Brownian motions on fractals

Takashi Kumagai *
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan
kumagai@kurims.kyoto-u.ac.jp
http://www.kurims.kyoto-u.ac.jp/~kumagai

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Abstract

We summarize various properties for Brownian motions on fractals, especially on the Sierpinski gaskets. The heat kernel for the process enjoys sub-Gaussian estimates and parabolic Harnack inequalities hold for the corresponding self-adjoint operator, where the time scale is the d_w-th power of the space scale for some $d_w \geq 2$. We give an overview of recent developments on stability of the inequalities under perturbations of the associated Dirichlet forms.
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1 Introduction

1.1 Overview

We first give an overview of the history and the current directions of research concerning stochastic processes on fractals. The pioneering work in this field is by mathematical physicists who have tried to analyse properties of disordered media such as heat transfer and wave transfer (a good survey is [H-BA*] – throughout the text, references with stars indicate books, lecture notes or surveys). Examples of disordered media are: polymers and networks where the objects are deterministic; phenomena like growth of molds and crystals where the objects are random. In each case, one cannot expect any invariance property under the action of some continuous group such as the group of translations. Mathematical physicists found some self-similar property, i.e. similarity between global parts and local parts, of the objects and tried to analyse them using this property. Motivated by these works, probabilists started to investigate in this field since the middle eighties. The first works are by Goldstein ([Gol]), Kusuoka ([Kus3]) and Barlow-Perkins ([B-P]), who constructed a diffusion process on the Sierpinski gasket which is a

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typical fractal. After that, Kigami ([Kig4]) constructed a Laplacian on the gasket as a limit of difference operators. This analytical approach motivated a work by Fukushima-Shima ([Fu-S]), which made it clear that the theory of Dirichlet forms was well-applicable to this area.

These results were then extended to various classes of fractals. Lindstrøm ([Lind]) introduced a class of finitely ramified fractals (i.e. self-similar sets that can be disconnected by removing a specific finite number of points), called nested fractals and constructed diffusion processes on them. Nested fractals are a class of fractals that have spatial symmetry, and include the gasket as an example. Then, Kigami introduced post critically finite self-similar sets (cf. [Kig*]), which roughly correspond to finitely ramified fractals. He constructed Dirichlet forms on them assuming the existence of 'harmonic structures'. As we will see in the next section for the gasket case, there is a finite dimensional recursive structure when constructing processes on such fractals. On the other hand, there is no such structure for infinitely ramified fractals, because the recursive structure is infinite dimensional. So the construction is much harder. Barlow-Bass ([B-B2, B-B3], etc.) constructed diffusion processes and obtained various properties for Sierpinski carpets which are typical infinitely ramified fractals. We note that for post critically finite self-similar sets, 'typical' diffusions are point recurrent (even when they are naturally embedded in high dimensional Euclidean spaces), whereas for the high dimensional Sierpinski carpets, typical diffusions are transient. While these diffusion processes take values on deterministic fractals, there are also works which construct diffusion processes on random Sierpinski gaskets by Hambly (cf. [Ham*]).

In this way, analytical properties of the diffusion processes (and the corresponding self-adjoint operators) have been studied, and it is getting clear that the diffusions on fractals have completely different properties from diffusions on Euclidean spaces. For instance, it is understood that such processes typically have sub-diffusive behaviour and heat kernels for Brownian motion on 'nice' fractals enjoy sub-Gaussian estimates (2.5). Through substantial amount of work, stochastic processes on fractals have been shown to be related to various other fields. To me, the following two directions of research are very attractive and seem to be promising.

(1) <u>Stability of parabolic Harnack inequalities and heat kernel estimates</u>
It is shown by Barlow-Bass ([B-B]) that the sub-Gaussian heat kernel estimates (which are equivalent to parabolic Harnack inequalities) are stable under perturbations of the Dirichlet forms when the state spaces are graphs. This type of stability is applicable to homogenization problems. It is relevant to global analysis on graphs/manifolds and further to the study of heat transfer on stochastic models such as percolation clusters.

(2) <u>Function spaces and stochastic processes on fractals</u>
Recently, it was discovered that the domains of Dirichlet forms on fractals are Besov-Lipschitz spaces (cf. [Jons, Kum2] etc.). This motivated further study of Besov-type function spaces (that has been done by researchers working in harmonic analysis, PDE, etc.) using probability and potential theory. For instance, detailed estimates of heat kernels can be obtained for operators naturally defined from norms of Besov spaces.

In this note, we shall restrict ourselves to the Sierpinski gasket and explain the construction of Brownian motion (using Dirichlet forms) and various properties in Section 2. In Section 3, we shall briefly explain the direction (1) above. Note that there is a nice recent survey of this direction [Bar1*]. We do not discuss about the direction (2) here. Readers interested in this direction may refer to surveys [Gri*, Kum*].

For Dirichlet forms and diffusion processes on nested fractals and post critically finite self-similar sets, there are the following lecture notes and a book [Bar2*, Kig*, Kus*], see also surveys [Lap*, Str*]. For diffusions on the Sierpinski carpets, [Bass*] is a good survey.

1.2 A quick view of the theory of Dirichlet forms

We briefly recall the definition of Dirichlet forms and the correspondence with processes following [FOT*]. Let X be a locally compact separable metric space and ν be a positive Radon measure on X whose support is X. Let $\mathcal{E}$ be a symmetric bilinear closed form with a domain $\mathcal{F}$ which is dense in $\mathbb{L}^2(X,\nu)$. $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form if it is Markovian, i.e., for each $u\in\mathcal{F}$, $v:=(0\vee u)\wedge 1\in\mathcal{F}$ and $\mathcal{E}(v,v)\le\mathcal{E}(u,u)$. A Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular if there exists $C\subset\mathcal{F}\cap C_0(X)$ such that C is dense in $\mathcal{F}$ with $\mathcal{E}_1$-norm and C is dense in $C_0(X)$ under the uniform norm, where $C_0(X)$ is a space of continuous compactly supported functions on X and $\mathcal{E}_1(\cdot,\cdot)=\mathcal{E}(\cdot,\cdot)+\|\cdot\|^2_{\mathbb{L}^2}$. $(\mathcal{E},\mathcal{F})$ is local if for each $u,v\in\mathcal{F}$ whose supports are disjoint compact sets, $\mathcal{E}(u,v)=0$. There is a one to one correspondence between regular Dirichlet forms on $\mathbb{L}^2(X,\nu)$ and ν-symmetric Hunt processes (i.e., strong Markov processes whose paths are right continuous and quasi-left continuous w.r.t. some filtration) on X except for some exceptional set of starting points. Further, if the regular Dirichlet form is local, then the corresponding process is a diffusion process (i.e. Hunt process with continuous paths).

2 Dirichlet forms on the Sierpinski gaskets

Let D_n be a n-dimensional simplex whose vertices are $\{p_0,p_1,\cdots,p_n\}$ where p_0 is the origin. For $i=1,2,\cdots,n+1$, let $F_i(z)=(z-p_i)/2+p_i$, $z\in\mathbb{R}^n$. Then, there exists a unique non-void compact set K such that $K=\cup_{i=1}^{n+1}F_i(K)$. This K is called a (n-dimensional) *Sierpinski gasket*. When $n=1$, $K=[p_0,p_1]$.

For simplicity, we will consider the 2-dimensional Sierpinski gasket (Figure 1) and we will abbreviate it as S.G.. Denote $V_0=\{p_0,p_1,p_2\}$, $V_n=\cup_{i_1,\cdots,i_n\in I}F_{i_1\cdots i_n}(V_0)$ where $I:=\{1,2,3\}$ and $F_{i_1\cdots i_n}:=F_{i_1}\circ\cdots\circ F_{i_n}$. Let $V_*=\cup_{n\in\bar{\mathbb{N}}}V_n$, where $\bar{\mathbb{N}}:=\mathbb{N}\cup\{0\}$. Then $K=Cl(V_*)$.

The Hausdorff dimension of K (w.r.t. the Euclidean metric) is $\log 3/\log 2=: d_f$. Let μ be a Borel measure on K such that

$$\mu(F_{i_1\cdots i_n}(K))=3^{-n}\qquad \forall i_1,\cdots,i_n\in I.$$

μ is a (normalized) Hausdorff measure on K.

2.1 Construction of Dirichlet forms on the Sierpinski gaskets

For $f,g\in\mathbb{R}^{V_n}:=\{h: h$ is a real-valued function on $V_n\}$, define quadratic forms as follows.

$$\mathcal{E}_n(f,g)=\frac{b_n}{2}\sum_{i_1\cdots i_n\in I}\sum_{x,y\in V_0}(f\circ F_{i_1\cdots i_n}(x)-f\circ F_{i_1\cdots i_n}(y))(g\circ F_{i_1\cdots i_n}(x)-g\circ F_{i_1\cdots i_n}(y)),$$

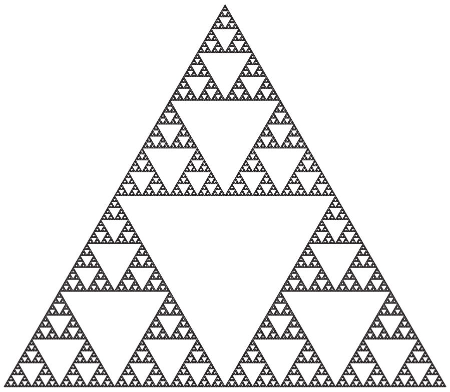

Figure 1: 2-dimensional Sierpinski gasket

where $\{b_n\}$ is a sequence of positive numbers with $b_0 = 1$. We want to choose $\{b_n\}$ so that there are some nice relations between the $\mathcal{E}_n$'s. Elementary computations yield

$$\inf\{\mathcal{E}_1(f,f) : f \in \mathbb{R}^{V_1},\ f|_{V_0} = u\} = \frac{3}{5}\cdot b_1\mathcal{E}_0(u,u) \qquad \forall u \in \mathbb{R}^{V_0}. \tag{2.1}$$

(One may refer to the background of the theory of electric networks, see [Bar2*] p.56–58.) So, taking $b_n = (5/3)^n$, we have

$$\mathcal{E}_n(f|_{V_n}, f|_{V_n}) \le \mathcal{E}_{n+1}(f,f) \quad \forall f \in \mathbb{R}^{V_{n+1}}$$

(equality holds when f is 'harmonic' on $V_{n+1} \setminus V_n$). Define

$$\mathcal{F}_* = \{f \in \mathbb{R}^{V_*} : \lim_{n\to\infty} \mathcal{E}_n(f,f) < \infty\}, \qquad \mathcal{E}(f,g) = \lim_{n\to\infty} \mathcal{E}_n(f,g) \qquad \forall f,g \in \mathcal{F}_*.$$

$(\mathcal{E}, \mathcal{F}_*)$ is a quadratic form on $\mathbb{R}^{V_*}$. Further, for each $f \in \mathbb{R}^{V_m}$, there exists a unique $P_m f \in \mathcal{F}_*$ such that $\mathcal{E}(P_m f, P_m f) = \mathcal{E}_m(f,f)$.

In order to extend this form to a form on $\mathbb{L}^2(K,\mu)$, we define the following.

$$R(p,q)^{-1} = \inf\{\mathcal{E}(f,f) : f \in V_*, f(p) = 1, f(q) = 0\} \qquad \forall p,q \in V_*,\ p \neq q.$$

$R(p,q)$ is an effective resistance between p and q. We set $R(p,p) = 0$ for $p \in V_*$.

Proposition 2.1 *1) $R(\cdot,\cdot)$ is a metric on V_*. It can be extended to a metric on K, which gives the same topology on K as the one from the Euclidean metric.*
2) For $p \neq q \in V_*$, $R(p,q) = \sup\{|f(p) - f(q)|^2/\mathcal{E}(f,f) : f \in \mathcal{F}_*, f(p) \neq f(q)\}$.

We can further prove the following. Let $d_w = \log 5/\log 2$ (we will mention the meaning of d_w later), we then have $R(p,q) \asymp \|p-q\|^{d_w - d_f}$. Here $\|\ \cdot\ \|$ is a Euclidean norm, $f(x) \asymp g(x)$ means $f(x)/g(x)$ are bounded from above and below by some positive constants.

From 2), we have $|f(p) - f(q)|^2 \le R(p,q)\mathcal{E}(f,f)$ for $f \in \mathcal{F}_*, p,q \in V_*$. Therefore $f \in \mathcal{F}_*$ can be extended continuously to K. Denote by $\mathcal{F}$ the set of functions in $\mathcal{F}_*$ extended to K. We have $\mathcal{F} \subset C(K) \subset \mathbb{L}^2(K,\mu)$.

Theorem 2.2 *$(\mathcal{E},\mathcal{F})$ is a local regular Dirichlet form on $\mathbb{L}^2(K,\mu)$ with the following property.*

$$|f(p)-f(q)|^2 \le R(p,q)\mathcal{E}(f,f) \qquad \forall f\in\mathcal{F},\ \forall p,q\in K \tag{2.2}$$

$$\mathcal{E}(f,g)=\frac{5}{3}\sum_{i\in I}\mathcal{E}(f\circ F_i, g\circ F_i) \qquad \forall f,g\in\mathcal{F} \tag{2.3}$$

Further, for each $\lambda>0$, there exists a symmetric positive continuous Green kernel $g_\lambda(\cdot,\cdot)$ with

$$Cap_1(\{y\})=\frac{1}{g_1(y,y)} \qquad \forall y\in K,$$

where $Cap_1(A)$ is a 1-capacity of A.

Thanks to the general theory mentioned in the last section, we have a corresponding diffusion process on K and a corresponding self-adjoint operator Δ on $\mathbb{L}^2(K,\mu)$. The diffusion process can be regarded as a strong Markov process $\{P^x\}_{x\in K}$ on $C([0,\infty),K)$. For $\omega\in C([0,\infty),K)$, set $X_t(\omega)=\omega(t)$ and define a μ-symmetric semigroup P_t as $P_tf(x)=E^x[f(X_t)]$. Since each point has positive capacity, there is no ambiguity concerning starting points. Roughly saying, this process is constructed from the simple random walk X_n on V_n by speeding up the time by 5^n, (i.e. considering $X_n([5^nt])$) and taking $n\to\infty$.

Any self-similar diffusion process on K whose law is invariant under local translations and reflections of each small triangle is a constant time change of this diffusion ([B-P]). Thus this process is called a Brownian motion on K.

For Brownian motion on the S.G., there is an associated branching process. Let $\{X_t^0\}$ be Brownian motion starting at the origin. Let $W=\inf\{t\in\bar{\mathbb{N}}\colon X_t^0\in V_0\setminus\{0\}\}$. For each $n\in\bar{\mathbb{N}}$, let $T_0^{(n)}(X^0)\equiv 0$ and define inductively $T_i^{(n)}$, W_n as follows,

$$T_{i+1}^{(n)}(X^0)=\inf\{t>T_i^{(n)}(X^0):X_t^0\in V_n\setminus\{X^0_{T_i^{(n)}}\}\},\quad W_n=\max\{i\in\mathbb{N}\colon T_i^{(n)}\le W\}. \tag{2.4}$$

Then $\{W_n\}_n$ is a branching process. Its off-spring distribution η has the generating function $f(s):=E[s^\eta]=s^2/(4-3s)$, especially $f'(1)=E[\eta]=5$. By the martingale convergence theorem, $W_n/5^n$ converges almost surely and it turns out that the limit random variable is equal in law to W. This associated branching process is useful to study detailed properties of Brownian motion.

We can also consider unbounded fractals and their associated Brownian motion. Define $\hat{K}=\cup_{n\in\bar{\mathbb{N}}}2^nK$; it is called an unbounded Sierpinski gasket. By essentially the same argument as above, we can construct a local regular Dirichlet form on $\mathbb{L}^2(\hat{K},\hat{\mu})$ ($\hat{\mu}$ is a Hausdorff measure on $\hat{K}$ with $\hat{\mu}(K)=1$). We call the corresponding diffusion process a Brownian motion on $\hat{K}$.

2.2 Heat kernel estimates and parabolic Harnack inequalities

One of the most important properties of this Brownian motion is that its heat kernel enjoys the following sub-Gaussian estimates. In order to state the theorem, we first define a metric called the shortest path metric. Define $d:V_*\times V_*\to\mathbb{R}_+$ as follows; if $x,y\in V_n$, then let $d(x,y)=$ (minimum number of steps from x to y on V_n)$/2^n$. d is well-defined and can be extended to a metric on K. This shortest path metric is a geodesic in the sense that for each $p,q\in K$, there is a continuous function $g : [0, d(p,q)] \to K$ such that $d(g(s), g(t)) = t - s$ for $0 \le s \le t \le d(p,q)$. In this case, $d(x,y) \asymp \|x - y\|$.

We have the following heat kernel estimates.

Theorem 2.3 [B-P] *There exists a jointly continuous symmetric function $p_t(x,y)$ such that*
(1) p_t is a transition density of $\{X_t\}$ w.r.t. μ; $P_t f(x) = \int_K p_t(x,y) f(y)\mu(dy)$ for $x \in K$, $f \in \mathbb{L}^2(K,\mu)$.
(2) $p_t(x_0,x)$ is a fundamental solution of the heat equation; $\frac{\partial}{\partial t} p_t(x_0,x) = \Delta_x p_t(x_0,x)$.
(3) There exist $c_1, \cdots, c_4 > 0$ such that the following holds for $x, y \in K, 0 < t \le 1$.

$$c_1 t^{-d_s/2} \exp(-c_2 (\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w - 1}}) \le p_t(x,y) \le c_3 t^{-d_s/2} \exp(-c_4 (\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w - 1}}). \qquad (2.5)$$

By integrating (2.5), we have $E^0[d(0, X_t)] \asymp t^{1/d_w}$; $d_w = \log 5/\log 2 > 2$ is called a random walk dimension. As $d_w > 2$, we say the process is sub-diffusive. $d_s = 2\log 3/\log 5 = 2d_f/d_w$ is called a spectral dimension (we will explain more about this exponent later). As we will see later, the energy measure of the form is singular with respect to the Hausdorff measure. So the well-known Davies method (see [Dav*]) for heat kernel estimates cannot be applied. Using detailed hitting time estimates instead, (2.5) is obtained (see [Bar2*] for details).

(2.5) is a very useful estimate; various properties of Brownian motion (some of them are discussed later) can be deduced from this estimate. (2.5) holds for the transition density of Brownian motion on $\hat{K}$ with $0 < t < \infty$.

From the heat kernel estimates, the following (generalized) parabolic Harnack inequality can be obtained.

Theorem 2.4 *For $s, r \in (0,1]$ and $x_0 \in K$, set*

$$Q_- = (s + r^{d_w}, s + 2r^{d_w}) \times B(x_0, r), \quad Q_+ = (s + 3r^{d_w}, s + 4r^{d_w}) \times B(x_0, r).$$

There exists $c_1 > 0$ such that, for any $s, r \in (0,1]$ and $x_0 \in K$, if u is a non-negative function on $(s, s + 4r^{d_w}) \times B(x_0, 2r)$ with $\frac{\partial u}{\partial t} = \Delta u$, then

$$\sup_{(t,x)\in Q_-} u(t,x) \le c_1 \inf_{(t,x)\in Q_+} u(t,x).$$

We will abbreviate this property as $(PHI(d_w))$. Once we have (2.5), $(PHI(d_w))$ can be deduced through well-known arguments (see, for instance, Section 3 in [Fe-S]). In fact, they are equivalent. We will discuss more about the equivalence condition in Section 3.

2.3 Domains of the Dirichlet forms

For $1 \le p < \infty,\ 1 \le q \le \infty,\ \beta \ge 0$ and $m \in \bar{\mathbb{N}}$, set

$$a_m(\beta, f) := L^{m\beta} (L^{md} \int\int_{|x-y| < c_0 L^{-m}} |f(x) - f(y)|^p d\mu(x) d\mu(y))^{1/p}, \quad f \in \mathbb{L}^p(K,\mu),$$

where $1 < L < \infty,\ 0 < c_0 < \infty$. Define a *Lipschitz space* $\mathrm{Lip}(\beta, p, q)(K)$ as a set of $f \in \mathbb{L}^p(K,\mu)$ such that $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^{\infty} \in l^q$. $\mathrm{Lip}(\beta, p, q)(K)$ is a Banach space with the norm

$\|f\|_{\mathrm{Lip}} := \|f\|_{\mathbb{L}^p} + \|\bar{a}(\beta, f)\|_{l^q}$. The Lipschitz space is determined independently of the choice of L and c_0 as long as the former is greater than 1 and the latter is positive.

We are interested in $p = 2$. When $K = \mathbb{R}^n$, $\mathrm{Lip}(\beta, 2, q)(\mathbb{R}^n) = B_\beta^{2,q}(\mathbb{R}^n)$ if $0 < \beta < 1$, $= \{0\}$ if $\beta > 1$. Here $B_\beta^{2,q}(\mathbb{R}^n)$ is the Besov space on $\mathbb{R}^n$. When $\beta = 1$, $\mathrm{Lip}(1, 2, \infty)(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$, i.e. the Sobolev space, and $\mathrm{Lip}(1, 2, 2)(\mathbb{R}^n) = \{0\}$.

Theorem 2.5 [Jons] *Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on the S.G.. Then,*

$$\mathcal{F} = Lip(\frac{d_w}{2}, 2, \infty)(K).$$

2.4 Other properties

Spectral properties By Theorem 2.2, $-\Delta$ has a compact resolvent. Set $\rho(x) = \sharp\{\lambda \leq x : \lambda \text{ is an eigenvalue of } -\Delta\}$. Then

$$0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty, \tag{2.6}$$

([Fu-S]). Remark that on (2.6), the limit sup and the limit inf do not coincide. This is because there exist 'many' localized eigenfunctions that produce eigenvalues with high multiplicities ([Bar-K]). Here, a localized eigenfunction is an eigenfunction u of $-\Delta$ such that $\mathrm{Supp}\ u \subset O$ on some open set $O \subset \mathrm{Int}\ K$. The corresponding eigenvalues are called localized eigenvalues. Let $\rho_L(x)$ be the number of localized eigenvalues of $-\Delta$ less than x. It is known that ρ_L satisfies (2.6) by the same d_s ([Kig2]). These properties of the Laplacian on the gasket clearly differ from those on Euclidean spaces. Δ on K consists only on point spectra.

Let $\hat{\Delta}$ be the Laplacian on $\hat{K}$. Eigenfunctions with compact supports are complete in $\mathbb{L}^2(\hat{K}, \hat{\mu})$ ([Tep]). Especially $\hat{\Delta}$ on $\hat{K}$ consists only on point spectra.

Energy measure By the general theory ([FOT*]), for each $f \in \mathcal{F}$, there is a uniquely determined Borel measure $\mu_{<f>}$ on K (called an energy measure) satisfying

$$\int_K g d\mu_{<f>} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g) \qquad \forall g \in \mathcal{F}.$$

The signed measure $\mu_{<f>}$ on the S.G. is singular with respect to the Hausdorff measure, and the martingale dimension of Brownian motion is 1 ([Kus2]).

For $V_0 = \{p_0, p_1, p_2\}$, let ϕ_i be a function on K which is 1 at p_i, 0 on $V_0 \setminus \{p_i\}$ and harmonic outside V_0. Let ϕ be a map from K to $\{(y_0, y_1, y_2) : y_0 + y_1 + y_2 = 0\}(=: H)$ such that $\phi(x) = \frac{1}{\sqrt{2}} P(\phi_0(x), \phi_1(x), \phi_2(x))$ (P is an orthogonal projection onto H). Denote the image of K by this map (called harmonic coordinates) as E_H. As ϕ is a homeomorphism, we identify functions on K and those on E_H. Fix orthogonal coordinates (x_1, x_2) on H and let $C^1(E_H)$ be an equivalent class of functions on E_H which have C^1-extensions to some open neighborhoods in H. Then, using a probability measure ν on E_H (which is singular to μ) and a matrix $Z(\cdot)$ the rank of which is ν-a.e. 1, $\mathcal{E}$ can be expressed as

$$\mathcal{E}(f, g) = \int_{E_H} {}^t(\nabla f(x)) Z(x) \nabla g(x) \nu(dx) \quad \forall f, g \in C^1(E_H),$$

([Kig3]). Here ∇ is an ordinary gradient w.r.t. the above orthogonal coordinates. (The gradients of elements in $C^1(E_H)$ are well-defined.) Further, it is proved that $C^1(E_H)$ is densely embedded in $\mathcal{F}$.

Short time asymptotics and large deviations For each $z \in [2/5, 1)$, let $\epsilon_{n,z} = (2/5)^n z$. Then

$$-\lim_{n\to\infty} \epsilon_{n,z}^{\frac{1}{d_w-1}} \log p_{\epsilon_{n,z}}(x,y) = d(x,y)^{\frac{d_w}{d_w-1}} F(\frac{z}{d(x,y)}) \qquad \forall x, y \in K,$$

where F is a periodic non-constant positive continuous function with period $5/2$ ([Kum3]). This F is related to the hitting time W in (2.4) as follows. Let $L(s) = -s^{1/d_w} \log E[e^{-sW}]$. Then the limit $k(s) := \lim_{n\to\infty} L(5^n s) > 0$ exists and it is non-constant, i.e. there is a 'tiny' oscillation. $F(y) = -y^{1/(d_w-1)} \inf_s (ys - k(s)s^{1/d_w})$. F is also non-constant from some property of the Legendre transform. Because F is non-constant, $\lim_{t\to 0} G(t)t^{1/(d_w-1)} \log p_t(x,y)$ does not exist for any choice of bounded functions G.

We next explain about large deviation results on the path space. Let P^x_ϵ be the law of $X_{\epsilon\cdot}$ starting at x. For each $z \in [2/5, 1)$, $A \subset \Omega_x := \{f \in C([0,T] \to K : f(0) = x\}$,

$$\begin{aligned} -\inf_{\phi\in \mathrm{Int}A} I^z_x(\phi) &\le \liminf_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log P^x_{\epsilon_{n,z}}(A) \\ &\le \limsup_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log P^x_{\epsilon_{n,z}}(A) \le -\inf_{\phi\in \mathrm{Cl}A} I^z_x(\phi), \end{aligned}$$

([Be-K]). Here $\{I^z_x\}_{z\in[2/5,1)}$ is a sequence of rate functions defined as follows for each $\phi \in \Omega_x$,

$$I^z_x(\phi) = \begin{cases} \int_0^T (\dot\phi(t))^{d_w/(d_w-1)} F(z/\dot\phi(t))dt & \text{if } \phi \text{ is absolutely continuous,} \\ \infty & \text{otherwise,} \end{cases} \tag{2.7}$$

where F is the same periodic function as above and $\dot\phi(t) := \lim_{s\to t} \frac{d(\phi(s),\phi(t))}{|s-t|}$ for $t \in [0,T]$. This result tells us that the classical Schilder-type large deviation does not hold when $\epsilon \to 0$. Instead, for each fixed z, it holds via the sequence $\epsilon_{n,z}$ as $n \to \infty$.

When $A = \{f \in \Omega_x : f(T) = y\}$, $\inf\{I^z_x(\phi) : \phi \in A\}$ is attained independently of z by the path(s) which moves on the geodesic(s) between x and y homogeneously. Thus 'the most probable path' should be this path, but the energy (action functional) of the path depends on time sequences determined by z.

Sample paths properties Brownian motion on the gasket is point recurrent, i.e. $P^x(\inf\{t > 0 : X_t = x\} < \infty) = 1$. Further, setting $t^n_i = i5^{-n}$, $\sum_{i=1}^{[t5^n]} d(X_{t^n_i}, X_{t^n_{i-1}})^{d_w} \asymp t$ as $n \to \infty$ for all $t > 0$ with probability 1. Thus this Brownian motion does not have finite quadratic variation so that it is not a semi-martingale ([B-P]).

The following Hölder continuity ([B-P]) and laws of iterated integrals ([FST]) hold. The exact values of the constants are not known.

$$c_1 \le \lim_{\delta\to 0} \sup_{0\le s\le t\le 1, |s-t|\le\delta} \frac{d(X_t, X_s)}{|s-t|^{1/d_w}(\log 1/|s-t|)^{1-1/d_w}} \le c_2 \quad P_x\text{ -a.e. }, \ \forall x \in K.$$

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{1-1/d_w}} = c_3, \quad \liminf_{t\to\infty} \frac{\sup_{0\le s\le t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{-1/d_w}} = c_4 \quad P_x\text{ -a.e.}, \ \forall x \in K.$$

Define the range of Brownian motion as $R_t = \mu(\{x : X(s) = x, \text{ for some } s \le t\})$. Then,

$$\limsup_{t\to\infty} \frac{R_t}{t^{d_s/2}(\log\log t)^{1-d_s/2}} = c_5, \quad \liminf_{t\to\infty} \frac{R_t}{t^{d_s/2}(\log\log t)^{-d_s/2}} = c_6 \quad P^x - \text{a.e.}, \ \forall x \in K,$$

([Bass-K]). From these results, we can observe some interesting behaviour of sample paths. If $R_t = (\sup_{s\le t} d(X_s, X_0))^{d_f}$ holds, then the loglog order of the limit sup of R_t would be $d_f - d_s/2$ instead of $1-d_s/2$. On the other hand, the order of loglog for the limit inf of R_t is $-d_s/2$, which is what one would expect if $R_t = (\sup_{s\le t} d(X_s, X_0))^{d_f}$ holds. This suggests that the trajectory of the process is essentially 1-dimensional at times when the limit sup of R_t is attained whereas it is more like a uniform covering at times when the limit inf of R_t is attained.

Local times, Green functions, the domain of Δ $\{X_t\}$ admits a jointly continuous local time L_t^x $(x \in K,\ t \ge 0)$ with the following estimate ([B-P]),

$$\lim_{\delta\to 0} \sup_{0\le s\le t, d(x,y)\le\delta} \frac{|L_s^x - L_s^y|}{d(x,y)^{(d_w-d_f)/2}(\log 1/d(x,y))^{1/2}} \le c_7(\sup_{z\in K} L_t^z)^{1/2}.$$

The laws of iterated logarithms for the local time are also known.

The λ-order Green kernel $g_\lambda(\cdot,\cdot)$ in Theorem 2.2 satisfies the following ([B-P]).

$$\begin{aligned} g_\lambda(x,x) &\asymp \lambda^{d_s/2-1} && \forall x \in K, \\ |g_\lambda(x,y) - g_\lambda(x',y)| &\asymp d(x,x')^{d_w-d_f} && \forall x, x', y \in K. \end{aligned}$$

We next give some remark about the domain $\mathcal{D}(\Delta)$ of the generator Δ. As we easily see from the above Green density estimates, $f \in \mathcal{D}(\Delta)$ has $d_w - d_f$ order Hölder continuity w.r.t. the Euclidean metric. On the other hand, as $f(X_t)$ is a semi-martingale, d_w-variational paths are mapped to quadratic variational paths by f. Thus, for $\mu\times\mu$-a.e. $y, z \in K$, if these points are sufficiently close, $|f(y) - f(z)| \le |y-z|^{d_w/2}$. As $d_w/2 > 1$, we have in particular that if f has C^1 extension to an open set containing K, then f is constant ([B-P]). For $f \in \mathcal{D}(\Delta)$, $f^2 \in \mathcal{D}(\Delta)$ if and only if f is constant ([BST]).

Homogenization Let $\Gamma_{\text{S.G.}} = \cup_{n\in\bar{\mathbb{N}}} 2^n V_n$ (cf. the left of Figure 2) be the S.G. graph and put conductances randomly on each edge of triangles in $\Gamma_{\text{S.G.}}$. We assume that the conductances are i.i.d. on each triangle and they are bounded from above and below by some positive constants (this corresponds to the uniform ellipticity condition). Let m be a probability measure which governs the randomness of conductances. Define $Y^\omega(n)$ as the corresponding Markov chain. We have

$$2^{-n}Y^\omega([5^n t]) \overset{n\to\infty}{\longrightarrow} X_{ct} \qquad \text{in law},$$

where $\{X_t\}$ is Brownian motion on the S.G. ([K-K]). The convergence is in probability w.r.t. m. There is a naturally defined renormalization map behind the variational formula (2.1). Detailed studies of the renormalization map ([Sa, Me]) are relevant to this homogenization problem.

3 Stability of the heat kernel estimates

Since (2.5) and $(PHI(d_w))$ are very strong and useful properties, it is natural and important to ask whether such properties are stable under perturbations of the operators. Such directions of research are developed quite recently for $d_w > 2$. In this section, we will summarize them; since they are well studied for graphs so far, we will discuss the research on graphs.

3.1 History in brief

Before explaining the results for sub-diffusive cases, let us very briefly overview the history for diffusive cases. See [Dav*, SC*] etc. for details.

For any divergence operator $\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^n$ satisfying a uniform elliptic condition, Aronson ([Aro]) proved (2.5) with $d_s = n$ and $d_w = 2$ (so (2.5) is sometimes called the Aronson-type estimate). Later in the 20th century, there were various outstanding results in the field of global analysis on manifolds. Let Δ be the Laplace-Beltrami operator on a complete Riemannian manifold M with the Riemannian metric d and with the Riemannian measure μ. Li-Yau ([L-Y]) proved the remarkable fact that if M has non-negative Ricci curvature, then the heat kernel $p_t(x, y)$ satisfies

$$\frac{c_1}{\mu(B(x,t^{1/2}))}\exp(-\frac{d(x,y)^2}{c_1 t}) \le p_t(x,y) \le \frac{c_2}{\mu(B(x,t^{1/2}))}\exp(-\frac{d(x,y)^2}{c_2 t}). \qquad (3.1)$$

A few years later, Grigor'yan ([Gri]) and Saloff-Coste ([SC2]) elegantly refined the result and proved, in conjunction with the results by Fabes-Stroock ([Fe-S]) and Kusuoka-Stroock ([K-S]), that (3.1) is equivalent to a volume doubling condition (VD) plus Poincaré inequalities (PI(2)) –see Definition 3.1, 3.3 for definitions in the graph setting. The results were then extended to the framework of Dirichlet forms in [Stu1, Stu2, B-M], to the framework of graphs in [Del]. Detailed heat kernel estimates are strongly related to the control of harmonic functions, i.e. elliptic and parabolic Harnack inequalities (EHI), (PHI(2)) on M. The origin of ideas and techniques used in this field go back to Nash ([Nash]), Moser ([Mo1, Mo2]) and there are many other significant works in this area. Summarizing, the following equivalence holds.

$$(3.1) \Leftrightarrow (VD) + (PI(2)) \Leftrightarrow (PHI(2)). \qquad (3.2)$$

An important corollary of this fact is, since (VD) and (PI(2)) are stable under certain perturbations of the operator, that (3.1) and (PHI(2)) are also stable under these perturbations.

3.2 Framework

Let Γ be an infinite connected locally finite graph. Assume that the graph Γ is endowed with a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if x and y are connected by a bond (in which case we write $x \sim y$). We call the pair (Γ, μ) a weighted graph. We can regard it as an electrical network.

We define a quadratic form on (Γ, μ) as follows. Set

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{\substack{x,y\in\Gamma\\ x\sim y}}(f(x)-f(y))(g(x)-g(y))\mu_{xy} \qquad \forall f,g \in \mathbb{R}^{\Gamma}. \qquad (3.3)$$

We say (Γ, μ') is a bounded perturbation of (Γ, μ) if $\mu_{xy} \asymp \mu'_{xy}$ for all $x \sim y$.

Now, define $\mu_x = \sum_{y \in \Gamma} \mu_{xy}$ for each $x \in \Gamma$. Set $\mu(A) = \sum_{x \in A} \mu_x$ for each $A \subset \Gamma$; μ is then a measure on Γ. For each $x \sim y$, define $P(x, y) = \mu_{xy}/\mu_x$, which is the transition probability matrix of the Markov chain corresponding to $\mathcal{E}$. Let $\{X_n\}_{n \in \bar{\mathbb{N}}}$ be a discrete time Markov chain which moves at unit time intervals to any vertex y in the neighbourhood of x with probabilities given by $\{P(x, y)\}$.

The discrete Laplace operator corresponding to the Markov chain is

$$\mathcal{L}f(x) = \sum_y P(x, y)f(y) - f(x) = \frac{1}{\mu_x} \sum_y (f(y) - f(x))\mu_{xy}.$$

Define the heat kernel of $\mathcal{L}$ (transition density of X_n) by $p_n(x, y) := \mathbb{P}^x(X_n = y)/\mu_y$. Clearly, $p_n(x, y) = p_n(y, x)$. The natural metric on the graph obtained by counting the number of steps in the shortest path between points is written $d(x, y)$ for $x, y \in \Gamma$.

3.3 Heat kernel estimates and Harnack inequalities on graphs

For $x \in \Gamma$ and $r \geq 0$, denote $B(x, r) = \{y \in \Gamma : d(x, y) < r\}$, $V(x, r) = \mu(B(x, r))$.

Definition 3.1 *Let (Γ, μ) be a weighted graph and let $\beta > 0$.*
(1) We say (Γ, μ) satisfies a (p_0) condition if there exists $p_0 > 0$ such that

$$p_{xy} = \mu_{xy}/\mu_x \geq p_0 \qquad \forall \{x, y\} \in B.$$

(2) We say (Γ, μ) satisfies a volume doubling condition (VD) if there exists $c_1 > 1$ such that

$$V(x, 2R) \leq c_1 V(x, R) \qquad \forall x \in \Gamma, R \geq 1. \tag{3.4}$$

(3) We say (Γ, μ) satisfies an elliptic Harnack inequality (EHI) if there exists $c_2 > 0$ such that, whenever $x \in \Gamma, R \geq 1$ and $h : \Gamma \to \mathbb{R}$ is non-negative and harmonic in $B(x, 2R)$,

$$\sup_{y \in B(x,R)} h(y) \leq c_2 \inf_{y \in B(x,R)} h(y).$$

(4) We say (Γ, μ) satisfies $(PHI(\beta))$, a parabolic Harnack inequality of order β if whenever $u(n, x) \geq 0$ is defined on $[0, 4N] \times \bar{B}(y, 2R)$ and satisfies

$$u(n + 1, x) - u(n, x) = \mathcal{L}u(n, x), \qquad (n, x) \in [0, 4N] \times B(y, 2R),$$

then

$$\max_{\substack{N \leq n \leq 2N \\ x \in B(y,R)}} u(n, x) \leq c_3 \min_{\substack{3N \leq n \leq 4N \\ x \in B(y,R)}} (u(n, x) + u(n + 1, x)), \tag{3.5}$$

where $N \geq 2R$ and $N \asymp R^\beta$.
(5) We say (Γ, μ) satisfies sub-Gaussian heat kernel estimates if for $x, y \in \Gamma, n \geq d(x, y)$, $p_n(\cdot, \cdot)$ satisfies

$$p_n(x, y) \leq \frac{c_4}{V(x, n^{1/\beta})} \exp[-(\frac{d(x, y)^\beta}{c_4 n})^{1/(\beta - 1)}], \tag{$UE(\beta)$}$$

$$p_n(x, y) + p_{n+1}(x, y) \geq \frac{c_5}{V(x, n^{1/\beta})} \exp[-(\frac{d(x, y)^\beta}{c_5 n})^{1/(\beta - 1)}]. \tag{$LE(\beta)$}$$

Theorem 3.2 [G-T1, G-T2] *Let (Γ,μ) be a weighted graph satisfying the (p_0) condition. Then,*

$$(PHI(\beta)) \Leftrightarrow (VD) + (EHI) + (\alpha) \Leftrightarrow (UE(\beta)) + (LE(\beta)).$$

Condition (α) is a condition for either hitting times to balls or resistances of annuli. When the above conditions hold, then $\beta \geq 2$.

These equivalence conditions are very useful. But it is not clear whether they are stable under bounded perturbations of forms or not. (Especially, it is still a big open problem whether (EHI) is stable under bounded perturbations or not.)

Recently, equivalent conditions to the parabolic Harnack inequality which are stable under bounded perturbations have been given by Barlow-Bass ([B-B]).

Definition 3.3 (1) *Denote $B_R := B(x_0,R)$. We say (Γ,μ) satisfies $(PI(\beta))$, a scaled Poincaré inequality with parameter $\beta \geq 2$, if there exists a constant $c_1 > 0$ such that for any ball $B_R \subset \Gamma$ with $R \geq 1$ and $f : B_R \to \mathbb{R}$,*

$$\sum_{x\in B_R} (f(x) - \bar{f}_{B_R})^2 \mu_x \leq c_1 R^\beta \sum_{x,y\in B_R} \mu_{xy}(f(x) - f(y))^2, \tag{3.6}$$

where $\bar{f}_{B_R} = \mu(B_R)^{-1} \sum_{y\in B_R} f(y)\mu_y$.
(2) *Let $\beta \geq 2$. We say (Γ,μ) satisfies $(CS(\beta))$, a cut-off Sobolev inequality with exponent β, if there exist constants $c_2, c_3 > 0$ and $\theta \in (0,1]$ such that for every $x_0 \in \Gamma, R \geq 1$, there exists a cut-off function $\varphi(= \varphi_{x_0,R})$ satisfying the following properties.*
(a) *$\varphi(x) \geq 1$ for $x \in B_{R/2}$.*
(b) *$\varphi(x) = 0$ for $x \in B_R^c$.*
(c) *$|\varphi(x) - \varphi(y)| \leq c_2(d(x,y)/R)^\theta$ for all $x,y \in \Gamma$.*
(d) *For any ball B_s with $1 \leq s \leq R$ and $f : B_{2s} \to \mathbb{R}$,*

$$\sum_{x\in B_s} f(x)^2 \sum_{y\in\Gamma} \mu_{xy}|\varphi(x) - \varphi(y)|^2 \leq c_3(\frac{s}{R})^{2\theta}(\sum_{x,y\in B_{2s}} \mu_{xy}|f(x) - f(y)|^2 + s^{-\beta} \sum_{y\in B_{2s}} f(y)^2\mu_y). \tag{3.7}$$

Theorem 3.4 [B-B] *Let (Γ,μ) be a weighted graph satisfying the (p_0) condition. Then,*

$$(VD) + (PI(\beta)) + (CS(\beta)) \Leftrightarrow (PHI(\beta)).$$

Remark 3.5 $(CS(2))$ always holds. Indeed, one can take $\varphi(x) = 2d(x, B(x_0,R)^c)/R$, then $|\varphi(x) - \varphi(y)| \leq 2/R$ if $\mu_{xy} > 0$, and (3.7) follows easily. Thus Theorem 3.4 is a nice extension of (3.2) to the cases of $\beta > 2$ for graphs.

Clearly, (VD), $(PI(\beta))$ and $(CS(\beta))$ are stable under bounded perturbations of Dirichlet forms. Moreover, they are stable under rough isometries, as we discuss in the next subsection.

On the other hand, in general it is not easy to check $(CS(\beta))$. Very recently, simpler equivalent conditions are given by Barlow-Coulhon-Kumagai ([BCK]) under a stronger volume growth condition.

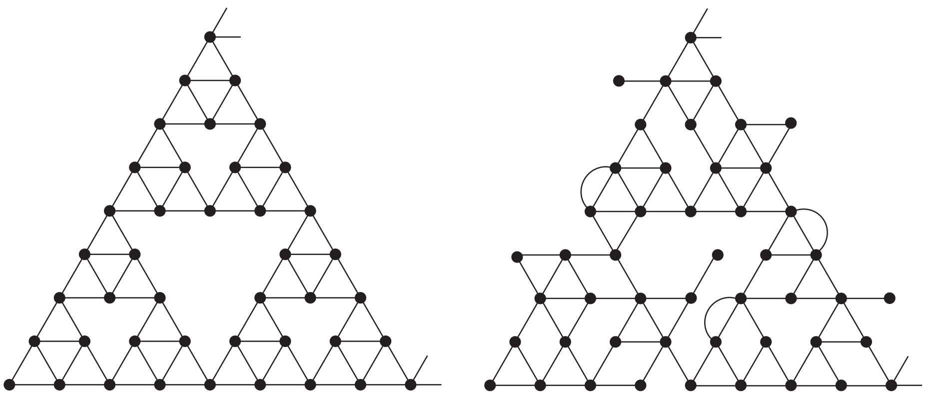

Figure 2: S.G. graph and modified S.G. graph

Definition 3.6 (1) *We say* (Γ,μ) *satisfies a volume growth condition* $(VG(\beta))$ *if there exist* $K>1$, $c_1>0$ *with* $\alpha=\log c_1/\log K<\beta$ *such that*

$$V(x,KR)\le c_1V(x,R)\qquad \forall x\in\Gamma, R\ge 1.$$

(2) *We say* (Γ,μ) *satisfies* $(RUE(\beta))$, $(RLE(\beta))$, *resistance upper and lower bounds of order* β, *if there exist* $c_2,c_3>0$ *such that for all* $x,y\in\Gamma$,

$$R(x,y)\le c_2\frac{d(x,y)^\beta}{V(x,d(x,y))}\quad (RUE(\beta)),\qquad R(x,y)\ge c_3\frac{d(x,y)^\beta}{V(x,d(x,y))}.\qquad (RLE(\beta))$$

Note that $(VG(\beta))$ is stronger than (VD) and implies, for $\theta>1$, $V(x,\theta R)\le c_4\theta^\alpha V(x,R)$.

Theorem 3.7 [BCK] *Let* (Γ,μ) *be a weighted graph satisfying the* (p_0) *condition and assume* $(VG(\beta))$*. Then,*

$$(UE(\beta))+(LE(\beta))\Leftrightarrow (RUE(\beta))+(RLE(\beta)).$$

When the above conditions hold, then the Markov chain is recurrent.

(Γ,μ) is called a *tree* if the graph has no loop; for any $\{x_i\}_{i=0}^L\subset\Gamma$ such that $x_i\sim x_{i+1}$ $(0\le i\le L-1)$, $x_L\sim x_0$ and $x_i\ne x_0$ $(1\le i\le L)$, it holds that $x_1=x_L$. If (Γ,μ) is a tree, then $R(x,y)=d(x,y)$ for all $x,y\in\Gamma$. Thus, we have the following corollary to Theorem 3.7.

Corollary 3.8 *Let* (Γ,μ) *be a tree. Then,*

$$(VG(\beta))+(UE(\beta))+(LE(\beta))\Leftrightarrow [V(x,d(x,y))\asymp d(x,y)^{\beta-1}\ \ \forall x,y\in\Gamma].$$

We note that measure metric space versions of these results are now under study.

3.4 Stability under rough isometries

Finally, we will discuss stability of parabolic Harnack inequalities under rough isometries.

Definition 3.9 *Let* $(\Gamma^{(1)}, \mu^{(1)})$, $(\Gamma^{(2)}, \mu^{(2)})$ *be weighted graphs satisfying the* (p_0) *condition . A map* $T : \Gamma^{(1)} \to \Gamma^{(2)}$ *is called a rough isometry if there exist positive constants* $a, c > 1, b > 0$ *and* $M > 0$ *such that*

$$a^{-1}d^{(1)}(x,y) - b \le d^{(2)}(T(x),T(y)) \le ad^{(1)}(x,y) + b \qquad \forall x, y \in \Gamma^{(1)}, \tag{3.8}$$
$$d^{(2)}(T(\Gamma^{(1)}), y') \le M \qquad \forall y' \in \Gamma^{(2)}, \tag{3.9}$$
$$c^{-1}\mu_x^{(1)} \le \mu_{T(x)}^{(2)} \le c\mu_x^{(1)} \qquad \forall x \in \Gamma^{(1)}, \tag{3.10}$$

where $\mu^{(i)}$, $d^{(i)}(\cdot,\cdot)$ *are the measure and the graph distance of* $(\Gamma^{(i)}, \mu^{(i)})$ *respectively for* $i = 1, 2$. *If there exists a rough isometry between two spaces, they are said to be roughly isometric. (One can check this is an equivalence relation.)*

$(PHI(\beta))$ is stable under rough isometries.

Theorem 3.10 [H-K] *Let* $(\Gamma^{(1)}, \mu^{(1)})$, $(\Gamma^{(2)}, \mu^{(2)})$ *be weighted graphs satisfying the* (p_0) *condition. If* $(\Gamma^{(1)}, \mu^{(1)})$ *satisfies* $(PHI(\beta))$ *w.r.t. the graph distance and* $(\Gamma^{(1)}, \mu^{(1)})$, $(\Gamma^{(2)}, \mu^{(2)})$ *are roughly isometric, then* $(\Gamma^{(2)}, \mu^{(2)})$ *also satisfies* $(PHI(\beta))$ *w.r.t. the graph distance.*

Likewise the case of Brownian motion on the S.G., the simple random walk on the S.G. graph (the left of Figure 2) satisfy $(UE(\log 5/\log 2))$ and $(LE(\log 5/\log 2))$ – see [J]. The graph on the right of Figure 2 is made by locally modifying the 2-dimensional S.G. graph so that (3.8) and (3.9) are satisfied. The two networks in Figure 2 are roughly isometric if the conductance on each edge is bounded from above and below by positive constants. In that case, using Theorem 3.10, the network on the right of Figure 2 also satisfies two sided sub-Gaussian estimates and thus satisfies $(PHI(\log 5/\log 2))$ w.r.t. the graph distance.

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