha$, we have UJS and PI(α') for all $\alpha' \ge \alpha$. However, by the remark following Lemma 4.5 we cannot have PHI(α') for any $\alpha' > \alpha$.
2. If we drop PI(α) then we have no lower bound on J. We can set $J(x,y) = d(x,y)^{-d-\alpha}$, and note that UJ($\alpha'$) and UJS hold for any $\alpha' < \alpha$.

3. We do not have an example to prove that UJS is independent of PI(α) and UJ(α). Note that since PI(α) implies a Nash inequality, (3.10) does hold, and so gives some kind of lower bound on J. We 'only just' needed to use UJS in the proof of Proposition 3.3, to control $Jw_r(z)$ when z and y are far apart.

Remark 5.10 In the definition of PHI(α) we included a parameter $\lambda \in (0,1]$. Suppose we call PHI$(1,\alpha)$ the PHI just with $\lambda = 1$. Then PHI$(1,\alpha)$ is enough to obtain UJ(α), PI(α) and UJS, so Theorem 1.6 gives that PHI$(1,\alpha)$ and PHI(α) are equivalent.

6 Counterexamples.

VD and PI(α) do not imply PHI(α).

Let $G = \mathbb{Z}^d$, $\alpha \in (0,2)$, and $\mu_x = 1$ for all x. So VD, and indeed V(d), hold. Let $J(x,y) = |x-y|^{-d-\alpha}$ for all $x, y \in \mathbb{Z}^d$. Then J(α) holds, so using Theorems 1.5 and 1.6 PHI(α) and hence PI(α) hold. Now let $\alpha < \alpha' < 2$; then PI(α') also holds. Thus we have VD and PI(α'), while by Lemma 4.6 PHI(α') cannot hold.

PHI(α) does not imply HKP(α).

Let $G = \mathbb{Z}^d$, $\alpha \in (0,2)$ and let $J_1(x,y) = |x-y|^{-d-\alpha}$ for $x \neq y$. Note that $V(d)$ and J(α) hold for J_1. So by Theorem 1.2 we have that HKP(α) and PHI(α) hold for J_1. (Of course, for this example this was already well known.) So, by Theorem 1.6, $PI(\alpha)$, UJS, and UJ(α) hold for J_1.

Choose $R \in 2\mathbb{N}$, with $R \gg 1$, and let $y_0 = (R, 0, \dots, 0)$ be on the x_1–axis with $|y_0| = d(0, y_0) = R$. Then let

$$J(x,y) = \begin{cases} J_1(x,y), & \text{if } \{x,y\} \neq \{0,y_0\}, \\ 0, & \text{if } \{x,y\} = \{0,y_0\}. \end{cases} \tag{6.1}$$

(So we just suppress jumps between 0 and y_0.) Since J(α) fails for J, by Theorem 1.5 HKP(α) must fail.

However, PHI(α) does hold. To see this we use Theorem 1.6, and verify that UJ(α), UJS and $PI(\alpha)$ all hold. First, as $J \le J_1$, UJ(α) is immediate. Since $J(0,\cdot)$ has only been modified from $J_1(0,\cdot)$ at one point, it is straightforward to verify that UJS still holds for J.

Finally, to verify $PI(\alpha)$, let $B = B(x_0, r)$ be a ball in $\mathbb{Z}^d$, and $f : B \to \mathbb{R}$. If B does not contain both 0 and y_0 then

$$\sum_{x\in B}\sum_{y\in B}(f(x)-f(y))^2 J_1(x,y) = \sum_{x\in B}\sum_{y\in B}(f(x)-f(y))^2 J(x,y),$$

so the Poincaré inequality for J follows from that for J_1. Now suppose that $0, y_0 \in B$. Then let y_1 be the mid-point of the line between 0 and y_0. We have

$$\begin{aligned}(f(0)-f(y_0))^2 J_1(0,y_0) &\le 2((f(0)-f(y_1))^2 + (f(y_1)-f(y_0))^2)R^{-d-\alpha} \\ &= 2^{d+\alpha+1}\Big(((f(0)-f(y_1))^2 J(0,y_1) + (f(y_1)-f(y_0))^2)J(y_1,y_0)\Big) \\ &\le 2^{d+\alpha+1}\sum_{x\in B}\sum_{y\in B}(f(x)-f(y))^2 J(x,y).\end{aligned}$$

Thus
$$\sum_{x\in B}\sum_{y\in B}(f(x)-f(y))^2 J_1(x,y) \le c\sum_{x\in B}\sum_{y\in B}(f(x)-f(y))^2 J(x,y),$$
and this implies that the Poincaré inequality holds for J.

EHI + E_α + $V(d)$ does not imply PHI(α).

We only give an outline of this example. Let $G=\mathbb{Z}$, $\alpha\in(1,2)$ and $J_0(x,y)=|x-y|^{-1-\alpha}$. Let $R_1 \gg 1$, and set $J_1(x,y)=(\log R_1)R_1^{-1-\alpha}1_{(|x-y|=R_1)}$. Let $X_t^{(i)}$, $i=0,1$ be independent processes associated with the jump kernels J_i. Let $X=X^{(0)}+X^{(1)}$; this is the process with jump kernel $J=J_0+J_1$. We take $\mu_x=1$ for all x.

We begin by remarking that HKP(α) does hold for $X^{(0)}$. In addition this process is 'strongly recurrent': one has
$$\mathbb{P}^x(T_y^{(0)} \le \tau_R^{(0)}) \ge p_0 > 0 \quad \text{ for } x,y\in[-R/2,R/2], \tag{6.2}$$
where $\tau_R^{(0)}$ is the exit time from $B(0,R)=[-R,R]$ by $X^{(0)}$, and $T_y^{(0)}$ is the hitting time of y by $X^{(0)}$.

We now show that X satisfies E_α. The upper bound is easy. Since $J\ge J_0$, the Nash inequality (3.1) holds for X. Hence, by [CKS], the transition density of X satisfies
$$p_t(x,y)\le c_1 t^{-1/\alpha}, \quad t>0.$$
So taking c_2 large enough, if $t=c_2r^\alpha$ then $\mathbb{P}^x(X_t\in B(0,r))\le \frac12$ for any $x\in\mathbb{Z}$, and the upper bound $E^x\tau_{B(0,r)}\le c_3r^\alpha$ follows.

For the lower bound, note that the condition E_α for $X^{(0)}$ implies that
$$\mathbb{P}^0(\tau_R^{(0)}\le\lambda R^\alpha)\ge c_4\lambda.$$
Thus there exists $c_5>0$ such that, $\mathbb{P}^0(\tau_R^{(0)}\ge c_5R^\alpha)\ge c_5$.

Let $\delta=R_1^{-1-\alpha}\log R_1$. By Doob's inequality, writing $Y_t=\sup_{s\le t}|X_s^{(1)}|$,
$$\mathbb{E}^0Y_T^2\le 4R_1^2\delta T, \tag{6.3}$$
and so
$$\mathbb{P}^0(Y_T\ge\lambda)\le\frac{4T\log R_1}{\lambda^2R_1^{\alpha-1}}. \tag{6.4}$$
So
$$\mathbb{P}^0(\tau_R\ge c_5R^\alpha)\ge\mathbb{P}^0(\tau_{R/2}^{(0)}\ge c_5R^\alpha, Y_T\le R/2)\ge\tfrac12 c_5,$$
provided R_1 is large enough. This establishes the lower bound in E_α for X.

To prove EHI it is enough to prove (6.2) for X, and using translation invariance and chaining it is enough to prove that
$$\mathbb{P}^x(T_0\le\tau_R)\ge p_1>0 \quad \text{ for } x\in[-R/4,R/4]. \tag{6.5}$$

As the whole argument is more lengthy than this counterexample deserves, we only sketch the main ideas. We note that there exists $\theta\in(0,1)$ such that
$$\mathbb{P}^x(T_0^{(0)}>\tau_R^{(0)})\le c_6(|x|/r)^\theta, \quad \text{ for } x\in[-R/4,R/4]. \tag{6.6}$$

Fix an interval R, and let $x \in [-R/2, R/2]$. We concentrate on the case when $R_1 \ll |x| \ll R$. Choose $\varepsilon > 0$ small, and let $r = r(x) = |x|^{1+\varepsilon}$, $t = t(x) = x^{\alpha+\alpha\varepsilon-\varepsilon\theta}$. Let $F = \{T_0^{(0)} < t, \tau_R^{(0)} > t\}$. Then

$$\begin{aligned}\mathbb{P}^x(F^c) &\leq \mathbb{P}^x(T_0^{(0)} \geq \tau_r^{(0)}) + \mathbb{P}^x(\tau_r^{(0)} > t) + \mathbb{P}^x(\tau_R^{(0)} < t) \\ &\leq c(|x|/r)^\theta + cr^\alpha/t + ctR^{-\alpha}.\end{aligned}$$

With the choices of r and t as above, one obtains $\mathbb{P}(F^c) \leq 3|x|^{-\varepsilon\theta}$ provided $|x| \leq R^{1/(1+\varepsilon)}$. Let $G = \{Y_t \leq x^{\alpha(1+\varepsilon)/2}\}$. Then, using (6.4), we have $\mathbb{P}^x(G^c) \leq |x|^{-\varepsilon\theta}$ also.

Suppose first that $|x| \leq R^{1/(1+\varepsilon)}$. Then run X and $X^{(0)}$ until $S_1 = T^{(0)}$. We declare the run a success if both F and G occur, so that success has a probability greater than $1 - c|x|^{-\varepsilon\theta}$. If the run is a success then we have $X_{S_1} = V_1$, where $|V_1| \leq x^{\alpha(1+\varepsilon)/2}$. We now repeat from the new starting point, and (if all the runs are successful) continue until we obtain $X_{S_N} = V_N$ with $|V_N| \leq R_1$. Summing the probabilities of failures, we find that, by choosing R_1 large enough, this can be made as small as we like.

If we start at a point in $(-R_1, R_1)$, a variant of the argument above gives that, with probability $p_1 > 0$, $X^{(0)}$ hits 0 before the first jump of $X^{(1)}$. Finally, if $R^{1/(1+\varepsilon)} < |x| \leq R/4$ then running $X^{(0)}$ until $S_0 = T^{(0)}$ we find with probability $p_2 > 0$ that $|X_{S_0}| \leq R^{1/(1+\varepsilon)}$.

We deduce from this that X satisfies E_α and EHI with constants which do not depend on R_1. On the other hand, X only satisfies UJ(α) with a constant of order $\log R_1$. This is enough to prove that the 'strong' form of the implication "VD+ EHI + $E_\alpha \Rightarrow$ PHI(α)" is false. That is (see Remark 1.7), we cannot have PHI(α) with a constant C_P depending only on the constants in VD, EHI and E_α.

To actually obtain a single graph which satisfies VD, EHI and E_α but not PHI(α), one needs to modify the example above as follows. Take a rapidly increasing sequence R_n, define J_n analogously to J_1, and let $J = J_0 + \sum_{n\geq 1} J_n$. This clearly fails to satisfy UJ(α), and so PHI(α) must also fail. However, arguments similar to the above show that E_α and EHI still hold.

Remark 6.1 A recent paper [BS] gives necessary and sufficient conditions for EHI to hold for α-stable processes in $\mathbb{R}^d$ with Lévy measure of the form

$$\nu(dx) = |x|^{-d-\alpha} f(x/|x|)dx,$$

where $f : S^{d-1} \to \mathbb{R}_+$ is bounded and symmetric. The condition in [BS] appears rather weaker than UJS. If (as one may expect) the results of [BS] hold also for processes on $\mathbb{Z}^d$, this would give another class of examples when VD, EHI and E_α hold, but PHI(α) fails.

References

[Ba] M.T. Barlow. Some remarks on the elliptic Harnack inequality. *Bull. Lon. Math. Soc.* **37** (2005), 200-208.

[BB] M.T. Barlow, R.F. Bass. Stability of parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** (2003) no. 4, 1501–1533.

[BBCK] M.T. Barlow, R.F. Bass, Z.-Q. Chen, M. Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.*, to appear.

[BGK] M.T. Barlow, A. Grigor'yan, T. Kumagai. Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.*, to appear.

[BL] R.F. Bass, D.A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.* **354**, no. 7 (2002), 2933–2953.

[BG] R.M. Blumenthal, R.K. Getoor. *Markov Processes and Potential Theory.* Academic Press, Reading, MA, 1968.

[BS] K. Bogdan, P. Sztonyk. Harnack's inequality for stable Lévy processes. *Potential Anal.* **22** (2005), no. 2, 133–150.

[CKS] E.A. Carlen, S. Kusuoka, D.W. Stroock. Upper bounds for symmetric Markov transition functions. *Ann. Inst. H. Poincaré Probab. Statist.* **23** (2 suppl.) (1987), 245–287.

[CK1] Z.-Q. Chen, T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. Stochastic Process Appl. **108** (2003), 27-62.

[CK2] Z.-Q. Chen, T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008), 277–317.

[Cou] T. Coulhon. Heat kernel and isoperimetry on non-compact Riemannian manifolds. *Contemporary Mathematics* **338**, pp. 65–99, Amer. Math. Soc. 2003.

[Del] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Math. Iberoamericana* **15** (1999), 181–232.

[Fo] M. Foondun. Heat kernel estimates and Harnack inequalities for some Dirichlet forms with non-local part. Preprint 2006.

[Gr] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik.* **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.

[GT] A. Grigor'yan, A. Telcs. Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Annalen* **324** (2002), 521–556.

[HSC] W. Hebisch, L. Saloff-Coste. On the relation between elliptic and parabolic Harnack inequalities. *Ann. Inst. Fourier (Grenoble)* **51** (2001), 1437–1481.

[SC1] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* **2** (1992), 27–38.

[SC2] L. Saloff-Coste. *Aspects of Sobolev-Type inequalities.* Lond. Math. Soc. Lect. Notes **289**, Cambridge Univ. Press, 2002.

[SS] L. Saloff-Coste, D.W. Stroock. Opérateurs uniformément sous-elliptiques sur les groupes de Lie. *J. Funct. Anal.* **98** (1991) 97–121.

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