Quenched Invariance Principles for Random Walks and Elliptic Diffusions in Random Media with Boundary

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Dedicated to Professors Martin T. Barlow and Ed Perkins on the occasion of their 60th birthdays.

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Abstract

Via a Dirichlet form extension theorem and making full use of two-sided heat kernel estimates, we establish quenched invariance principles for random walks in random environments with a boundary. In particular, we prove that the random walk on a supercritical percolation cluster or amongst random conductances bounded uniformly from below in a half-space, quarter-space, etc., converges when rescaled diffusively to a reflecting Brownian motion, which has been one of the important open problems in this area. We establish a similar result for the random conductance model in a box, which allows us to improve existing asymptotic estimates for the relevant mixing time. Furthermore, in the uniformly elliptic case, we present quenched invariance principles for domains with more general boundaries.

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1 Introduction

Invariance principles for random walks in d-dimensional reversible random environments date back to the 1980s [27, 40, 42, 43]. The most robust of the early results in this area concerned scaling limits for the annealed law, that is, the distribution of the random walk averaged over the possible realizations of the environment, or possibly established a slightly stronger statement involving some

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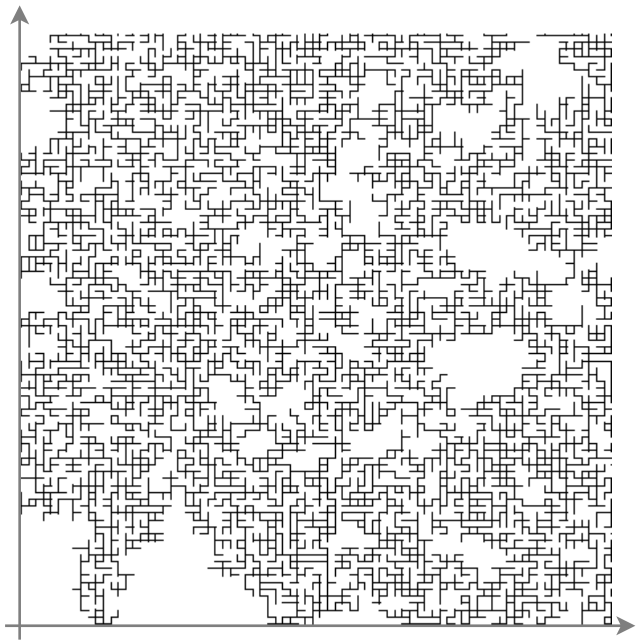

Figure 1: A section of the unique infinite cluster for supercritical percolation on $\mathbb{Z}_+^2$ with parameter $p = 0.52$.

form of convergence in probability. Studying the behavior of the random walks under the quenched law, that is, for a fixed realization of the environment, has proved to be a much more difficult task, especially when there is some degeneracy in the model. This is because it is often the case that a typical environment has 'bad' regions that need to be controlled. Nevertheless, over the last decade significant work has been accomplished in this direction. Indeed, in the important case of the random walk on the unique infinite cluster of supercritical (bond) percolation on $\mathbb{Z}^d$, building on the detailed transition density estimates of [6], a Brownian motion scaling limit has now been established [15, 46, 52]. Additionally, a number of extensions to more general random conductance models have also been proved [3, 8, 17, 45].

Whilst the above body of work provides some powerful techniques for overcoming the technical challenges involved in proving quenched invariance principles, such as studying 'the environment viewed from the particle' or the 'harmonic corrector' for the walk (see [16] for a survey of the recent developments in the area), these are not without their limitations. Most notably, at some point, the arguments applied all depend in a fundamental way on the translation invariance or ergodicity under random walk transitions of the environment. As a consequence, some natural variations of the problem are not covered. Consider, for example, supercritical percolation in a half-space $\mathbb{Z}_+ \times \mathbb{Z}^{d-1}$, or possibly an orthant of $\mathbb{Z}^d$. Again, there is a unique infinite cluster (Figure 1 shows a simulation of such in the first quadrant of $\mathbb{Z}^2$), upon which one can define a random walk. Given the invariance principle for percolation on $\mathbb{Z}^d$, one would reasonably expect that this process would converge, when rescaled diffusively, to a Brownian motion reflected at the boundary. After all, as is illustrated in the figure, the 'holes' in the percolation cluster that are in contact with the boundary are only on the same scale as those away from it. However, the presence of a boundary means that the translation invariance/ergodicity properties necessary for applying the existing arguments are lacking. For this reason, it has been one of the important open problems in this area to prove the quenched invariance principle for random walk on a percolation cluster, or amongst random conductances more generally, in a half-space (see [15, Section B] and [16, Problem 1.9]). Our aim is to provide a new approach for overcoming this issue, and thereby establish invariance principles within some general framework that includes examples such as those just described.

The approach of this paper is inspired by that of [19, 20], where the invariance principle for random walk on grids inside a given Euclidean domain D is studied. It is shown first in [19] for a class of bounded domains including Lipschitz domains and then in [20] for *any* bounded domain D that the simple random walk converges to the (normally) reflecting Brownian motion on D when the mesh size of the grid tends to zero. Heuristically, the normally reflecting Brownian motion is a continuous Markov process on $\overline{D}$ that behaves like Brownian motion in D and is 'pushed back' instantaneously along the inward normal direction when it hits the boundary. See Section 2 for a precise definition and more details. The main idea and approach of [19, 20] is as follows: (i) show that random walk killed upon hitting the boundary converges weakly to the absorbing Brownian motion in D, which is trivial; (ii) establish tightness for the law of random walks; (iii) show any sequential limit is a symmetric Markov process and can be identified with reflecting Brownian motion via a Dirichlet form characterization. In [19, 20], (ii) is achieved by using a forward-backward martingale decomposition of the process and the identification in (iii) is accomplished by using a result from the boundary theory of Dirichlet form, which says that the reflecting Brownian motion on D is the maximal Silverstein's extension of the absorbing Brownian motion in D; see [19, Theorem 1.1] and [23, Theorem 6.6.9].

For quenched invariance principles for random walks in random environments with a boundary, step (i) above can be established by applying a quenched invariance principle for the full-space case. For step (ii), i.e. establishing tightness, the forward-backward martingale decomposition method does not work well with unbounded random conductances. To overcome this difficulty, as well as for the desire to establish an invariance principle for every starting point, we will make the full use of detailed two-sided heat kernel estimates for random walk on random clusters. In particular, we provide sufficient conditions for the subsequential convergence that involve the Hölder continuity of harmonic functions (see Section 2.2). This continuity property can be verified in examples by using existing two-sided heat kernel bounds. We remark that the corrector-type methods for full-space models, such as the approach of [15], often require only upper bounds on the heat kernel. Using the Hölder regularity, we can further show that any subsequential limit of random walks in random environments is a conservative symmetric Hunt process with continuous sample paths. In step (iii), we can identify the subsequential limit process with the reflecting Brownian motion by a Dirichlet form argument (see Theorem 2.1). In summary, our approach for proving quenched invariance principles for random walks in random environments with a boundary encompasses two novel aspects: a Dirichlet form extension argument and the full use of detailed heat kernel estimates.

The full generality of the random conductance model to which we are able to apply the above argument is presented in Section 3. As an illustrative application of Theorem 2.1, though, we state here a theorem that verifies the conjecture described above concerning the diffusive behavior of the random walk on a supercritical percolation cluster on a half-space, quarter-space, etc. We recall that the variable speed random walk (VSRW) on a connected (unweighted) graph is the continuous time Markov process that jumps from a vertex at a rate equal to its degree to a uniformly chosen neighbor (see Section 3 for further details). In this setting, similar results to that stated can be obtained for the so-called constant speed random walk (CSRW), which has mean one exponential holding times, or the discrete time random walk (see Remark 3.18 below).

Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and $\mathbb{R}_+ := [0, \infty)$. Then the following is our main theorem.

Theorem 1.1. *Fix $d_1, d_2 \in \mathbb{Z}_+$ such that $d_1 \geq 1$ and $d := d_1 + d_2 \geq 2$. Let $\mathcal{C}_1$ be the unique infinite cluster of a supercritical bond percolation process on $\mathbb{Z}_+^{d_1} \times \mathbb{Z}^{d_2}$, and let $Y = (Y_t)_{t\geq 0}$ be the associated VSRW. For almost-every realization of $\mathcal{C}_1$, it holds that the rescaled process $Y^n = (Y_t^n)_{t\geq 0}$, as defined by*

$$Y_t^n := n^{-1} Y_{n^2 t},$$

started from $Y_0^n = x_n \in n^{-1}\mathcal{C}_1$, where $x_n \to x \in \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$, converges in distribution to $\{X_{ct}; t \geq 0\}$, where $c \in (0, \infty)$ is a deterministic constant and $\{X_t; t \geq 0\}$ is the (normally) reflecting Brownian motion on $\mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$ started from x.

As an alternative to unbounded domains, one could consider compact limiting sets, replacing Y^n in the previous theorem by the rescaled version of the variable speed random walk on the largest percolation cluster contained in a box $[-n, n]^d \cap \mathbb{Z}^d$, for example. As presented in Section 4.1, another application of Theorem 2.1 allows an invariance principle to be established in this case as well, with the limiting process being Brownian motion in the box $[-1, 1]^d$, reflected at the boundary. Consequently, we are able to refine the existing knowledge of the mixing time asymptotics for the sequence of random graphs in question from a tightness result [13] to an almost-sure convergence one (see Corollary 4.4 below).

Although in the percolation setting we only consider relatively simple domains with 'flat' boundaries, this is mainly for technical reasons so that deriving the percolation estimates in Section 3.1 required for our proofs is manageable. Indeed, in the case when we restrict to uniformly elliptic random conductances, so that controlling the clusters of extreme conductances is no longer an issue, we are able to derive from Theorem 2.1 quenched invariance principles in any uniform domain, the collection of which forms a large class of possibly non-smooth domains that includes (global) Lipschitz domains and the classical van Koch snowflake planar domain as special cases. These applications are discussed in Section 4.2.

Homogenization of reflected SDE/PDE on half-planes and more general domains has been studied in various contexts (see, for example, [5, 14, 41, 51, 53]; we refer to [38, 41] and the references therein for the history of homogenization for diffusions in random environments). In a recent paper [51], Rhodes proves homogenization (as a convergence in product measure in environment and state space of quenched distribution, which implies an annealed invariance principle) for symmetric reflected diffusions in upper half spaces. His method is based on the Girsanov formula and a use of subsidiary diffusions with an invariant probability measure, which is very different from ours. Although we can also only handle symmetric cases, our methods contribute to this field as well. This is because the analytical part of our results (namely Section 2) holds for the entire class of uniform domains. Moreover, our results are on the level of quenched invariance principles. The presentation of how our techniques can be applied in the uniformly elliptic random divergence form setting appears in Section 4.3. Note further that in this setting we resolve the open problem on the quenched invariance principle starting from arbitrary starting points posed in [51, pp. 1004–1005].

The remainder of the paper is organized as follows. In Section 2, we introduce an abstract framework for proving invariance principles for reversible Markov processes in a Euclidean domain. This is applied in Section 3 to our main example of a random conductance model in half-spaces, quarter-spaces, etc. The details of the other examples discussed above are presented in Section 4. Our results for the random conductance model depend on a number of technical percolation estimates, some of the proofs of which are contained in the appendix that appears at the end of this article. The appendix also contains a proof of a generalization of existing quenched invariance principles that allows for arbitrary starting points (previous results have always started the relevant processes from the origin, which will not be enough for our purposes).

Finally, in this paper, for a locally compact separable metric space E, we use $C_b(E)$ and $C_\infty(E)$ to denote the space of bounded continuous functions on E and the space of continuous functions on E that vanish at infinity, respectively. The space of continuous functions on E with compact support will be denoted by $C_c(E)$. For real numbers a, b, we use $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

2 Framework

The following definition is taken from Väisälä [55], where various equivalent definitions are discussed. An open connected subset D of $\mathbb{R}^d$ is called *uniform* if there exists a constant C such that for every $x, y \in D$ there is a rectifiable curve γ joining x and y in D with $\mathrm{length}(\gamma) \le C|x-y|$ and moreover $\min\{|x-z|, |z-y|\} \le C\,\mathrm{dist}(z, \partial D)$ for all points $z \in \gamma$. Here $\mathrm{dist}(z, \partial D)$ is the Euclidean distance between the point z and the set ∂D. Note that a uniform domain with respect to an inner metric is called *inner uniform* in [35, Definition 3.6].

For example, the classical van Koch snowflake domain in the conformal mapping theory is a uniform domain in $\mathbb{R}^2$. Every (global) Lipschitz domain is uniform, and every *non-tangentially accessible domain* defined by Jerison and Kenig in [37] is a uniform domain (see (3.4) of [37]). However, the boundary of a uniform domain can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred (besides the easy fact that the Hausdorff dimension of the boundary is strictly less than d).

It is known (see Example 4 on page 30 and Proposition 1 in Chapter VIII of [39]) that any uniform domain in $\mathbb{R}^d$ has $m(\partial D)=0$ and there exists a positive constant $c>0$ such that

$$m(D\cap B_E(x,r))\geq c\,r^n \qquad \text{for all } x\in\overline{D} \text{ and } 0<r\leq 1, \tag{2.1}$$

where m denotes the Lebesgue measure in $\mathbb{R}^d$ and $B_E(x,r)$ denotes the Euclidean ball of radius r centered at x.

Let D be a uniform domain in $\mathbb{R}^d$. Suppose $(A(x))_{x\in\overline{D}}$ is a measurable symmetric $d\times d$ matrix-valued function such that

$$c^{-1}I\leq A(x)\leq cI \qquad \text{for a.e. } x\in\overline{D}, \tag{2.2}$$

where I is the d-dimensional identity matrix and c is a constant in $[1,\infty)$. Let

$$\mathcal{E}(f,g):=\frac{1}{2}\int_D \nabla f(x)\cdot A(x)\nabla g(x)dx \qquad \text{for } f,g\in W^{1,2}(D), \tag{2.3}$$

where

$$W^{1,2}(D):=\left\{f\in L^2(D;m):\ \nabla f\in L^2(D;m)\right\}.$$

An important property of a uniform domain $D\subset\mathbb{R}^d$ is that there is a bounded linear extension operator $T:W^{1,2}(D)\to W^{1,2}(\mathbb{R}^d)$ such that $Tf=f$ a.e. on D for $f\in W^{1,2}(D)$. It follows that $(\mathcal{E},W^{1,2}(D))$ is a regular Dirichlet form on $L^2(\overline{D};m)$ and so there is a continuous diffusion process $X=(X_t,t\geq 0;\mathbb{P}_x,x\in\overline{D})$ associated with it, starting from $\mathcal{E}$-quasi-every point. Here a property is said to hold $\mathcal{E}$-quasi-everywhere means that there is a set $\mathcal{N}\subset\overline{D}$ having zero capacity with respect to the Dirichlet form $(\mathcal{E},W^{1,2}(D))$ so that the property holds for points in $\mathcal{N}^c$. According to [35, Theorem 3.10] and (2.1) (see also [12, (3.6)]), X admits a jointly continuous transition density function $p(t,x,y)$ on $\mathbb{R}_+\times\overline{D}\times\overline{D}$ and

$$c_1t^{-d/2}\exp\left(-\frac{c_2|x-y|^2}{t}\right)\leq p(t,x,y)\leq c_3t^{-d/2}\exp\left(-\frac{c_4|x-y|^2}{t}\right) \tag{2.4}$$

for every $x,y\in\overline{D}$ and $0<t\leq 1$. Here the constants $c_1,\dots,c_4>0$ depend on the diffusion matrix $A(x)$ only through the ellipticity bound c in (2.2). Consequently, X can be refined so that it can start from every point in $\overline{D}$. The process X is called a symmetric reflecting diffusion on $\overline{D}$. We refer to [21] for sample path properties of X. When $A=I$, X is the (normally) reflecting Brownian motion on $\overline{D}$. Reflecting Brownian motion X on $\overline{D}$ in general does not need to be semimartingale. When ∂D locally has finite lower Minkowski content, which is the case when D is a Lipschitz domain, X is a semimartingale and admits the following Skorohod decomposition (see [22, Theorem 2.6]):

$$X_t=X_0+W_t+\int_0^t \vec{n}(X_s)dL_s, \quad t\geq 0. \tag{2.5}$$

Here W is the standard Brownian motion in $\mathbb{R}^d$, $\vec{n}$ is the unit inward normal vector field of D on ∂D, and L is a positive continuous additive functional of X that increases only when X is on the boundary, that is, $L_t = \int_0^t 1_{\{X_s \in \partial D\}} dL_s$ for $t \geq 0$. Moreover, it is known that the reflecting Brownian motion spends zero Lebesgue amount of time at the boundary ∂D. These together with (2.5) justify the heuristic description we gave in the introduction for the reflecting Brownian motion in D.

2.1 Convergence to reflecting diffusion

In this subsection, D is a uniform domain in $\mathbb{R}^d$ and X is a reflecting diffusion process on $\overline{D}$ associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(D))$ on $L^2(D;m)$ given by (2.3). Denote by $(X^D, \mathbb{P}_x^D, x \in \overline{D})$ the subprocess of X killed on exiting D. It is known (see, e.g., [23]) that the Dirichlet form of X^D on $L^2(D;m)$ is $(\mathcal{E}, W_0^{1,2}(D))$, where

$$W_0^{1,2}(D) := \left\{ f \in W^{1,2}(D) : \ f = 0 \ \ \mathcal{E}\text{-quasi-everywhere on } \partial D \right\}.$$

Suppose that $\{D_n;\, n \geq 1\}$ is a sequence of Borel subsets of $\overline{D}$ such that each D_n supports a measure m_n that converges vaguely to the Lebesgue measure m on $\overline{D}$. The following result plays a key role in our approach to the quenched invariance principle for random walks in random environments with boundary.

Theorem 2.1. *For each $n \in \mathbb{N}$, let $(X^n, \mathbb{P}_x^n, x \in D_n)$ be an m_n-symmetric Hunt process on D_n. Assume that for every subsequence $\{n_j\}$, there exists a sub-subsequence $\{n_{j(k)}\}$ and a continuous conservative m-symmetric strong Markov process $(\widetilde{X}, \widetilde{\mathbb{P}}_x, x \in \overline{D})$ such that the following three conditions are satisfied:*

(i) *for every $x_{n_{j(k)}} \to x$ with $x_{n_{j(k)}} \in D_{n_{j(k)}}$, $\mathbb{P}^{n_{j(k)}}_{x_{n_{j(k)}}}$ converges weakly in $\mathbb{D}([0,\infty), \overline{D})$ to $\widetilde{\mathbb{P}}_x$;*

(ii) *$\widetilde{X}^D$, the subprocess of $\widetilde{X}$ killed upon leaving D, has the same distribution as X^D;*

(iii) *the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ of $\widetilde{X}$ on $L^2(D;m)$ has the properties that*

$$\mathcal{C} \subset \tilde{\mathcal{F}} \qquad \text{and} \quad \tilde{\mathcal{E}}(f,f) \leq C_0\, \mathcal{E}(f,f) \qquad \text{for every } f \in \mathcal{C}, \tag{2.6}$$

where $\mathcal{C}$ is a core for the Dirichlet form $(\mathcal{E}, W^{1,2}(D))$ and $C_0 \in [1,\infty)$ is a constant.

It then holds that for every $x_n \to x$ with $x_n \in D_n$, $(X^n, \mathbb{P}^n_{x_n})$ converges weakly in $\mathbb{D}([0,\infty), \overline{D})$ to $(X, \mathbb{P}_x)$.

Proof. With both $\widetilde{X}$ and X being m-symmetric Hunt processes on $\overline{D}$, it suffices to show that their corresponding (quasi-regular) Dirichlet forms on $L^2(D;m)$ are the same; that is $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}, W^{1,2}(D))$. Condition (iii) immediately implies that $W^{1,2}(D) \subset \widetilde{\mathcal{F}}$ and

$$\widetilde{\mathcal{E}}(f,f) \leq C_0 \mathcal{E}(f,f) \qquad \text{for every } f \in W^{1,2}(D).$$

Next, observe that since $\widetilde{X}$ is a diffusion process admitting no killings, its associated Dirichlet form is strongly local. Thus for every $u \in \widetilde{\mathcal{F}}$, $\widetilde{\mathcal{E}}(u,u) = \frac{1}{2}\widetilde{\mu}_{\langle u\rangle}(\overline{D})$, where $\widetilde{\mu}_{\langle u\rangle}$ is the energy measure corresponding to u. By the proof of [47, Proposition on page 389] ,

$$\widetilde{\mu}_{\langle u\rangle}(dx) \le C_0 \nabla u(x) A(x) \nabla u(x) dx \le c C_0 |\nabla u(x)|^2 dx \quad \text{on } \overline{D} \qquad \text{for every } u \in W^{1,2}(D).$$

This in particular implies that

$$\widetilde{\mu}_{\langle u\rangle}(\partial D) = 0 \qquad \text{for } u \in W^{1,2}(D). \tag{2.7}$$

On the other hand, by the strong local property of $\widetilde{\mu}_{\langle u\rangle}$ and the fact that $\widetilde{X}^D$ has the same distribution as X^D, we have that every bounded function in $\widetilde{\mathcal{F}}$ – the collection of which we denote by $\widetilde{\mathcal{F}}_b$ – is locally in $W^{1,2}_0(D)$ and

$$\mathbf{1}_D(x)\widetilde{\mu}_{\langle u\rangle}(dx) = \mathbf{1}_D(x)\nabla u(x) A(x) \nabla u(x) dx \qquad \text{for } u \in \widetilde{\mathcal{F}}_b. \tag{2.8}$$

This together with (2.7) implies that $\widetilde{\mathcal{E}}(u,u) = \mathcal{E}(u,u)$ for every bounded $u \in W^{1,2}(D)$, and hence for every $u \in W^{1,2}(D)$. Furthermore, (2.8) implies that for $u \in \widetilde{\mathcal{F}}_b$, $\int_D |\nabla u(x)|^2 dx < \infty$ and so $u \in W^{1,2}(D)$. Consequently we have $\widetilde{\mathcal{F}} \subset W^{1,2}(D)$ and thus $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}, W^{1,2}(D))$. □

Remark 2.2. (i) Note that if $(X^n_t)_{t\ge 0}$ is conservative for each $n \in \mathbb{N}$ and $\{\mathbb{P}^{n_{j(k)}}_{x_{n_{j(k)}}}\}$ is tight, then $\tilde{X}$ is conservative.
(ii) Theorem 2.1 can be viewed as a variation of [19, Theorem 1.1]. The difference is that in [19, Theorem 1.1], the constant C_0 in (2.6) is assumed to be 1 but the limiting process $\widetilde{X}$ only need to be Markov and does not need to be continuous a priori, while for Theorem 2.1, the condition on the constant C_0 is weaker but we need to assume a priori that the limit process $\widetilde{X}$ is continuous. □

2.2 Sufficient condition for subsequential convergence

In this subsection, we give some sufficient conditions for the subsequential convergence of $\{X^n\}$; in other words, sufficient conditions for (i) in Theorem 2.1. For simplicity, we assume that $0 \in D_n$ for all $n \ge 1$ throughout this section, though note this restriction can easily be removed.

We start by introducing our first main assumption, which will allow us to check an equicontinuity property for the λ-potentials associated with the elements of $\{X^n\}$ (see Proposition 2.4 below). In the statement of the assumption, we suppose that $(\delta_n)_{n\ge 1}$ is a decreasing sequence in $[0,1]$ with $\lim_{n\to\infty} \delta_n = 0$ and such that $|x-y| \ge \delta_n$ for all distinct $x, y \in D_n$. (When $\delta_n \equiv 0$, this condition always holds. However, our assumption will give an additional restriction.) We denote by $\tau_A(X^n)$ the first exit time of the process X^n from the set A.

Assumption 2.3. *There exist $c_1, c_2, c_3, \beta, \gamma \in (0,\infty)$, $N_0 \in \mathbb{N}$ such that the following hold for all $n \ge N_0$, $x_0 \in B_E(0, c_1 n^{1/2})$, and $\delta_n^{1/2} \le r \le 1$.*

(i) *For all $x \in B_E(x_0, r/2) \cap D_n$,*

$$\mathbb{E}^n_x \left[\tau_{B_E(x_0,r)\cap D_n}(X^n)\right] \le c_2 r^\beta.$$

(ii) *If h_n is bounded in D_n and harmonic (with respect to X^n) in a ball $B_E(x_0, r)$, then*

$$|h_n(x) - h_n(y)| \le c_3 \left(\frac{|x-y|}{r} \right)^{\gamma} \|h_n\|_\infty \qquad \text{for } x, y \in B_E(x_0, r/2) \cap D_n.$$

Define for $\lambda > 0$ the λ-potential

$$U_n^\lambda f(x) = \mathbb{E}_x^n \int_0^\infty e^{-\lambda t} f(X_t^n)\, dt \qquad \text{for } x \in D_n.$$

Proposition 2.4. *Under Assumption 2.3 there exist $C = C_\lambda \in (0, \infty)$ and $\gamma' \in (0, \infty)$ such that the following holds for any bounded function f on D_n, for any $n \ge N_0$ and any $x, y \in D_n$ such that $x \in B_E(0, c_1 n^{1/2})$ and $|x - y| < 1/4$:*

$$|U_n^\lambda f(x) - U_n^\lambda f(y)| \le C|x-y|^{\gamma'} \|f\|_\infty. \tag{2.9}$$

In particular, we have

$$\lim_{\delta \to 0} \sup_{n \ge N_0} \sup_{\substack{x,y \in D_n \cap B_E(0, c_1 n^{1/2}): \\ |x-y|<\delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| = 0. \tag{2.10}$$

Proof. The proof is similar to that of [11, Proposition 3.3]. Fix $x_0 \in B_E(0, c_1 n^{1/2}) \cap D_n$, let $1 \ge r \ge \delta_n^{1/2}$, and suppose $x, y \in B_E(x_0, r/2)$. Set $\tau_r^n := \tau_{B_E(x_0,r) \cap D_n}(X^n)$. By the strong Markov property,

$$\begin{aligned} U_n^\lambda f(x) &= \mathbb{E}_x^n \int_0^{\tau_r^n} e^{-\lambda t} f(X_t^n)\, dt + \mathbb{E}_x^n \left[(e^{-\lambda \tau_r^n} - 1) U_n^\lambda f(X_{\tau_r^n}^n) \right] + \mathbb{E}_x^n \left[U_n^\lambda f(X_{\tau_r^n}^n) \right] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

and similarly when x is replaced by y. We have by Assumption 2.3(i) that

$$|I_1| \le \|f\|_\infty \mathbb{E}_x^n \tau_r^n \le c_2 r^\beta \|f\|_\infty,$$

and by noting $\|U_n^\lambda f\|_\infty \le \frac{1}{\lambda} \|f\|_\infty$ that

$$|I_2| \le \lambda \mathbb{E}_x^n \tau_r^n \|U_n^\lambda f\|_\infty \le c_2 r^\beta \|f\|_\infty.$$

Similar statements also hold when x is replaced by y. So,

$$\left| U_n^\lambda f(x) - U_n^\lambda f(y) \right| \le 4 c_2 r^\beta \|f\|_\infty + \left| \mathbb{E}_x^n U_n^\lambda f(X_{\tau_r^n}^n) - \mathbb{E}_y^n U_n^\lambda f(X_{\tau_r^n}^n) \right|. \tag{2.11}$$

But $z \to \mathbb{E}_z^n U_n^\lambda f(X_{\tau_r^n}^n)$ is bounded in $\mathbb{R}^d$ and harmonic in $B_E(x_0, r)$, so by Assumption 2.3(ii), the second term in (2.11) is bounded by $c_3(|x-y|/r)^\gamma \|U_n^\lambda f\|_\infty$. So by $\|U_n^\lambda f\|_\infty \le \frac{1}{\lambda} \|f\|_\infty$ again, we have

$$\left| U_n^\lambda f(x) - U_n^\lambda f(y) \right| \le c \left(r^\beta + \lambda^{-1} \left(\frac{|x-y|}{r} \right)^\gamma \right) \|f\|_\infty \qquad \text{for } x, y \in B_E(x_0, r/2). \tag{2.12}$$

Now, for distinct $x, y \in D_n$ with $x \in B_E(0, c_1 n^{1/2})$ and $(\delta_n^{1/2})^2 \le |x-y| < 1/4$ (note that since $|x-y| \ge \delta_n$ for distinct x and y, the first inequality always hold), let $x_0 = x$ and $r = |x-y|^{1/2}$. Then $\delta_n \le r < 1/2$ and $y \in B_E(x_0, r/2)$ (because $|x_0 - y| = r^2 < r/2$). Thus we can apply (2.12) to obtain

$$\begin{aligned} \left| U_n^\lambda f(x) - U_n^\lambda f(y) \right| &\le c\left(|x-y|^{\beta/2} + \lambda^{-1}|x-y|^{\gamma/2} \right) \|f\|_\infty \\ &\le c(1+\lambda^{-1})|x-y|^{(\beta\wedge\gamma)/2}\|f\|_\infty. \end{aligned}$$

So (2.9) holds with $C = c(1+\lambda^{-1})$ and $\gamma' = (\beta \wedge \gamma)/2$. The result at (2.10) is immediate from (2.9). $\square$

We note that with an additional mild condition, we can further obtain equi-Hölder continuity of the associated semigroup. (The next proposition will only be used in the proof of Theorem 3.13 below.) Set $B_R := B_E(0, R) \cap D_n$ for $R \in [2, \infty)$. Denote by X^{n,B_R} the subprocess of X^n killed upon exiting B_R, and $\{P_t^{n,B_R}; t \ge 0\}$ the transition semigroup of X^{n,B_R}. (When $R = \infty$, we set $(P_t^n)_{t\ge 0} := (P_t^{n,B_\infty})_{t\ge0}$, i.e. the semigroup of X^n itself.) For $p \in [1, \infty]$, we use $\|\cdot\|_{p,n,R}$ to denote the L^p-norm with respect to m_n on B_R.

Proposition 2.5. *Let $R \in [2, \infty]$ and $t > 0$. Suppose there exist $c_1 > 0$ and $N_1 \in \mathbb{N}$ (that may depend on R and t) such that for every $g \in L^1(B_R, m_n)$,*

$$\|P_t^{n,B_R} g\|_{\infty,n,R} \le c_1 \|g\|_{1,n,R}, \qquad \text{for all } n \ge N_1.$$

Suppose in addition that Assumption 2.3 holds with X^{n,B_R} and B_R in place of X^n and D_n, respectively. It then holds that there exist constants $c \in (0,\infty)$ and $N_2 \ge 1$ (that also may depend on R and t) such that

$$\left| P_t^{n,B_R} f(x) - P_t^{n,B_R} f(y) \right| \le c_2 |x-y|^{\gamma'} \|f\|_{2,n,R},$$

for every $n \ge N_2$, $f \in L^2(B_R; m_n)$, and m_n-a.e. $x, y \in B_{R/2}$ with $|x-y| < 1/4$. Here γ' is the constant of Proposition 2.4.

Proof. We follow [11, Proposition 3.4]. For notational simplicity, we drop the suffices $n \ge N_0$ and B_R throughout the proof. Using spectral representation theorem for self-adjoint operators, there exist projection operators $E_\mu = E_\mu^{n,R}$ on the space $L^2(B_R; m_n)$ such that

$$f = \int_0^\infty dE_\mu(f), \quad P_t f = \int_0^\infty e^{-\mu t}\, dE_\mu(f), \quad U^\lambda f = \int_0^\infty \frac{1}{\lambda+\mu}\, dE_\mu(f). \tag{2.13}$$

Define

$$h = \int_0^\infty (\lambda+\mu) e^{-\mu t}\, dE_\mu(f).$$

Since $\sup_\mu (\lambda+\mu)^2 e^{-2\mu t} \le c$, we have

$$\|h\|_2^2 = \int_0^\infty (\lambda+\mu)^2 e^{-2\mu t}\, d\langle E_\mu(f), E_\mu(f)\rangle \le c\int_0^\infty d\langle E_\mu(f), E_\mu(f)\rangle = c\|f\|_2^2,$$

where for $f, g \in L^2$, $\langle f, g\rangle$ is the inner product of f and g in L^2. Thus h is a well defined function in L^2.

Now, suppose $g \in L^1$. By the assumption, $\|P_t g\|_\infty \le c\|g\|_1$, from which it follows that $\|P_t g\|_2 \le c\|g\|_1$. Since $\sup_\mu (\lambda+\mu)e^{-\mu t/2} \le c$, using Cauchy-Schwarz we have

$$\begin{aligned}
\langle h, g\rangle &= \int_0^\infty (\lambda+\mu)e^{-\mu t}\, d\langle E_\mu(f), g\rangle \\
&\le \left(\int_0^\infty (\lambda+\mu)e^{-\mu t}\, d\langle E_\mu(f), f\rangle\right)^{1/2} \left(\int_0^\infty (\lambda+\mu)e^{-\mu t}\, d\langle E_\mu(g), g\rangle\right)^{1/2} \\
&\le c\left(\int_0^\infty d\langle E_\mu(f), f\rangle\right)^{1/2} \left(\int_0^\infty e^{-\mu t/2}\, d\langle E_\mu(g), g\rangle\right)^{1/2} \\
&= c\|f\|_2 \|P_{t/2} g\|_2 \le c' \|f\|_2 \|g\|_1.
\end{aligned}$$

Taking the supremum over $g \in L^1$ with L^1 norm less than 1, this yields $\|h\|_\infty \le c\|f\|_2$. Finally, by (2.13),

$$U^\lambda h = \int_0^\infty e^{-\mu t}\, dE_\mu(f) = P_t f, \qquad a.e.,$$

and so the Hölder continuity of $P_t f$ follows from Proposition 2.4.) $\square$

Let $\mathbb{D}(\mathbb{R}_+, \overline{D})$ be the space of right continuous functions on $\mathbb{R}_+$ having left limits and taking values in $\overline{D}$ that is equipped with the Skorohod topology. For $t \ge 0$, we use X_t to denote the coordinate projection map on $\mathbb{D}(\mathbb{R}_+, \overline{D})$; that is, $X_t(\omega) = \omega(t)$ for $\omega \in \mathbb{D}(\mathbb{R}_+, \overline{D})$. For subsequential convergence to a diffusion, we need the following.

Assumption 2.6. (i) *For any sequence $x_n \to x$ with $x_n \in D_n$, $\{\mathbb{P}^n_{x_n}\}$ is tight in $\mathbb{D}(\mathbb{R}_+, \overline{D})$.*

(ii) *For any sequence $x_n \to x$ with $x_n \in D_n$ and any $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{n\to\infty} \mathbb{P}^n_{x_n} \left(J(X^n, \delta) > \varepsilon\right) = 0,$$

where $J(X, \delta) := \int_0^\infty e^{-u} \left(1 \wedge \sup_{\delta \le t \le u} |X_t - X_{t-\delta}|\right) du$.

We need the following well-known fact (see, for example [7, Lemma 6.4]) in the proof of Proposition 2.8. For readers' convenience, we provide a proof here.

Lemma 2.7. *Let K be a compact subset of $\mathbb{R}^d$. Suppose f and f_k, $k \in \mathbb{N}$, are functions on K such that $\lim_{k\to\infty} f_k(y_k) = f(y)$ whenever $y_k \in K$ converges to y. Then f is continuous on K and f_k converges to f uniformly on K.*

Proof. We first show that f is continuous on K. Fix $x_0 \in K$. Let x_k be any sequence in K that converges to it. Since $\lim_{i\to\infty} f_i(x) = f(x)$ for every $x \in K$, there is a sequence $n_k \in \mathbb{N}$ that increases to infinity so that $|f_{n_k}(x_k) - f(x_k)| \le 2^{-k}$ for every $k \ge 1$. Since $\lim_{k\to\infty} f_{n_k}(x_k) = f(x_0)$, it follows that $\lim_{k\to\infty} |f(x_0) - f(x_k)| = 0$. This shows that f is continuous at x_0 and hence on K.

We next show that f_k converges uniformly to f on K. Suppose not. Then there is $\varepsilon > 0$ so that for every $k \geq 1$, there are $n_k \geq k$ and $x_{n_k} \in K$ so that $|f_{n_k}(x_{n_k}) - f(x_{n_k})| > \varepsilon$. Since K is compact, by selecting a subsequence if necessary, we may assume without loss of generality that $x_{n_k} \to x_0 \in K$. As $\lim_{k\to\infty} f_{n_k}(x_{n_k}) = f(x_0)$ by the assumption, we have $\liminf_{k\to\infty} |f(x_0) - f(x_{n_k})| \geq \varepsilon$. This contradicts to the fact that f is continuous on K. $\square$

Now, applying the argument in [7, Section 6], we can prove that any subsequential limit of the laws of X^n under $\mathbb{P}^n_{x_n}$ is the law of a symmetric diffusion. For this, we need to introduce a projection map from $\overline{D}$ to D_n. For each $n \geq 1$, let $\phi_n : \overline{D} \to D_n$ be a map that projects each $x \in \overline{D}$ to some $\phi_n(x) \in D_n$ that minimizes $|x - y|$ over $y \in D_n$ (if there is more than one such point that does this, we choose and fix one). If needed, we extend a function f defined on D_n to be a function on $\overline{D}$ by setting $f(x) = f(\phi_n(x))$. Note that each D_n supports the measure m_n that converges vaguely to m. This implies that for each $x \in \overline{D}$ and $r > 0$, there is an $N \geq 1$ so that $\phi_n(x) \in B_E(x, r)$ for every $n \geq N$. From this, one concludes that

$$\phi_n(x_n) \to x_0 \quad \text{for every sequence } x_n \in \overline{D} \text{ that converges to } x_0. \tag{2.14}$$

Proposition 2.8. *Suppose that Assumptions 2.3 and 2.6 hold and that $\{X^n, \mathbb{P}^n_x, x \in D_n\}$ is conservative for sufficiently large n. For every subsequence $\{n_j\}$, there exists a sub-subsequence $\{n_{j(k)}\}$ and a continuous conservative m-symmetric Hunt process $(\widetilde{X}, \widetilde{\mathbb{P}}_x, x \in \overline{D})$ such that for every $x_{n_{j(k)}} \to x$, $\mathbb{P}^{n_{j(k)}}_{x_{n_{j(k)}}}$ converges weakly in $\mathbb{D}([0,\infty), \overline{D})$ to $\widetilde{\mathbb{P}}^x$.*

Proof. For notational simplicity, let us relabel the subsequence as $\{n\}$. We first claim that there exists a (sub-)subsequence $\{n_j\}$ such that $U^\lambda_{n_j} f$ converges uniformly on compact sets for each $\lambda > 0$ and $f \in C_b(\overline{D})$. Indeed, let $\{\lambda_i\}$ be a dense subset of $(0, \infty)$ and $\{f_k\}$ a sequence of functions in $C_b(\overline{D})$ such that $\|f_k\|_\infty \leq 1$ and whose linear span is dense in $(C_b(\overline{D}), \|\cdot\|_\infty)$. For fixed m and i, by Proposition 2.4 and the Ascoli-Arzelà theorem, there is a subsequence of $U^{\lambda_i}_n f_k$ that converges uniformly on compact sets. By a diagonal selection procedure, we can choose a subsequence $\{n_j\}$ such that $U^{\lambda_i}_{n_j} f_k$ converges uniformly on compact sets for every m and i to a Hölder continuous function which we denote as $U^{\lambda_i} f_k$. Noting that

$$U^\lambda_n - U^\beta_n = (\beta - \lambda) U^\lambda_n U^\beta_n, \quad \|U^\lambda_n\|_{\infty\to\infty} \leq \frac{1}{\lambda}, \quad \|U^\lambda_n - U^\beta_n\|_{\infty\to\infty} \leq \frac{\beta - \lambda}{\lambda\beta}, \tag{2.15}$$

a careful limiting argument shows that $U^\lambda_{n_j} f$ converges uniformly on compact sets, say to $U^\lambda f$, for any $\lambda > 0$ and any continuous function f, and (2.15) holds as well for $\{U^\lambda\}$. By the equi-continuity of $U^\lambda_{n_j} f$, we also have $U^\lambda_{n_j} f(x_{n_j}) \to U^\lambda f(x)$ for each $x_{n_j} \in D_{n_j}$ that converges to $x \in \overline{D}$.

We next claim that $\mathbb{P}^{n_j}_{x_{n_j}}$ converges weakly, say to $\widetilde{\mathbb{P}}_x$. Indeed, by Assumption 2.6(i), $\{\mathbb{P}^{n_j}_{x_{n_j}}\}$ is tight, so it suffices to show that any two limit points agree. Let $\mathbb{P}'$ and $\mathbb{P}''$ be any two limit points. Then, one sees that

$$\mathbb{E}' \left[\int_0^\infty e^{-\lambda s} f(X_s) ds \right] = U^\lambda f(x) = \mathbb{E}'' \left[\int_0^\infty e^{-\lambda s} f(X_s) ds \right],$$

for any $f \in C_b(\overline{D})$. So, by the uniqueness of the Laplace transform,

$$\mathbb{E}'[f(X_s)] = \mathbb{E}''[f(X_s)]$$

for almost all $s \geq 0$ and hence for every $s \geq 0$ since $s \to X_s$ is right continuous. So the one-dimensional distributions of X_t under $\mathbb{P}'$ and $\mathbb{P}''$ are the same. Set $P_s f(x) := \mathbb{E}' f(X_s)$. We have $P_s^{n_j} f(x_{n_j}) \to P_s f(x)$ for every sequence $x_{n_j} \in D_{n_j}$ that converges to x. Recall the restriction map ϕ_n introduced proceeding the statement of this theorem. It follows from (2.14) that $P_s^{n_j}(f \circ \phi_{n_j})(y_{n_j}) \to P_s f(y)$ for every sequence $y_{n_j} \in \overline{D}$ that converges to y. Thus by Lemma 2.7, $P_s^{n_j}(f \circ \phi_{n_j})$ converges to $P_s f$ uniformly on compact subsets of $\overline{D}$ and $P_s f \in C_b(\overline{D})$ for every $f \in C_c(\overline{D})$. For $f, g \in C_c(\overline{D})$ and $0 \leq s < t$, by the Markov property of X under $\mathbb{P}_{x_{n_j}}^{n_j}$,

$$\begin{aligned}
\mathbb{E}_{x_{n_j}}^{n_j}\left[g(X_s) f(X_t)\right] &= \mathbb{E}_{x_{n_j}}^{n_j}\left[((P_{t-s}^n f) g)(X_s)\right] \\
&= \mathbb{E}_{x_{n_j}}^{n_j}\left[((P_{t-s} f) g)(X_s)\right] + \mathbb{E}_{x_{n_j}}^{n_j}\left[((P_{t-s}^n f - P_{t-s} f) g)(X_s)\right].
\end{aligned}$$

The first term of the right-hand side converges to $\mathbb{E}'[((P_{t-s} f) g)(X_s)]$ by the above proof, while the second term goes to 0 since $P_{t-s}^{n_j} f \to P_{t-s} f$ uniformly on compact sets. Repeating this, we conclude that for every $k \geq 1$ and every $0 < s_1 < s_2 < \cdots < s_k$ and $f_j \in C_c(\overline{D})$,

$$\mathbb{E}'\left[\prod_{j=1}^k f_j(X_{s_j})\right] = \mathbb{E}''\left[\prod_{j=1}^k f_j(X_{s_j})\right] = P_{s_1}\left(f_1 P_{s_2-s_1}(f_2 P_{s_3-s_2}(f_3 \ldots))\right)(x). \tag{2.16}$$

This proves that the finite dimensional distributions of X under $\mathbb{P}'$ and $\mathbb{P}''$ are the same. Consequently, $\mathbb{P}' = \mathbb{P}''$, which we now denote as $\widetilde{\mathbb{P}}_x$. Moreover, (2.16) shows that $(X, \widetilde{\mathbb{P}}_x, x \in \overline{D})$ is a Markov process with transition semigroup $\{P_t, t \geq 0\}$.

Next we show that $\{\widetilde{\mathbb{P}}_x : x \in \overline{D}\}$ is a strong Markov process. Note that X is conservative with $\widetilde{\mathbb{P}}_x(X_0 = x) = 1$, and under Assumption 2.6(ii), X_t is continuous a.s. under $\widetilde{\mathbb{P}}_x$. We also have $P_t f \in C_b(\overline{D})$ for $f \in C_c(\overline{D})$. It is easy to deduce from these properties and (2.16) that for every $f \in C_c(\overline{D})$ and every stopping time T,

$$\mathbb{E}_x\left[f(X_{T+t}) | \mathcal{F}_{T+}\right] = P_t f(X_T), \qquad x \in \overline{D}.$$

See the proof of Theorem 2.3.1 on p.56 of [24]. From it, one gets the strong Markov property of X by a standard measure-theoretic argument (see p.57 of [24]). Since X is continuous and has infinite lifetime, this in fact shows that X is a continuous conservative Hunt process.

Finally, for $f, g \in C_c(\overline{D})$, by the convergence of semigroups and vague convergence of measures, it holds that, for every $t > 0$,

$$\int_{D_n} (P_t^{n_j} f)(x) g(x) m_{n_j}(dx) \to \int_{\overline{D}} (P_t f)(x) g(x) m(dx).$$

Since X^{n_j} is m_n-symmetric, this readily yields the desired m-symmetry of $\tilde{X}$. $\square$

Remark 2.9. Note that we did not use any special properties of the Euclidean metric in this section, so that all the arguments in this section can be extended to a metric measure space without any changes. □

By Theorem 2.1, Remark 2.2 (i) and Proposition 2.8, we see that in order to prove $(X^n, \mathbb{P}^n_{x_n})$ converges weakly to $(X, \mathbb{P}_x)$ in $\mathbb{D}([0,\infty), \overline{D})$ as $n \to \infty$, it suffices to verify condition (ii), (iii) in Theorem 2.1, vague convergence of the measure m_n to m on $\overline{D}$, Assumptions 2.3 and 2.6, and the conservativeness of $(X^n, \mathbb{P}^n_{x_n})$ for each $n \in \mathbb{N}$.

3 Random conductance model in unbounded domains

In this section we will obtain, as a first application of our theorem, a quenched invariance principle for random walk amongst random conductances on half-spaces, quarter-spaces, etc. The assumptions we make on the random conductances include the supercritical percolation model, and random conductances bounded uniformly from below and with finite first moments. For the main conclusion, see Theorem 3.17.

Fix $d_1, d_2 \in \mathbb{Z}_+$ such that $d_1 \geq 1$ and $d := d_1 + d_2 \geq 2$. Define a graph $(\mathbb{L}, E_{\mathbb{L}})$ by setting $\mathbb{L} := \mathbb{Z}_+^{d_1} \times \mathbb{Z}^{d_2}$ and $E_{\mathbb{L}} := \{e = \{x, y\} : x, y \in \mathbb{L}, |x - y| = 1\}$. Given $\mathcal{O} \subseteq E_{\mathbb{L}}$, let $\mathcal{C}_\infty(\mathbb{L}, \mathcal{O})$ be the infinite connected cluster of $(\mathbb{L}, \mathcal{O})$, provided it exists and is unique (otherwise set $\mathcal{C}_\infty(\mathbb{L}, \mathcal{O}) := \emptyset$).

Let $\mu = (\mu_e)_{e \in E_{\mathbb{L}}}$ be a collection of independent and identically distributed random variables on $[0, \infty)$, defined on a probability space $(\Omega, \mathbb{P})$ such that

$$p_1 := \mathbb{P}(\mu_e > 0) > p_c^{\mathrm{bond}}(\mathbb{Z}^d), \tag{3.1}$$

where $p_c^{\mathrm{bond}}(\mathbb{Z}^d) \in (0, 1)$ is the critical probability for bond percolation on $\mathbb{Z}^d$. We assume that there is $c > 0$ so that

$$\mathbb{P}(\mu_e \in (0, c)) = 0, \tag{3.2}$$

and

$$\mathbb{E}(\mu_e) < \infty. \tag{3.3}$$

This framework includes the special cases of supercritical percolation (where $\mathbb{P}(\mu_e = 1) = p_1 = 1 - \mathbb{P}(\mu_e = 0)$) and the random conductance model with conductances bounded from below (that is, $\mathbb{P}(c \leq \mu_e < \infty) = 1$ for some $c > 0$) and having finite first moments. For each $x \in \mathbb{L}$, set $\mu_x = \sum_{y \sim x} \mu_{xy}$. Set

$$\mathcal{O}_1 := \{e \in E_{\mathbb{L}} : \mu_e > 0\}, \qquad \mathcal{C}_1 := \mathcal{C}_\infty(\mathbb{L}, \mathcal{O}_1).$$

Note that for the random conductance model bounded from below, $\mathcal{C}_1 = \mathbb{L}$.

For each realization of $\mathcal{C}_1$, there is a continuous time Markov chain $Y = (Y_t)_{t \geq 0}$ on $\mathcal{C}_1$ with transition probabilities $P(x, y) = \mu_{xy}/\mu_x$, and the holding time at each $x \in \mathcal{C}_1$ being the exponential distribution with mean μ_x^{-1}. Such a Markov chain is sometimes called a variable speed random walk (VSRW). The corresponding Dirichlet form is $(\mathcal{E}, L^2(\mathcal{C}_1;\nu))$, where ν is the counting measure on $\mathcal{C}_1$ and

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y\in\mathcal{C}_1,\, x\sim y} (f(x)-f(y))(g(x)-g(y))\mu_{xy} \qquad \text{for } f,g\in L^2(\mathcal{C}_1;\nu).$$

The corresponding discrete Laplace operator is $\mathcal{L}_V f(x) = \sum_y (f(y)-f(x))\mu_{xy}$. For each f, g that have finite support, we have

$$\mathcal{E}(f,g) = -\sum_{x\in\mathcal{C}_1} (\mathcal{L}_V f)(x) g(x).$$

We will establish a quenched invariance principle for Y in Section 3.3, but we first need to derive some preliminary estimates regarding the geometry of $\mathcal{C}_1$ and the heat kernel associated with Y.

3.1 Percolation estimates

In this section, we derive a number of useful properties of the underlying percolation cluster $\mathcal{C}_1$. Most importantly, we introduce the concept of 'good' and 'very good' balls for the model and provide estimates for the probability of such occurring, see Definition 3.4 and Proposition 3.5 below.

Since a variety of percolation models will appear in the course of this paper, let us now make explicit that the critical probability for bond/site percolation on an infinite connected graph containing the vertex 0 is

$$p_c := \inf\left\{p\in[0,1] : \mathbb{P}_p\left(0 \text{ is in an infinite connected cluster of open bonds/sites}\right) > 0\right\},$$

where $\mathbb{P}_p$ is the law of parameter p bond/site percolation on the graph in question. Note in particular that the critical probability for bond percolation on $\mathbb{L}$ is identical to $p_c^{\text{bond}}(\mathbb{Z}^d)$, see [32, Theorem 7.2] (which is the bond percolation version of a result originally proved as [33, Theorem A]). Recall

$$\mathcal{O}_1 := \{e\in E_{\mathbb{L}} : \mu_e > 0\}, \qquad \mathcal{C}_1 := \mathcal{C}_\infty(\mathbb{L}, \mathcal{O}_1).$$

That $\mathcal{C}_1$ is non-empty almost-surely is guaranteed by [10, Corollary to Theorem 1.1] (this covers $d\geq 3$, and, as is commented there, the case $d=2$ can be tackled using techniques from [36]).

Now, suppose that μ is actually a restriction of independent and identically distributed (under $\mathbb{P}$) random variables $(\mu_e)_{e\in E_{\mathbb{Z}^d}}$, where $E_{\mathbb{Z}^d}$ are the usual nearest-neighbor edges for the integer lattice $\mathbb{Z}^d$, and define

$$\tilde{\mathcal{O}}_1 := \{e\in E_{\mathbb{Z}^d} : \mu_e > 0\}, \qquad \tilde{\mathcal{C}}_1 := \mathcal{C}_\infty(\mathbb{Z}^d, \tilde{\mathcal{O}}_1).$$

For sufficiently large K so that

$$q = q(K) := \mathbb{P}(0<\mu_e<K^{-1}) + \mathbb{P}(\mu_e > K) < p_1 - p_c^{\text{bond}}(\mathbb{Z}^d),$$

and writing $\tilde{\mathcal{O}}_I := \{e \in E_{\mathbb{Z}^d} : \mu_e \in I\}$ for $I \subseteq [0,\infty)$, we let

$$\begin{aligned}\tilde{\mathcal{O}}_R &:= \tilde{\mathcal{O}}_{(0,K^{-1})\cup(K,\infty)},\\ \tilde{\mathcal{O}}_S &:= \left\{e \in \tilde{\mathcal{O}}_1 : e \cap e' \neq \emptyset \text{ for some } e' \in \tilde{\mathcal{O}}_R\right\},\\ \tilde{\mathcal{O}}_2 &:= \tilde{\mathcal{O}}_1 \backslash \tilde{\mathcal{O}}_S.\end{aligned}$$

We will also define $\mathcal{O}_2 := \tilde{\mathcal{O}}_2 \cap E_{\mathbb{L}}$, and set $\mathcal{C}_2 := \mathcal{C}_\infty(\mathbb{L}, \mathcal{O}_2)$ – the next lemma will guarantee that this set is non-empty almost-surely.

To represent the set of 'holes', let $\mathcal{H} := \mathcal{C}_1 \backslash \mathcal{C}_2$. Moreover, for $x \in \mathcal{C}_1$, let $\mathcal{H}(x)$ be the connected component of $\mathcal{C}_1 \backslash \mathcal{C}_2$ containing x. The following lemma provides control on the size of these components. Since its proof is a somewhat technical adaptation to our setting of that used to establish [3, Lemma 2.3], which dealt with the whole $\mathbb{Z}^d$ model, we defer this to the appendix. Note, though, that in the percolation case (i.e. when μ_e are Bernoulli random variables) or the uniformly elliptic random conductor case, the proof of the result is immediate; indeed, for large enough K, we have that $\mathcal{O}_2 = \mathcal{O}_1$, and so $\mathcal{H} = \emptyset$.

Lemma 3.1. *For sufficiently large K, the following holds.*
(i) All the connected components of $\mathcal{H}$ are finite. Furthermore, there exist constants c_1, c_2 such that: for each $x \in \mathbb{L}$,

$$\mathbb{P}\left(x \in \mathcal{C}_1 \text{ and } \operatorname{diam}(\mathcal{H}(x)) \geq n\right) \leq c_1 e^{-c_2 n},$$

where diam *denotes the diameter with respect to the ℓ_∞ metric on $\mathbb{Z}^d$.*
(ii) There exists a constant α such that, $\mathbb{P}$-a.s., for large enough n, the volume of any hole intersecting the box $[-n,n]^d \cap \mathbb{L}$ is bounded above by $(\log n)^\alpha$.

In what follows, we will need to make comparisons between two graph metrics on $(\mathcal{C}_1, \mathcal{O}_1)$, and the Euclidean metric. The first of these, d_1, will simply be defined to be the shortest path metric on $(\mathcal{C}_1, \mathcal{O}_1)$, considered as an unweighted graph. To define the second metric, $\overline{d}_1$, we follow [8] by defining edge weights

$$t(e) := C_A \wedge \mu_e^{-1/2},$$

where $C_A < \infty$ is a deterministic constant, and then letting $\overline{d}_1$ be the shortest path metric on $(\mathcal{C}_1, \mathcal{O}_1)$, considered as a weighted graph (in [8], the analogous metric was denoted $\tilde{d}$). We note that the latter metric on $\mathcal{C}_1$ satisfies

$$\left(C_A^{-2} \vee \mu_{\{y,z\}}\right) \left|\overline{d}_1(x,y) - \overline{d}_1(x,z)\right|^2 \leq 1,$$

for any $x, y, z \in \mathcal{C}_1$ with $\{y,z\} \in \mathcal{O}_1$. Observe that, since the weights $t(e)$ are bounded above by C_A, we immediately have that $\overline{d}_1$ is bounded above by $C_A d_1$, Hence the following lemma establishes both $\overline{d}_1$ and d_1 are comparable to the Euclidean one. An easy consequence of this is the comparability of balls in the different metrics, see Lemma 3.3. The proofs of both these results are deferred to the appendix.

Lemma 3.2. *There exist constants c_1, c_2, c_3 such that: for $R \geq 1$,*

$$\sup_{\substack{x,y\in\mathbb{L}:\\ |x-y|\leq R}} \mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } d_1(x,y) \geq c_1 R\right) \leq c_2 e^{-c_3 R}, \tag{3.4}$$

and also, for every $x, y \in \mathbb{L}$,

$$\mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } \overline{d}_1(x,y) \leq c_1^{-1}|x-y|\right) \leq c_2 e^{-c_3|x-y|}. \tag{3.5}$$

Lemma 3.3. *There exist constants c_1, c_2, c_3, c_4 such that: for every $x \in \mathbb{L}$, $R \geq 1$,*

$$\mathbb{P}\left(\{x \in \mathcal{C}_1\} \cap \left\{\mathcal{C}_1 \cap B_E(x, c_1 R) \subseteq B_1(x,R) \subseteq \overline{B}_1(x, C_A R) \subseteq B_E(x, c_2 R)\right\}^c\right) \leq c_3 e^{-c_4 R},$$

where $B_1(x,R)$ is a ball in the metric space $(\mathcal{C}_1, d_1)$, $\overline{B}_1(x,R)$ is a ball in $(\mathcal{C}_1, \overline{d}_1)$, and $B_E(x,R)$ is a Euclidean ball.

We continue by adapting a definition for 'good' and 'very good' balls from [3]. In preparation for this, we define $\mu_e^0 := \mathbf{1}_{\{e\in\mathcal{O}_1\}}$, set $\mu_x^0 := \sum_{y\in\mathbb{L}} \mu^0_{\{x,y\}}$ for $x \in \mathbb{L}$, and then extend μ^0 to a measure on $\mathbb{L}$. Moreover, we set $\beta := 1 - 2(1+d)^{-1}$.

Definition 3.4. *(i) Let C_V, C_P, C_W, C_R, C_D be fixed strictly positive constants. We say the pair $(x,R) \in \mathcal{C}_1 \times \mathbb{R}_+$ is* good *if:*

$$B_1(x, C_A^{-1} r) \subseteq \overline{B}_1(x,r) \subseteq B_1(x, C_D r), \qquad \forall r \geq R, \tag{3.6}$$

$$|y-z| \geq C_R^{-1} R, \qquad \forall y \in \overline{B}_1(x, R/2),\ z \in \overline{B}_1(x, 8R/9)^c, \tag{3.7}$$

$$C_V R^d \leq \mu^0(\overline{B}_1(x,R)), \tag{3.8}$$

$$\operatorname{diam}(\mathcal{H}(y)) \leq R^\beta, \qquad \forall y \in B_E(x,R) \cap \mathcal{C}_1, \tag{3.9}$$

and the weak Poincaré inequality

$$\sum_{y\in B_1(x,R)} \left(f(y) - \check{f}_{B_1(x,R)}\right)^2 \mu_y^0 \leq C_P R^2 \sum_{\substack{y,z\in B_1(x,C_W R):\\ \{y,z\}\in\mathcal{O}_1}} |f(y) - f(z)|^2 \tag{3.10}$$

holds for every $f : B_1(x, C_W R) \to \mathbb{R}$. (Here $\check{f}_{B_1(x,R)}$ is the value which minimizes the left-hand side of (3.10)).
(ii) We say a pair $(x,R) \in \mathcal{C}_1 \times \mathbb{R}_+$ is very good *if: there exists $N = N_{(x,R)}$ such that (y,r) is good whenever $y \in \overline{B}_1(x,R)$ and $N \leq r \leq R$. We can always assume that $N \geq 2$. Moreover, if $N \leq M$, we will say that (x,R) is M-very good.*
(iii) Let $\alpha \in (0,1]$. For $x \in \mathcal{C}_1$, we define $R_x^{(\alpha)}$ to be the smallest integer M such that (x,R) is R^α-very good for all $R \geq M$. We set $R_x^{(\alpha)} = 0$ if $x \notin \mathcal{C}_1$.

The following proposition, which is an adaptation of [3, Proposition 2.8], provides bounds for the probabilities of these events and for the distribution of $R_x^{(\alpha)}$.

Proposition 3.5. *There exist $c_1, c_2, C_V, C_P, C_W, C_R, C_D$ (depending on the law of μ and the dimension d) such that the following holds. For $x \in \mathbb{L}$, $R \geq 1$, $\alpha \in (0,1]$,*

$$\mathbb{P}\left(x \in \mathcal{C}_1,\ (x,R) \text{ is not good}\right) \leq c_1 e^{-c_2 R^\beta}, \tag{3.11}$$

$$\mathbb{P}\left(x \in \mathcal{C}_1,\ (x,R) \text{ is not } R^\alpha\text{-very good}\right) \leq c_1 e^{-c_2 R^{\alpha\beta}}. \tag{3.12}$$

Hence

$$\mathbb{P}\left(x \in \mathcal{C}_1,\ R_x^{(\alpha)} \geq n\right) \leq c_1 e^{-c_2 n^{\alpha\beta}}. \tag{3.13}$$

Proof. That

$$\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.6) \text{ does not hold}\right) \leq c_3 e^{-c_4 R}$$

is a straightforward consequence of Lemma 3.3.

For the second property, we have

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.7) \text{ does not hold}\right)\\
&\quad= \mathbb{P}\left(x \in \mathcal{C}_1,\ \exists y \in \overline{B}_1(x,R/2),\ z \in \overline{B}_1(x,8R/9)^c :\ |y-z| < C_R^{-1}R\right)\\
&\quad\leq \mathbb{P}\left(x \in \mathcal{C}_1,\ \exists y \in B_E(x,c_5R) \cap \overline{B}_1(x,R/2),\ z \in \overline{B}_1(x,8R/9)^c :\ |y-z| < C_R^{-1}R\right)\\
&\qquad+c_6 e^{-c_7 R}\\
&\quad\leq c_6 e^{-c_7 R} + \sum_{y \in B_E(x,c_5R)} \sum_{z \in B_E(y,C_R^{-1}R)} \mathbb{P}\left(y,z \in \mathcal{C}_1,\ \overline{d}_1(y,z) > R/3\right)\\
&\quad\leq c_8 e^{-c_9 R}
\end{aligned}$$

where we apply Lemma 3.3 to deduce the first inequality, and (3.4) to obtain the final one.

For (3.8), applying Lemma 3.3 again yields

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.8) \text{ does not hold}\right)\\
&\quad= \mathbb{P}\left(x \in \mathcal{C}_1,\ \mu^0(\overline{B}_1(x,R)) < C_V R^d\right)\\
&\quad\leq c_{10} e^{-c_{11}R} + \mathbb{P}\left(x \in \mathcal{C}_1,\ |\mathcal{C}_1 \cap B_E(x,c_{12}R)| < C_V R^d\right).
\end{aligned}$$

Now, let $Q \subseteq B_E(x,c_{12}R) \cap \mathbb{L}$ be a cube of side-length $c_{13}R$ such that $\inf_{y \in \mathbb{L}\backslash Q} |x-y| \geq c_{13}R/2$. Moreover, if we let $\mathcal{C}^+(Q)$ be the largest connected component of the graph $(Q, \mathcal{O}_1)$, then

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.8) \text{ does not hold}\right)\\
&\quad\leq \mathbb{P}\left(x \in \mathcal{C}_1 \cap \mathcal{C}^+(Q),\ |\mathcal{C}_1 \cap Q| < C_V R^d\right) + \mathbb{P}\left(x \in \mathcal{C}_1 \backslash \mathcal{C}^+(Q)\right) + c_{10}e^{-c_{11}R}\\
&\quad\leq \mathbb{P}\left(|\mathcal{C}^+(Q)| < C_V R^d\right) + \mathbb{P}\left(x \in \tilde{\mathcal{C}}_1 \backslash \mathcal{C}^+(Q)\right) + c_{10}e^{-c_{11}R}.
\end{aligned}$$

This bound is now expressed in terms of the full $\mathbb{Z}^d$ model, for which appropriate estimates already exist. In particular, the first term here is bounded above by $\mathbb{P}(G(Q)^c)$, where $G(Q)$ is the event that $|\mathcal{C}^+(Q)| \geq \frac{1}{2}\mathbb{P}(0 \in \tilde{\mathcal{C}}_1)|Q|$ (recall that $\tilde{\mathcal{C}}_1 := \mathcal{C}_\infty(\mathbb{Z}^d, \tilde{\mathcal{O}}_1)$). Consequently, by simply translating the relevant part of [3, Lemma 2.6] to our setting (taking $K = \infty$), we obtain that it is bounded above by $c_{13}e^{-c_{14}R^\beta}$. That the second term is bounded above by $c_{15}e^{-c_{16}R}$ can be established by applying [6, Lemma 2.8].

To check the fourth property, we simply note

$$\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.9) \text{ does not hold}\right) \leq \sum_{y \in B_E(x,R)\cap\mathbb{L}} \mathbb{P}\left(y \in \mathcal{C}_1,\ \mathrm{diam}(\mathcal{H}(y)) > R^\beta\right),$$

which may be bounded above by $c_{17}e^{-c_{18}R^\beta}$ by applying Lemma 3.1.

Finally, for the Poincaré inequality, we will apply [6, Proposition 2.12]. In particular, this result yields that if Q is a cube of side-length $2R$ contained in $\mathbb{L}$, $\mathcal{C}^+(Q)$ is the largest connected component of the graph $(Q, \mathcal{O}_1)$, and $H(Q)$ is the event that

$$\min_a \sum_{y \in \mathcal{C}^+(Q)} \left(f(y) - a\right)^2 \mu_y^0 \leq CR^2 \sum_{\substack{y,z \in \mathcal{C}^+(Q):\\ \{y,z\} \in \mathcal{O}_1}} |f(y) - f(z)|^2$$

for every $f : \mathcal{C}^+(Q) \to \mathbb{R}$, then $\mathbb{P}(H(Q)^c) \leq c_{19}e^{-c_{20}R^\beta}$. Furthermore, it is clear that if $H(Q)$ holds and also $B_1(x,R) \subseteq \mathcal{C}^+(Q) \subseteq B_1(x, c_{21}R)$, then (3.10) holds. This means that

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.10) \text{ does not hold}\right)\\
&\quad\leq\quad c_{22}e^{-c_{23}R^\beta} + \mathbb{P}\left(\{x \in \mathcal{C}_1\} \cap \left\{B_1(x,R) \subseteq \mathcal{C}^+(Q) \subseteq B_1(x, c_{21}R)\right\}^c\right),
\end{aligned}$$

where Q is chosen such that $B_E(x,R) \cap \mathbb{L} \subseteq Q$. Noting as above that $\mathbb{P}(x \in \mathcal{C}_1 \backslash \mathcal{C}^+(Q)) \leq c_{24}e^{-c_{25}R}$, at the expense of adjusting constants, we may replace $\{x \in \mathcal{C}_1\}$ by $\{x \in \mathcal{C}_1 \cap \mathcal{C}^+(Q)\}$ in the above bound. On the event $\{x \in \mathcal{C}_1 \cap \mathcal{C}^+(Q)\} \cap \{B_1(x,R) \subseteq \mathcal{C}^+(Q)\}^c$, it is elementary to check that $B_1(x,R) \not\subseteq B_E(x,R)$, which is impossible. Since $\mathcal{C}^+(Q) \subseteq B_E(x,2R)$, we have thus shown that

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1,\ (3.10) \text{ does not hold}\right)\\
&\quad\leq\quad c_{26}e^{-c_{27}R^\beta} + \mathbb{P}\left(\{x \in \mathcal{C}_1\} \cap \{B_E(x,2R) \subseteq B_1(x,cR)\}^c\right).
\end{aligned}$$

By applying Lemma 3.3 once again, this expression is bounded above by $c_{28}e^{-c_{29}R^\beta}$, and so we have completed the proof of (3.11).

Given (3.11), a simple union bound subsequently yields (3.12), and the inequality at (3.13) is a straightforward consequence of this. □

Remark 3.6. It only requires a simple argument to check that if (x,R) is good and $y \in \mathcal{C}_1$ satisfies $d_1(x,y) \geq C_D R$, then

$$C_D^{-1} d_1(x,y) \leq \overline{d}_1(x,y) \leq C_A d_1(x,y)$$

(cf. [8, Lemma 2.10(a)]). □

Finally, we state a bound that allows us to compare ν with $\tilde{\nu}$, which is the measure defined similarly from the whole $\mathbb{Z}^d$ model, i.e. uniform measure on $\tilde{\mathcal{C}}_1$. Its proof can be found in the appendix.

Lemma 3.7. *There exists a constant c such that if $Q \subseteq \mathbb{L}$ is a cube of side length n, then*

$$\mathbb{P}\left(\tilde{\nu}(Q) - \nu(Q) \geq n^{d-1}(\log n)^{d+1}\right) \leq cn^{-2}.$$

3.2 Heat kernel estimates

Let $D_n = n^{-1}\mathcal{C}_1$, $\overline{D} = \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$ (recall $\mathbb{R}_+ := [0,\infty)$). Let Y be the VSRW on $\mathcal{C}_1$, and for a given realization of $\mathcal{C}_1 = \mathcal{C}_1(\omega)$, $\omega \in \Omega$, write P_x^ω for the law of Y started from $x \in \mathcal{C}_1$. Moreover, define Z to be the trace of Y on $\mathcal{C}_2$, that is, the time change of Y by the inverse of $A_t = \int_0^t 1_{(Y_s \in \mathcal{C}_2)} ds$. Specifically, writing $\mathfrak{a}_t = \inf\{s : A_s > t\}$ for the right-continuous inverse of A, we set

$$Z_t = Y_{\mathfrak{a}_t}, \qquad t \geq 0.$$

Note that unlike Y, the process Z may perform long jumps by jumping over the holes of $\mathcal{C}_2$. If $x \in \mathcal{C}_2(\omega)$ then we have $Z_0 = Y_0 = x$, P_x^ω-a.s., but otherwise $Z_0 = Y_{\mathfrak{a}_0}$.

Given the percolation estimates of Section 3.1, we can follow [3, Section 4] to establish the following theorems, which correspond to Proposition 4.7(c) and Theorem 4.11 in [3]. We remark that the second of the two results will be used in this paper only for the proof of Theorem 3.12. Since it is the case that, given Proposition 3.5, the proofs are a simple modification of those in [3], we omit them. For the statement of the first result, we set

$$\Psi(R,t) = \begin{cases} e^{-R^2/t} & \text{if } t > e^{-1}R, \\ e^{-R\log(R/t)} & \text{if } t < e^{-1}R. \end{cases}$$

Proposition 3.8. *Write $\tau_A^Z = \inf\{t : Z_t \notin A\}$, $\tau_A^Y = \inf\{t : Y_t \notin A\}$. There exist constants $\delta, c_i \in (0,\infty)$ and random variables $(R_x, x \in \mathbb{L})$ with*

$$\mathbb{P}(R_x \geq n, x \in \mathcal{C}_1) \leq c_1 e^{-c_2 n^\delta}, \tag{3.14}$$

such that the following holds: for $x \in \mathcal{C}_1$, $t > 0$ and $R \geq R_x$,

$$P_x^\omega(\tau^Z_{B_E(x,R)} < t) \leq c_3 \Psi(c_4 R, t),$$
$$P_x^\omega(\tau^Y_{B_E(x,R)} < t) \leq c_3 \Psi(c_4 R, t).$$

Theorem 3.9. *There exist: constants $\delta, c_i \in (0,\infty)$; a set $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$; and random variables $(S_x, x \in \mathbb{L})$ satisfying $S_x(\omega) < \infty$ for each $\omega \in \Omega_1$ and $x \in \mathcal{C}_2(\omega)$, and*

$$\mathbb{P}(S_x \geq n, x \in \mathcal{C}_2) \leq c_1 e^{-c_2 n^\delta};$$

such that the following statements hold.
(a) For $x, y \in \mathcal{C}_2(\omega)$ the transition density of Z, as defined by setting $q_t^Z(x,y) := P_x^\omega(Z_t = y)$, satisfies

$$q_t^Z(x,y) \le c_3 t^{-d/2} \exp(-c_4|x-y|^2/t), \quad t \ge |x-y| \vee S_x,$$
$$q_t^Z(x,y) \ge c_5 t^{-d/2} \exp(-c_6|x-y|^2/t), \quad t \ge |x-y|^{3/2} \vee S_x.$$

(b) Further, if $x \in \mathcal{C}_2(\omega)$, $t \ge S_x$ and $B = B_2(x, 2\sqrt{t})$ then

$$q_t^{Z,B}(x,y) \ge c_7 t^{-d/2} \quad \text{for } y \in B_2(x, \sqrt{t}),$$

where $B_2(x,R)$ is a ball in the (unweighted) graph $(\mathcal{C}_2, \mathcal{O}_2)$, and $q^{Z,B}$ is the transition of Z killed on exiting B, i.e. $q_t^{Z,B}(x,y) := P_x^\omega(Z_t = y, \tau_B^Z > t)$.

Applying Proposition 3.8, we can establish the following, which corresponds to [3, Proposition 5.13 (b)]. To state the result, we introduce the rescaled process $Y^n = (Y_t^n)_{t\ge0}$ by setting

$$Y_t^n := n^{-1} Y_{n^2 t}.$$

Proposition 3.10 (Tightness). *Let $K, T, r > 0$. For $\mathbb{P}$-a.e. ω, the following is true: if $x_n \in D_n, n \ge 1$, $x \in B_E(0,K)$ are such that $x_n \to x$, then*

$$\lim_{R\to\infty} \limsup_{n\to\infty} P_{nx_n}^\omega \left(\sup_{s\le T} |Y_s^n| > R \right) = 0, \tag{3.15}$$
$$\lim_{\delta\to0} \limsup_{n\to\infty} P_{nx_n}^\omega \left(\sup_{|s_1-s_2|\le\delta, s_i\le T} |Y_{s_2}^n - Y_{s_1}^n| > r \right) = 0. \tag{3.16}$$

In particular, for $\mathbb{P}$-a.e. ω, if $x_n \in D_n, n \ge 1$, $x \in \overline{D}$ are such that $x_n \to x$, under $P_{nx_n}^\omega$, the family of processes $(Y_t^n)_{t\ge0}, n \in \mathbb{N}$ is tight in $\mathbb{D}([0,\infty), \overline{D})$.

Proof. Since the statement is slightly different from [3, Proposition 5.13], we sketch the proof. Note that since $x_n \to x \in B_E(0,K)$, then by setting $M = K+1$ we have that $nx_n \in B_E(0, nM)$ for all n suitably large. Let $nR > \sup_n R_{nx_n}$. Then, by Proposition 3.8,

$$P_{nx_n}^\omega \left(\sup_{s\le T} |Y_s^n| > R \right) = P_{nx_n}^\omega \left(\tau_{B_E(0,nR)}^Y < n^2 T \right) \le c_1 \Psi(c_2 nR, n^2 T).$$

Considering separately the cases $1/n < T/R$ and $1/n \ge T/R$, we deduce that

$$P_{nx_n}^\omega \left(\sup_{s\le T} |Y_s^n| > R \right) \le c_3 e^{-c_4 R^2/T} \vee e^{-R}.$$

Since $\limsup_n R_{nx_n}/n \le \limsup_n \sup_{x\in B_E(0,Mn)} R_x/n < \infty$, $\mathbb{P}$-a.s., due to the Borel-Cantelli argument using (3.14), we obtain (3.15).

We next prove (3.16). Write

$$p(x,T,\delta,r) = P_x^\omega\left(\sup_{|s_1-s_2|\le\delta, s_i\le T}|Y_{s_2}-Y_{s_1}|>r\right),$$

so that

$$P_{nx_n}^\omega\left(\sup_{|s_1-s_2|\le\delta, s_i\le T}|Y^n_{s_2}-Y^n_{s_1}|>r\right) = p(nx_n, n^2T, n^2\delta, nr).$$

Arguing similarly to the proof of [3, Proposition 5.13], we have

$$p(nx_n, n^2T, n^2\delta, 2nr) \le c\exp(-cnT^{1/2}) + c(T/\delta)\exp(-cr^2/\delta),$$

provided

$$T^{1/2} \ge n^{-1}R_x^{2/3}, \quad \delta > n^{-1}r, \quad r \ge n^{-1}\max_{y\in B_E(nx_n, n^{3/2}T^{3/4})} R_y. \tag{3.17}$$

Note that $B_E(nx_n, n^{3/2}T^{3/4}) \subset B_E(0, nM + n^{3/2}T^{3/4})$ for large n. If T, r and δ are fixed, due to the Borel-Cantelli argument using (3.14), each of conditions in (3.17) holds when n is large enough. So, for $\mathbb{P}$-a.e. ω,

$$\limsup_{n\to\infty} p(nx_n, n^2T, n^2\delta, 2nr) \le c(T/\delta)\exp(-cr^2/\delta),$$

and (3.16) follows.

Using (3.15) and (3.16), we have tightness for $\{P_{nx_n}^\omega\}$ by [31, Corollary 3.7.4]. □

We can further establish the following theorems, which correspond to [17, Lemma 5.6, Proposition 6.1] and [3, Theorem 7.3]. We denote by $(q_t^Y(x,y))_{x,y\in\mathcal{C}_1, t>0}$ the heat kernel associated with Y, i.e. for $x, y \in \mathcal{C}_1$, $t > 0$,

$$q_t^Y(x,y) := P_x^\omega(Y_t = y).$$

(We recall that the invariant measure of Y is the uniform measure ν on $\mathcal{C}_1$).

Proposition 3.11. *There exist $c_1, c_2, c_3, \gamma \in (0,\infty)$ (non-random) and random variables $(R_x, x \in \mathbb{L})$ with*

$$\mathbb{P}(R_x \ge n, x \in \mathcal{C}_1) \le \exp(-c_1 n^\gamma), \tag{3.18}$$

such that if $x, y \in \mathcal{C}_1$, then

$$q_t^Y(x,y) \le c_2 t^{-d/2} \quad \textit{for } t \ge (c_3 \vee 2d_1(x,y) \vee R_x)^{1/4}.$$

Proof. Note that the corresponding result for Z-process is given in [3, Corollary 4.3]. We need to obtain similar result for Y-process. First, note that because we have Proposition 3.5, the proof of [3, Proposition 4.1 and Corollary 4.3] (with $\varepsilon = 1/4$ for simplicity) goes through once (4.7) in [3] is verified. To check [3, (4.7)], we use [8, Theorem 2.3], which can be proved almost identically in our case. Note that in [8, Theorem 2.3], the metric $\tilde{d}$ is used, but thanks to Remark 3.6, we can obtain the same estimates using the metric d_1. Finally, using Cauchy-Schwarz, we obtain the desired inequality. □

For $G \subset \mathcal{C}_1$, we define $\partial^{\rm out}(G)$ to be the exterior boundary of G in the graph $(\mathcal{C}_1, \mathcal{O}_1)$, i.e. those vertices of $\mathcal{C}_1 \backslash G$ that are connected to G by an edge in $\mathcal{O}_1$, and set $cl(G) = G \cup \partial^{\rm out}(G)$. We say that a function h is Y–harmonic in $A \subset \mathcal{C}_1$ if h is defined on $cl(A)$ and $\mathcal{L}_V h(x) = 0$ for $x \in A$.

Theorem 3.12 (Elliptic Harnack inequality). *There exist random variables $(R'_x, x \in \mathbb{L})$ with*

$$\mathbb{P}(x \in \mathcal{C}_1, R'_x \geq n) \leq ce^{-c'n^\delta},$$

and a constant C_E such that if $x_0 \in \mathcal{C}_1$, $R \geq R'_{x_0}$ and $h : cl(B_1(x_0, R)) \to \mathbb{R}_+$ is Y–harmonic on $B_1 = B_1(x_0, R)$, then writing $B'_1 = B_1(x_0, R/2)$,

$$\sup_{B'_1} h \leq C_E \inf_{B'_1} h.$$

Proof. Given Lemma 3.1, the proof is almost identical to that of [3, Theorem 7.3]. □

3.3 Quenched invariance principle

To prove a quenched invariance principle for Y (see Theorem 3.17 below), we will check the conditions of Theorem 2.1 one by one. We choose $\delta_n = c_*^2/n$, where c_* is a constant that will be chosen later. First, since X^n is a continuous time Markov chain with holding time at x being an exponential random variable of mean μ_x^{-1}, it is conservative. Condition (ii) in Theorem 2.1 is a consequence of the quenched invariance principle for the whole space (cf. [3]) and the fact that $\mathcal{C}_1 \subseteq \tilde{\mathcal{C}}_1$, which is a consequence of the uniqueness of the infinite percolation clusters in the two settings. Since in the original papers quenched invariance principles are uniformly stated in terms of the random walk started from the origin, whereas we require such to hold from an arbitrary starting point, we state the following generalization of existing results. We will suppose that $\tilde{P}^\omega_y$ refers to the quenched law of the VSRW $\tilde{Y}$ on $\tilde{\mathcal{C}}_1$ started from $y \in \tilde{\mathcal{C}}_1$, and $\tilde{Y}^n$ refers to the rescaled process defined by setting $\tilde{Y}^n_t := n^{-1}\tilde{Y}_{n^2t}$. The proof of the result can be found in the appendix.

Theorem 3.13. *There exists a deterministic constant $c \in (0, \infty)$ such that, for $\mathbb{P}$-a.e ω, the laws of the processes $\tilde{Y}^n$ under $\tilde{P}^\omega_{nx_n}$, where $nx_n \in \mathcal{C}_1$ and $x_n \to x$, converge weakly to the laws of $(B_{ct})_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is standard Brownian motion on $\mathbb{R}^d$ started from x.*

Concerning Assumption 2.3(i), we will prove the following: there exist $c_*, c_1 \in (0, \infty)$ (non-random) and $N_0(\omega)$ such that for all $n \geq N_0(\omega)$, $x_0, x \in B_E(0, n^{1/2}) \cap D_n$ and $c_*/n^{1/2} \leq r' \leq 1$,

$$E^\omega_x \left(\tau_{B_E(x_0,r')\cap D_n}(Y^n)\right) \leq c_1 r'^2. \tag{3.19}$$

Applying Proposition 3.11, we have the following: for all $x_0, x \in B_E(0, n^{3/2}) \cap \mathcal{C}_1$ and $r \leq n$, if $c_2 r^2 \geq (c_3 \vee 2\sup_{z \in B_E(x_0,r)\cap \mathcal{C}_1} d_1(x, z) \vee R_x)^{1/4}$, then

$$\begin{aligned} P^\omega_x \left(\tau^Y_{B_E(x_0,r)\cap\mathbb{L}} \geq c_2 r^2\right) &\leq P^\omega_x \left(Y_{c_2r^2} \in B_E(x_0, r)\right) = \int_{B_E(x_0,r)} q^Y_{c_2r^2}(x, z)\mu^0(dz) \\ &\leq \frac{c}{(c_2r^2)^{d/2}}\mu^0(B_E(x_0, r)) \leq \frac{cc_4r^d}{c_2^{d/2}r^d} \leq 1/2, \end{aligned} \tag{3.20}$$

where we used $\mu^0(B_E(x_0,r)) \le c_4 r^d$ and we set $c_2 := (2cc_4)^{2/d} \vee c_3^{1/4}$. Now, using Lemma 3.3, there exists $N_1(\omega) \in \mathbb{N}$ that satisfies $\mathbb{P}(N_1(\omega) \ge m) \le c_1 e^{-c_2 m}$ such that $B_E(x, c_1 R) \subset B_1(x,R)$ for $x \in B_E(0,R) \cap \mathcal{C}_1$ and $R \ge N_1(\omega)$. On the other hand, $|x-z| \le |x| + |x_0| + |x_0 - z| \le 2n^{3/2} + r$ for $z \in B_E(x_0, r)$, so taking $r \ge c_* n^{1/2}$ with c_* large enough, there exists $N_2(\omega)$ that satisfies $\mathbb{P}(N_2(\omega) \ge m) \le c_1 e^{-c_2 m}$ such that $c_2 r^2 \ge (2 \sup_{z \in B_E(x_0,r)\cap \mathcal{C}_1} d_1(x,z))^{1/4}$ holds for $n \ge N_2(\omega)$. Next, by (3.18),

$$\mathbb{P}\left(\sup_{x \in B_E(0,n^{3/2}) \cap \mathcal{C}_1} R_x \ge n \right) \le n^{3d/2} e^{-c_1 n^\gamma}.$$

Summarizing, (3.20) holds for all $x_0, x \in B_E(0, n^{3/2}) \cap \mathcal{C}_1$, $c_* n^{1/2} \le r \le n$ and $n \ge N_0(\omega) := N_2(\omega) \vee N_3(\omega)$, where $N_3(\omega) := \sup_{x \in B_E(0,n^{3/2})\cap\mathcal{C}_1} R_x$. Moreover, the random variable $N_0(\omega)$ is almost-surely finite; in fact, we have the following tail bound for it, which will be useful in Example 4.1 below:

$$\mathbb{P}\left(N_0(\omega) \ge m\right) \le c_1 e^{-c_2 m^\gamma}. \tag{3.21}$$

Using the Markov property, we can inductively obtain

$$P_x^\omega \left(\tau^Y_{B_E(x_0,r)\cap \mathbb{L}} \ge k c_2 r^2 \right) \le (1/2)^k, \qquad \forall k \in \mathbb{N}.$$

So,

$$E_x^\omega \left(\tau^Y_{B_E(x_0,r)\cap \mathbb{L}} \right) \le \sum_k (k+1) c_2 r^2 P_x^\omega \left(k c_2 r^2 \le \tau^Y_{B_E(x_0,r)\cap\mathbb{L}} < (k+1) c_2 r^2 \right) \le 3 c_2 r^2.$$

For $Y^n = n^{-1} Y_{n^2 t}$, we therefore have: for $n \ge N_0(\omega)$, $x_0 \in B_E(0, n^{1/2}) \cap D_n$, $x \in B_E(0, n^{1/2}) \cap D_n$ and $c_*/n^{1/2} \le r' \le 1$,

$$E_{nx}^\omega \left(\tau_{B_E(x_0,r')\cap D_n}(Y^n) \right) \le \frac{1}{n^2} E_{nx}^\omega \left(\tau^Y_{B_E(nx_0,r)\cap\mathbb{L}} \right) \le \frac{1}{n^2} \cdot 3 c_2 r^2 = 3 c_2 r'^2,$$

where $r = nr'$. Thus (3.19) holds, and so Assumption 2.3(i) holds with $\delta_n = c_*^2/n$, $\beta = 2$.

Regarding Assumption 2.3(ii) we observe that, using Theorem 3.12, the relevant condition can be obtained similarly to [9, Proposition 3.2]. (Note that Proposition 3.2 in [9] is a parabolic version, whereas we just need an elliptic version.) Indeed, taking $(\log n)^{2/\delta}$ as n in Theorem 3.12,

$$\mathbb{P}\left(\sup_{x \in B_E(0,cn^2)\cap\mathcal{C}_1} R'_x \ge (\log n)^{2/\delta} \right) \le c^d n^{2d} e^{-c'(\log n)^2} \le c'/n^2.$$

Thus, by the Borel-Cantelli lemma, there exists $N_1(\omega) \in \mathbb{N}$ such that

$$\sup_{x \in B_E(0,cn^2)\cap\mathcal{C}_1} R'_x \le (\log n)^{2/\delta}, \quad \forall n \ge N_1(\omega),$$

so the elliptic Harnack inequality holds for Y-harmonic functions on balls $B_E(x_0, R)$ with $x_0 \in B_E(0, cn^2)$, $R \ge (\log n)^{2/\delta}$. By scaling $Y^n(t) = n^{-1} Y_{n^2 t}$, the elliptic Harnack inequality holds uniformly for Y^n-harmonic functions on $B_E(x_0, R)$ with $x_0 \in B_E(0, cn)$, $R \ge (\log n)^{2/\delta}/n$. Given the elliptic Harnack inequality, we can obtain the desired Hölder continuity in a similar way as in the proof of Proposition 3.2 of [9]. Thus, setting $\delta_n := c_*^2/n$, Assumption 2.3(ii) holds for $R \geq \delta_n^{1/2}$, since $\delta_n^{1/2} \geq (\log n)^{2/\delta}/n$.

Next, we remark that part (i) of Assumption 2.6 is direct from Proposition 3.10, and part (ii) follows from Proposition 3.10 (especially (3.16) implies the condition).

The following proposition gives the appropriate convergence for the sequence of measures $(m_n)_{n\geq 1}$ defined by setting $m_n := n^{-d}\nu(n\,\cdot)$. (Recall that ν is the invariant measure for Y, and so the measure m_n is invariant for Y^n.)

Proposition 3.14. *$\mathbb{P}$-a.s., the measures $(m_n)_{n\geq 1}$ converge vaguely to m, a deterministic multiple of Lebesgue measure on $\overline{D}$.*

Proof. First note that if $Q \subset \overline{D}$ is a cube of side length λ, then applying Lemma 3.7 in a Borel Cantelli argument yields that, $\mathbb{P}$-a.s.,

$$\frac{\tilde{\nu}(nQ) - \nu(nQ)}{n^d} \to 0. \tag{3.22}$$

Next, consider a rectangle of the form $R = [0, \lambda_1] \times \cdots \times [0, \lambda_d]$. Since the full $\mathbb{Z}^d$ model is ergodic under coordinate shifts, a simple application of a multidimensional ergodic theorem yields that, $\mathbb{P}$-a.s.,

$$\frac{\tilde{\nu}(nR)}{n^d} \to c_1 \prod_{i=1}^{d} \lambda_i,$$

where $c_1 := \mathbb{P}(0 \in \tilde{\mathcal{C}}_1) \in (0, 1]$. An inclusion-exclusion argument allows one to extend this result to any rectangle of the form $[x_1, x_1+\lambda_1] \times \cdots \times [x_d, x_d+\lambda_d]$, where $x_i \geq 0$ for $i = 1, \dots, d$. Clearly the particular orthant is not important, so the result can be further extended to cover any rectangle $R \subset \overline{D}$, and, in particular, we have that, $\mathbb{P}$-a.s.,

$$\frac{\tilde{\nu}(nQ)}{n^d} \to c_1 \lambda^d.$$

Applying (3.22), we obtain that the above limit still holds when $\tilde{\nu}$ is replaced by ν. The proposition follows. □

Finally, in order to verify Condition (iii) in Theorem 2.1, we first give a lemma.

Lemma 3.15. *Let $\{\eta_i\}_{i\geq 1}$ be independent and identically distributed with $E|\eta_1| < \infty$. Suppose $\{a_k^n\}_{k=1}^n$ is a sequence of real numbers with $|a_k^n| \leq M$ for all k, n (here $M > 0$ is some fixed constant) such that the following two limits exist:*

$$a := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k^n, \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |a_k^n|.$$

It then holds that $\frac{1}{n}\sum_{k=1}^{n} a_k^n \eta_k$ converges to $aE[\eta_1]$ almost surely as $n \to \infty$.

Proof. This can be proved similarly to Etemadi's proof of the strong law of large numbers (see [30]). □

Proposition 3.16. *If $C_0 := \mathbb{E}(\mu_e) < \infty$, then $\mathbb{P}$-a.s.,*

$$\tilde{\mathcal{E}}(f,f) \le \limsup_{n\to\infty} \mathcal{E}^{(n)}(f,f) \le C_0 \int_{\overline{D}} |\nabla f(x)|^2 dx, \qquad \forall f \in C_c^2(\overline{D}), \tag{3.23}$$

where $C_c^2(\overline{D})$ is the space of compactly supported functions on $\overline{D}$ with a continuous second derivative, and

$$\mathcal{E}^{(n)}(f,f) := \frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathcal{C}_1:\\ \{x,y\}\in\mathcal{O}_1}} (f(x/n) - f(y/n))^2 \mu_{x,y}$$

is the Dirichlet form on $L^2(D_n; m_n)$ corresponding to Y^n. In particular, condition (iii) in Theorem 2.1 holds.

Proof. The first inequality of (3.23) is standard. Indeed,

$$\begin{aligned}\tilde{\mathcal{E}}(f,f) &= \sup_{t>0} \frac{1}{t}(f - P_t f, f) = \sup_{t>0} \liminf_{k\to\infty} \frac{1}{t}(f - P_t^{n_{j(k)}} f, f)\\ &\le \liminf_{k\to\infty} \sup_{t>0} \frac{1}{t}(f - P_t^{n_{j(k)}} f, f) = \liminf_{k\to\infty} \mathcal{E}^{(n_{j(k)})}(f,f),\end{aligned}$$

where the first inner product is in $L^2(\overline{D}; m)$, and the other two are in $L^2(D_{n_{j(k)}}; m_{n_{j(k)}})$. Moreover, to establish the second inequality of (3.23) we apply the local uniform convergence of $P_t^{n_{j(k)}} f$ to $P_t f$ (cf. the proof of Proposition 2.8) and the vague convergence of $m_{n_{j(k)}}$ to m (Lemma 3.14). We now prove the second inequality. Suppose $\operatorname{Supp} f \subset B_E(0,M) \cap \overline{D}$ for some $M > 0$, then

$$\mathcal{E}^{(n)}(f,f) = \frac{n^{2-d}}{2} \sum_{(x,y)\in H_{n,M}} (f(x/n) - f(y/n))^2 \mu_{x,y},$$

where $H_{n,M} := \{(x,y) : x, y \in B_E(0,nM) \cap \mathcal{C}_1, \{x,y\} \in \mathcal{O}_1\}$. Clearly, this quantity increases when $H_{n,M}$ is replaced by $H'_{n,M} := \{(x,y) : x \in B_E(0,nM) \cap \mathbb{L}, \{x,y\} \in E_{\mathbb{Z}^d}\}$. Note that $\#H'_{n,M} \sim c_1 (nM)^d$. Moreover, set $a^n_{(x,y)} := n^2(f(x/n) - f(y/n))^2$ and $\eta_{(x,y)} := \mu_{x,y}$. Then, since $f \in C_c^2$, $0 \le a_{(x,y)} \le M'$ for some $M' > 0$, and further,

$$\lim_{n\to\infty} (2c_1(nM)^d)^{-1} \sum_{(x,y)\in H'_{n,M}} a^n_{(x,y)} = c_1^{-1} M^{-d} \int_{\overline{D}} |\nabla f(x)|^2 dx,$$

by applying Lemma 3.15 we obtain that, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} n^{-d} \sum_{(x,y)\in H'_{n,M}} a^n_{(x,y)} \eta_{(x,y)} = 2C_0 \int_{\overline{D}} |\nabla f(x)|^2 dx.$$

The result at (3.23) follows. $\square$

Putting together the above results, we conclude the following.

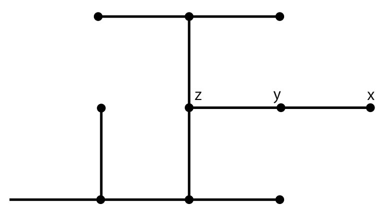

Figure 2: An example trap structure.

Theorem 3.17. *For the random conductance model on $\mathbb{L}$ with independent and identically distributed conductances $(\mu_e)_{e\in E_{\mathbb{L}}}$ satisfying (3.1), (3.2) and (3.3), there exists a deterministic constant $c \in (0,\infty)$ such that, for $\mathbb{P}$-a.e ω, the laws of the processes Y^n under $P^\omega_{nx_n}$, where $nx_n \in \mathcal{C}_1$ and $x_n \to x$, converge weakly to the laws of $\{X_{ct}; t \geq 0\}$, where $\{X_t; t \geq 0\}$ is the reflecting Brownian motion on $\overline{D}$ started from x.*

Remark 3.18. (i) The diffusion constant c is the same for the model restricted to $\mathbb{L}$ as for the full $\mathbb{Z}^d$ model.
(ii) When the conductance is not bounded from below, we cannot apply our theorem because Assumption 2.3(i) does not hold in general, and we do not know how to obtain the quenched invariance principle without this. Indeed, consider the realization of edge weights shown in Figure 2, where the conductance on $\{x,y\}$ is 1 and it is $O(n^{-\alpha})$ on $\{y,z\}$ where $\alpha > 2$. One can easily compute that $E^\omega_x \tau_{B_E(x,2)}(Y) \geq c_1 n^\alpha \gg n^2$. Let $p_0 = \mathbb{P}(\mu_e = 0)$ and $p_1 = \mathbb{P}(\mu_e = 1)$. Then the probability that such a trap configuration appears is $p_0^{4d-3} p_1 \mathbb{P}(0 < \mu_e \leq n^{-\alpha}) = c_2 \mathbb{P}(0 < \mu_e \leq n^{-\alpha})$. Now let $\Omega_n := \{\exists x_n \in B_E(0, n/2) \text{ such that } E^\omega_{x_n} \tau_{B_E(x_n,2)}(Y) \geq c_3 n^\alpha\}$. If we have $\mathbb{P}(0 < \mu_e \leq x) \geq c_4 x^{d/\alpha}$ for small $x > 0$, then $\mathbb{P}(\Omega_n) \geq 1 - (1 - c_4 n^{-d})^{n^d} \geq 1 - e^{-c_5}$ for large n. In particular, $\limsup_n \Omega_n$ occurs with positive probability. Set $X^n_t := n^{-1} Y_{n^2 t}$. Then, for $\omega \in \Omega_n$, we have

$$E^\omega_{n^{-1}x_n}\left(\tau_{B_E(0,1)\cap D_n}(X^n)\right) = E^\omega_{x_n}\left(\tau_{B_E(0,n)\cap D_0}(Y_{n^2\cdot})\right) \geq E^\omega_{x_n}\left(\tau_{B_E(x_n,2)}(Y_{n^2\cdot})\right) \geq c_3 n^{\alpha-2}.$$

Since Ω_n occurs infinitely often with positive probability, Assumption 2.3(i) does not hold for any choice of $\beta > 0$ (by choosing $x_0 = 0, r = 1$).
(iii) There is another natural continuous time Markov chain on $\mathcal{C}_1$, namely with transition probability $P(x,y) = \mu_{xy}/\mu_x$ and the holding time being the exponential distribution with mean one for each point. (Such a Markov chain is sometimes called a constant speed random walk (CSRW).) It is a time change of the VSRW; the corresponding Dirichlet form is $(\mathcal{E}, L^2(\mathcal{C}_1;\mu))$, and the corresponding discrete Laplace operator is $\mathcal{L}_C f(x) = \frac{1}{\mu_x}\sum_y (f(y) - f(x))\mu_{xy}$. For the whole space case, one can deduce the quenched invariance principle of CSRW from that of VSRW by an ergodic theorem. (See [3, Section 6.2] and [8, Section 5]. Note that the limiting process degenerates if $\mathbb{E}\mu_e = \infty$.) Since our state space $\mathbb{L}$ features a lack of translation invariance, we cannot use the ergodic theorem. So far, we do not know how to circumvent this issue to prove the quenched invariance principle for general CSRW on $\mathbb{L}$. (However, we do note that for the case of random walk on a supercritical

percolation cluster, the CSRW and VSRW behave similarly, and the quenched invariance principle for the CSRW can be proved in a similar way as for the VSRW case. Moreover, the quenched invariance principle for the discrete time simple random walk on $\mathcal{C}_1$ follows easily from that for the CSRW.)

(iv) To extend Theorem 3.17 to apply to more general domains, it will be enough to check the percolation estimates from which we deduced Assumptions 2.3 and 2.6 in these settings. Whilst we believe doing so should be possible, at least under certain smoothness assumptions on the domain boundary, we do not feel the article would benefit significantly by the increased technical complication of pursuing such results, and consequently omit to do so here. Instead we restrict our discussion of more general domains to the case of uniformly elliptic conductances, where the relevant estimates are straightforward to check (see Section 4.2 below). Similarly, given suitable full space quenched invariance principles and percolation estimates (namely the estimates given in Lemmas 3.1 – 3.3), our results should readily adapt to percolation models on other lattices.

(v) Given the various estimates we have established so far, it is possible to extend the quenched invariance principle of Theorem 3.17 to a local limit theorem, i.e. a result describing the uniform convergence of transition densities. More specifically, the additional ingredient needed for this is an equicontinuity result for the rescaled transition densities on $\mathcal{C}_1$, which can be obtained by applying an argument similar to that used to deduce Assumption 2.3(ii), together with the heat kernel upper bound estimate of Proposition 3.11. Since the proof of such a result is relatively standard (cf. [9, 25]), we will only write out the details in the compact box case (see Section 4.1 below), where convergence of transition densities is also useful for establishing convergence of mixing times. $\square$

4 Other Examples

4.1 Random conductance model in a box

The purpose of this section is to explain how to adapt the results of Section 3 to the compact space case. For $d \geq 2$ fixed, set $\mathbb{B}(n) := [-n, n]^d \cap \mathbb{Z}^d$, let $E_{\mathbb{B}(n)} = \{e = \{x, y\} : x, y \in \mathbb{B}(n), |x - y| = 1\}$ be the set of nearest neighbor bonds, and suppose $\mu = (\mu_e)_{e \in E_{\mathbb{Z}^d}}$ is a collection of independent random variables satisfying the assumptions made on the weights in Section 3, i.e. (3.1), (3.2) and (3.3). For each n and each realization of μ, let $\mathcal{C}_1(n)$ be the largest component of $\mathbb{B}(n)$ that is connected by edges satisfying $\mu_e > 0$, and let Y^n be the VSRW on $\mathcal{C}_1(n)$. We will write $P^\omega_{n,x}$ for the quenched law of Y^n started from $x \in \mathcal{C}_1(n)$. The aim of this section is to show, via another application of Theorem 2.1, that $X^n = (X^n_t)_{t \geq 0}$, defined by setting

$$X^n_t := n^{-1} Y^n_{n^2 t},$$

converges as $n \to \infty$ to reflecting Brownian motion on $D = [-1, 1]^d$, for almost-every realization of the random environment μ. We observe that, in the case of uniformly elliptic random conductances, this result was recently established using an alternative argument in [18]. Note also that, by

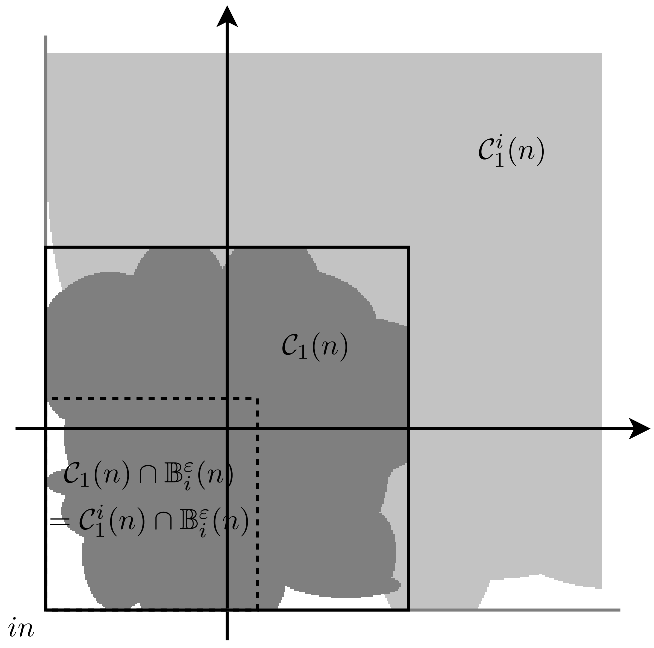

Figure 3: Schematic illustration of how, for large n, the largest cluster $\mathcal{C}_1(n)$ (dark-grey) in $\mathbb{B}(n)$ (square with solid boundary) and the largest cluster $\mathcal{C}_1^i(n)$ (light-grey) in the quarter-plane with corner in and containing 0 agree within $\mathbb{B}_i^\varepsilon(n)$ (square with dashed boundary).

applying a result from [26], the above functional scaling limit readily yields the corresponding convergence of mixing times (see Corollary 4.4 below for a precise statement).

To prove the results described in the previous paragraph, we start by considering a decomposition of $\mathbb{B}(n)$. In particular, fix $\varepsilon \in (0,1)$ and for $i = (i_1, \dots, i_d) \in \{-1,1\}^d$, let $\mathbb{B}_i^\varepsilon(n)$ be the cube of side-length $\lfloor n(1+\varepsilon) \rfloor$ which has a corner at ni and contains 0. Within each of the 2^d sets of the form $\mathbb{B}_i^\varepsilon(n)$, the random walk on $\mathcal{C}_1(n)$ reflects only at the faces of the box adjacent to the single corner vertex ni. As a consequence, we will be able to transfer a number of key estimates to the current framework from the unbounded case considered in Section 3 – note that the reason for taking $\varepsilon > 0$ is so that the boxes $\mathbb{B}_i^\varepsilon(n)$ overlap, which will allow us to 'patch' together results proved for different parts of the box. For the purpose of transferring results from Section 3, the following lemma will be useful. Its proof can be found in the appendix.

Lemma 4.1. *There exist constants c_1, c_2 such that if $Q_1, Q_2 \subseteq \mathbb{Z}_+^d$ are the cubes of side length $\lfloor n(1+\varepsilon) \rfloor$, $2n$ containing 0, respectively, $\mathcal{C}^+(Q_2)$ is the largest connected component of the graph $(Q_2, \mathcal{O}_1)$, and $\mathcal{C}_1$ is the unique infinite component of $(\mathbb{Z}_+^d, \mathcal{O}_1)$, then*

$$\mathbb{P}\left(\mathcal{C}^+(Q_2) \cap Q_1 \neq \mathcal{C}_1 \cap Q_1\right) \leq c_1 e^{-c_2 n}.$$

In particular, if $\mathcal{C}_1^i(n)$ is the unique infinite percolation cluster on the copy of $\mathbb{Z}_+^d$ that has corner vertex ni and contains 0, then the above result implies that with probability at least $1 - c_1 e^{-c_2 n}$ (or, by Borel-Cantelli, almost-surely for large n) we have that

$$\mathcal{C}_1(n) \cap \mathbb{B}_i^\varepsilon(n) = \mathcal{C}_1^i(n) \cap \mathbb{B}_i^\varepsilon(n) \subseteq \tilde{\mathcal{C}}_1, \qquad \forall i \in \{-1,1\}^d, \tag{4.1}$$

where the inclusion is a consequence of the uniqueness of the infinite percolation clusters in question. (This result is summarized by Figure 3.)

We now check the conditions listed at the end of Section 2 one by one. Since in light of (4.1), most of these are straightforward adaptations of the arguments given in Section 3, we will only provide a brief description of how to do this. Firstly, as was the case in the $\mathbb{L}$ setting, since X^n is a continuous time Markov chain with holding time at x being $\exp(\mu_x)$, it is conservative. Secondly, given (4.1), condition (ii) in Theorem 2.1 is a consequence of the quenched invariance principle for the whole space stated as Theorem 3.13. Moreover, since $\mathcal{C}_1(n)$ agrees with $\tilde{\mathcal{C}}_1$ up to a distance $c \log n$ of the boundary, at least for large n (see [13, Proposition 1.2]), by applying the full $\mathbb{Z}^d$ version of Proposition 3.14, we have that the measures m_n, defined analogously to the previous section, $\mathbb{P}$-a.s. converge weakly to (a suitably rescaled version of) Lebesgue measure on $[-1,1]^d$. Similarly, the Dirichlet form comparison of (iii) can be obtained by following the same argument used to prove the corresponding result in Section 3 – Proposition 3.16. Applying (4.1), we are also able to deduce the following tightness result, which is analogous to Proposition 3.10, and from which Assumption 2.6 is readily obtained.

Proposition 4.2. *For $\mathbb{P}$-a.e. ω, if $x_n \in n^{-1}\mathcal{C}_1(n)$, $n \geq 1$ is such that $x_n \to x \in \overline{D}$, then under P^ω_{n,nx_n}, the family of processes $(X^n_t)_{t\geq 0}, n \in \mathbb{N}$ is tight in $\mathbb{D}([0,\infty),[0,1]^d)$, and any convergent subsequence has limit in $C([0,\infty),[0,1]^d)$.*

Proof. Note that in the bounded case the limit corresponding to (3.15) is immediate, and hence it will suffice to check the limit corresponding to (3.16). To do this, the same argument can be applied, so long as one can check the following: for any $r \in (0,\varepsilon)$, there exist c_i and random variables $(R^n_x, x \in \mathbb{B}(n), n \geq 1)$ with

$$\mathbb{P}(R^n_x \geq rn, x \in \mathcal{C}_1(n)) \leq c_1 e^{-c_2 n^\delta},$$

such that if $x \in \mathcal{C}_1(n)$, $t > 0$ and $R \geq R^n_x$, then

$$P^\omega_{n,x}\left(\tau^{Y^n}_{B_E(x,R)} < t\right) \leq c_3 \Psi(c_4 R, t). \tag{4.2}$$

For this purpose, if $x \in \mathbb{B}^0_i(n)$, set $\tilde{R}^n_x$ to be equal to $R^{n,i}_x$, the quantity defined in Proposition 3.8 with $\mathcal{C}_1$ replaced by $\mathcal{C}^i_1(n)$. If $\tilde{R}^n_x \leq \varepsilon n$ and the part of $\mathcal{C}_1(n)$ contained in $B_E(x,\varepsilon n)$ is identical to the part of $\mathcal{C}^i_1(n)$ contained in this set, then set $R^n_x = \tilde{R}^n_x$. Else, set $R^n_x = 3n$. The required exponential decay for the distributional tail of R^n_x then follows from Proposition 3.8 and (4.1). Moreover, since the probability on the left-hand side of (4.2) is 0 for $R \geq 3n$, the bound at (4.2) follows. □

It remains to check Assumption 2.3. For part (i), we simply note that the combination of (3.19) (or more precisely, the exponential tail bound for N_0 that appears as (3.21)) and (4.1) in a standard Borel-Cantelli argument implies the following: there exist $c_*, c_1 > 0$ (non-random) and $N_0(\omega)$ such that if $n \geq N_0(\omega)$, then, for each $x_0, x \in n^{-1}\mathcal{C}_1(n)$ and $c_*/n^{1/2} \leq r' \leq 1$,

$$E^{\omega}_{n,x}\left(\tau_{B_E(x_0,r')}(X^n)\right) \leq c_1 r'^2,$$

as desired. For part (ii) of this assumption, first note that we can obtain the elliptic Harnack inequality uniformly for X^n-harmonic functions on $B_E(x_0, R)$, where $x_0 \in n^{-1}\mathcal{C}_1(n)$ and $(\log n)^{2/\delta}/n \leq R \leq 1$ for large n. (This can be proved similarly as before, namely when $x_0 \in \mathbb{B}^0_i(n)$, Theorem 3.12 can be applied by replacing $\mathcal{C}_1$ by $\mathcal{C}^i_1(n)$ due to (4.1).) Given the elliptic Harnack inequality, we can obtain Hölder continuity in a similar way as in the proof of [9, Proposition 3.2], for example. Hence we have established the following.

Theorem 4.3. *There exists a constant $c \in (0,\infty)$ such that, for $\mathbb{P}$-a.e. ω, the process X^n under P^{ω}_{n,nx_n}, where $nx_n \in \mathcal{C}_1$ and $x_n \to x \in [-1,1]^d$, converges in distribution to $(B_{ct})_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is the reflecting Brownian motion on $[-1,1]^d$ started from x.*

Next, for $p \in [1,\infty]$, define the L^p mixing time of the VSRW on $\mathcal{C}_1(n)$ to be

$$t^p_{\mathrm{mix}}(\mathcal{C}_1(n)) := \inf\left\{t > 0 : \sup_{x\in\mathcal{C}_1(n)} \left(\int_{\mathcal{C}_1(n)} |q^n_t(x,y) - 1|^p \,\pi^n(dy)\right)^{1/p} < \frac{1}{4}\right\}, \tag{4.3}$$

where we denote by q^n the transition density of the VSRW with respect to its (unique) invariant probability measure π^n. The above result then has the following corollary. Note that in the percolation setting, the obvious adaptation of this result to discrete time gives a refinement of the first statement of [13, Theorem 1.1].

Corollary 4.4. *Fix $p \in [1,\infty]$. For $\mathbb{P}$-a.e. ω, we have that*

$$n^{-2} t^p_{\mathrm{mix}}(\mathcal{C}_1(n)) \to c^{-1} t^p_{\mathrm{mix}}([-1,1]^d),$$

where c is the constant of Theorem 4.3, and $t^p_{\mathrm{mix}}([-1,1]^d)$ is the mixing time of reflecting Brownian motion on $[-1,1]^d$ (defined analogously to (4.3)).

Proof. First note that a simple rescaling yields that, $\mathbb{P}$-a.s., π^n converges weakly to a rescaled version of Lebesgue measure on $[-1,1]^d$. The $\mathbb{P}$-a.s. Hausdorff convergence of $n^{-1}\mathcal{C}_1(n)$ (equipped with Euclidean distance) to $[-1,1]^d$ is a straightforward consequence of this. To establish the corollary by applying [26, Theorem 1.4], it will thus be enough to extend the weak convergence result of Theorem 4.3 to a uniform convergence of transition densities (so as to satisfy [26, Assumption 1]). According to [26, Proposition 2.4] (cf. [25, Theorem 15]) and the quenched invariance principle mentioned above, it is enough to show [26, (2.11)], namely, for any $0 < a < b < \infty$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \sup_{\substack{x,y,z\in n^{-1}\mathcal{C}_1(n):\\ d_E(ny,nz)\leq n\delta}} \sup_{t\in[a,b]} \left|q^n_{n^2t}(nx,ny) - q^n_{n^2t}(nx,nz)\right| = 0. \tag{4.4}$$

To prove this, first we have the following Hölder continuity, which can be checked similarly to Assumption 2.3(ii):

$$\left|q^n_{n^2t}(nx, ny) - q^n_{n^2t}(nx, nz)\right| \le c_1|y-z|^\gamma \|q^n_{n^2t}(nx,\cdot)\|_\infty, \qquad \forall x, y, z \in n^{-1}\mathcal{C}_1(n). \tag{4.5}$$

For $0 < a \le t < 1$, say, a compact version of Proposition 3.11 and scaling gives that

$$\|q^n_{n^2t}(nx,\cdot)\|_\infty \le c_3|\mathcal{C}_1(n)|(n^2t)^{-d/2} \le c_3 a^{-d/2}$$

for large n. For $t \ge 1$, Cauchy-Schwarz and monotonicity of $q^n_{n^2t}(nx, nx)$ implies $\|q^n_{n^2t}(nx,\cdot)\|_\infty \le c_4$. In particular,

$$\|q^n_{n^2t}(nx,\cdot)\|_\infty \le c_2(a) \tag{4.6}$$

uniformly in $x \in D_n$, $t \ge a$, for large n, $\mathbb{P}$-a.s. Thus, for $t \in [a, b]$, the right-hand side of (4.5) is bounded from above by $c_3(a)|y-z|^\gamma$. Taking $n \to \infty$ and then $\delta \to 0$, we obtain (4.4). $\square$

Finally, as a corollary of the heat kernel continuity derived in the proof of the previous result, we obtain the following local central limit theorem. We let $g_n : [-1,1]^d \to \mathcal{C}_1(n)$ be such that $g_n(x)$ is a closest point in $\mathcal{C}_1(n)$ to nx in the $|\cdot|_\infty$-norm. (If there is more than one such point, we choose one arbitrarily.)

Corollary 4.5. *Let $q_t(\cdot,\cdot)$ be the heat kernel of the reflecting Brownian motion on $[-1,1]^d$. For $\mathbb{P}$-a.e. ω and for any $0 < a < b < \infty$, we have that*

$$\lim_{n\to\infty} \sup_{x,y\in[-1,1]^d} \sup_{t\in[a,b]} |q^n_{nt}(g_n(x), g_n(y)) - q_{ct}(x,y)| = 0, \tag{4.7}$$

where c is the constant of Theorem 4.3.

Proof. Given the above results, the proof is standard. By (4.5) and (4.6) and the Ascoli-Arzelà theorem (along with the Hausdorff convergence of $n^{-1}\mathcal{C}_1(n)$ to $[-1,1]^d$), there exists a subsequence of $q^n_{nt}(g_n(\cdot), g_n(\cdot))$ that converges uniformly to a jointly continuous function on $[a,b]\times[-1,1]^d\times[-1,1]^d$. Using Theorem 4.3, it can be checked that this function is the heat kernel of the limiting process. Since the limiting process is unique, we have the convergence of the full sequence of $q^n_{nt}(g_n(\cdot), g_n(\cdot))$. The uniform convergence in (4.7) is then another consequence of (4.5) and (4.6). $\square$

4.2 Uniformly elliptic random conductances in uniform domains

When the conductances are uniform elliptic, i.e. bounded from above and below by fixed positive constants, we can obtain quenched invariance principles for a much wider class of domains than those considered in the examples presented so far. In particular, let D be a uniform domain in $\mathbb{R}^d$, $d \ge 2$. For each $n \ge 1$, let $\hat{D}_n$ be the largest connected component of of $nD \cap \mathbb{Z}^d$, and set

$E_{\hat{D}_n} = \{e = \{x, y\} : x, y \in \hat{D}_n, |x - y| = 1\}$. Suppose $\mu = (\mu_e)_{e \in E_{\mathbb{Z}^d}}$ is a collection of independent random variables such that

$$\mathbb{P}(C_1 \leq \mu_e \leq C_2) = 1, \qquad \forall e \in E_{\mathbb{Z}^d},$$

where C_1, C_2 are non-random positive constants. Let Y^n be either VSRW or CSRW on $\hat{D}_n$. Moreover, set $D_n := n^{-1}\hat{D}_n$ and define $X^n_t := n^{-1}Y^n_{n^2 t}$. It is then the case that $X^n = (X^n_t)_{t \geq 0}$ converges as $n \to \infty$ to a (constant time change of) reflecting Brownian motion on D, for $\mathbb{P}$-almost-every realization of the random environment μ.

Since checking the details for this case is much simpler than for previous settings, we will not provide a full proof of the result described in the previous paragraph, but merely indicate how to establish the key estimates. Indeed, in the uniformly elliptic case, we have

$$c_1 R^d \leq n^{-d} |B_E(x, R) \cap D_n| \leq c_2 R^d, \qquad c_3 R^d \leq n^{-d} \mu(B_E(x, nR) \cap \hat{D}_n) \leq c_4 R^d$$

for all large n and all $n^{-1} \leq R \leq \operatorname{diam}(D)$, $x \in D$, where c_i are non-random positive constants. Furthermore, the weak Poincaré inequality (3.10) holds both for the counting measure and $n^{-d}\mu$ uniformly for $n^{-1} \leq R \leq \operatorname{diam}(D)$, in the sense that the constants do not depend on n. Given these, we can apply [28] (and the natural relations between heat kernels of discrete and continuous time Markov chains) to deduce

$$c_5 t^{-d/2} \exp(-c_6 |x - y|^2 / t) \leq q^n_t(x, y) \leq c_7 t^{-d/2} \exp(-c_8 |x - y|^2 / t), \quad |x - y| \leq t \leq \operatorname{diam}(D),$$

where $q^n_t(x, y)$ is defined as $n^{-d} P^\omega_x(X^n_t = y)$ for the VSRW and $n^{-d}\mu(ny)^{-1} P^\omega_x(X^n_t = y)$ for the CSRW. Given these heat kernel estimates, it is then straightforward to verify the conditions required for the quenched invariance principle by applying similar arguments to those of Sections 3 and 4.1.

4.3 Uniform elliptic random divergence form in domains

In this section, we explain how Theorem 2.1 can be applied in the random divergence form setting. Let D be a uniform domain in $\mathbb{R}^d$, $d \geq 2$. Assume that we have a random divergence form as follows. There exists a probability space $(\Omega, \mathbb{P})$ with shift operators $(\tau_x)_{x \in \mathbb{R}^d}$ that are ergodic, and a symmetric $d \times d$ matrix $A^\omega(x)$ for each $x \in \mathbb{R}^d$ and $\omega \in \Omega$ such that $A^\omega(x) = A^{\tau_x \omega}(0)$ and

$$\mathbb{P}(c_1 I \leq A^\omega(x) \leq c_2 I) = 1, \qquad \forall x \in \mathbb{R}^d,$$

where $c_1, c_2 \in (0, \infty)$ are deterministic constants. For $n \geq 1$, let

$$\mathcal{E}^n(f, f) = \frac{1}{2} \int_{nD} \nabla f(x) A^\omega(x) \nabla f(x) dx.$$

Let $(Y^n_t)_{t \geq 0}$ be the reflected diffusion process on $n\overline{D}$ associated with the regular Dirichlet form $(\mathcal{E}^n, W^{1,2}(nD))$ on $L^2(nD; dx)$, and set $X^n_t := n^{-1} Y^n_{n^2 t}$. (A natural setting would be to take D to be a cone. In this case $nD = D$, so the random diffusion matrix $A^\omega(x)$ only needs to be defined for $x \in D$ rather than for $x \in \mathbb{R}^d$.) Observe that process X^n takes value in $\overline{D}$. It is then the case that $X^n = (X^n_t)_{t\geq 0}$ converges as $n \to \infty$ to a reflecting Brownian motion on D with some strictly positive covariance matrix B, for $\mathbb{P}$-almost-every realization of the random environment ω. (Note that B is determined by the invariance principle on the whole space $\mathbb{R}^d$.) Indeed, the Dirichlet form of X^n on $L^2(D; dx)$ is

$$n^{2-d}\mathcal{E}^n(f_n, f_n) = \frac{1}{2}\int_D \nabla f(x)A(nx)\nabla f(x)dx,$$

where $f_n(x) := f(n^{-1}x)$. In view of Section 2.1, the transition density function of X^n has estimates (2.4) with constants $c_1, \dots, c_4$ independent of n. In this case, the quenched invariance principle on $\mathbb{R}^d$ is proved in [48]. (To be precise, in the paper the author assumed C^2 smoothness for the coefficients. However, this was to apply the Itô formula, and could be avoided by using the Fukushima decomposition instead.) Given these, one can easily verify the conditions required for the quenched invariance principle in D. (Note that because of the uniform ellipticity, Condition (iii) in Theorem 2.1 is trivial in this case. Moreover, one can extend the quenched invariance principle of [48] to arbitrary starting points by applying the argument of Theorem 3.13.) Thus for $\mathbb{P}$-almost-every realization of the random environment ω and for every starting point $x \in \overline{D}$, the reflecting diffusion X^n converges weakly to a reflecting Brownian motion on D. This gives an affirmative answer to the open problem of [51, pp. 1004–1005].

As we mentioned briefly in the introduction, homogenization of reflected SDE/PDE on half-planes has been studied for periodic coefficients in [5, 14, 53] etc., and for random divergence forms in [51]. (Note that their equations contain additional reflection terms, though the precise framework varies in each paper.) Homogenization for random divergence forms without extra reflection terms on bounded C^2 domains is discussed in [41, Section 14.4]. Although we can only handle symmetric cases, our results hold for general uniform domains.

A Appendix

A.1 Proofs for percolation estimates

The aim of this section is to verify the percolation estimates stated as Lemmas 3.1-3.3, 3.7 and 4.1.

For the purpose of proving Lemma 3.1, it will be useful to note that for large K the collection of edges $\tilde{\mathcal{O}}_2$ stochastically dominates $\tilde{\mathcal{O}}_3$, the edges of a bond percolation process on $E_{\mathbb{Z}^d}$ with probability $p_3 = p_3(K)$, where the parameter p_3 can be chosen to satisfy $\lim_{K\to\infty} p_3 = p_1$ (see [3, Proposition 2.2]). In fact, the proof of this result from [3] further shows that, for a given value of K (that is suitably large), it is possible to couple all the relevant random variables in such a way that $\tilde{\mathcal{O}}_3 \subseteq \tilde{\mathcal{O}}_2$ almost-surely. We will henceforth assume that this is the case, where K is fixed large enough to ensure that $p_3 > p_c^{\mathrm{bond}}(\mathbb{Z}^d)$. We will also define $\mathcal{O}_3 := \tilde{\mathcal{O}}_3 \cap E_{\mathbb{L}}$ and $\mathcal{C}_3 := \mathcal{C}_\infty(\mathbb{L}, \mathcal{O}_3)$. Note that $\mathcal{C}_3$ is non-empty by the uniqueness of infinite supercritical bond percolation clusters on $\mathbb{L}$.

Proof of Lemma 3.1. First observe that $\mathcal{O}_3 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_1$. It follows that there exists an infinite connected component $\mathcal{C}$ of $(\mathbb{L}, \mathcal{O}_2)$ such that $\mathcal{C}_3 \subseteq \mathcal{C} \subseteq \mathcal{C}_1$. For such a $\mathcal{C}$ (at the moment, we do not know its uniqueness), we have that $\mathcal{C}_1 \backslash \mathcal{C} \subseteq \mathcal{H}_3 \subseteq \mathbb{L} \backslash \mathcal{C}_3$. Denote by $\mathcal{G}(x)$ the connected component of $\mathbb{L} \backslash \mathcal{C}_3$ containing x (if $x \in \mathcal{C}_3$, we set $\mathcal{G}(x) = \emptyset$). To prove part (i) of the lemma, it suffice to show that there exist constants c_1, c_2 such that: for each $x \in \mathbb{L}$,

$$\mathbb{P}\left(\operatorname{diam}(\mathcal{G}(x)) \geq n\right) \leq c_1 e^{-c_2 n}. \tag{A.1}$$

For this, we will follow the renormalization argument used in the proof of [17, Proposition 2.3], making the adaptations necessary to deal with the boundary issues that arise in our setting.

We start by coupling a finite range-dependent site percolation model with our bond percolation process. For $L \in \mathbb{N}$, $x \in \mathbb{Z}^d$, define

$$Q_L(x) := L(x + e_1 + \cdots + e_{d_1}) + [0, L]^d \cap \mathbb{Z}^d, \ \tilde{Q}_{3L}(x) := L(x + e_1 + \cdots + e_{d_1}) + [-L, 2L]^d \cap \mathbb{Z}^d,$$

where $e_1, \dots, e_d$ are the standard basis vectors for $\mathbb{Z}^d$. Given these sets, let $G_L(x)$ be the event such that

- there exists an $\tilde{\mathcal{O}}_3$-crossing cluster for $Q_L(x)$ in $\tilde{Q}_{3L}(x)$, i.e. there is a $\tilde{\mathcal{O}}_3$-connected cluster in $\tilde{Q}_{3L}(x)$ that, for all d directions, joins the 'left face' to the 'right face' of $Q_L(x)$,
- all paths along edges of $\tilde{\mathcal{O}}_3$ that are contained in $\tilde{Q}_{3L}(x)$ and have diameter greater than L are connected to the (necessarily unique) crossing cluster.

We will say that $x \in \mathbb{Z}^d$ is 'white' if $G_L(x)$ holds and 'black' otherwise, and make the important observation that if two neighboring vertices are white, then their crossing clusters must be connected. Since $p_3 > p_c^{\mathrm{bond}}(\mathbb{Z}^d)$, we can apply (the bond percolation version of) [49, Theorem 5] for $d = 2$ and [50, Theorem 3.1] for $d \geq 3$ to deduce that

$$\lim_{L \to \infty} \mathbb{P}\left(G_L(x)\right) = 1, \tag{A.2}$$

(cf. [4, (2.24)]). Moreover, although $(\mathbf{1}_{G_L(x)})_{x \in \mathbb{Z}^d}$ are not independent, the dependence between these random variables is of finite range. Thus, by [44, Theorem 0.0], one can suppose that, for suitably large L, the collection $(\mathbf{1}_{G_L(x)})_{x \in \mathbb{Z}^d}$ dominates a site percolation process on $\mathbb{Z}^d$ of density arbitrarily close to 1. Noting that for any infinite connected graph G with maximal vertex degree Δ the critical site and bond percolation probabilities satisfy

$$p_c^{\mathrm{site}}(G) \leq 1 - (1 - p_c^{\mathrm{bond}}(G))^{\Delta - 1},$$

([34, Theorem 3]), we have that $p_c^{\mathrm{site}}(\mathbb{L})$ is bounded above by $1 - (1 - p_c^{\mathrm{bond}}(\mathbb{Z}^d))^{2d-1} < 1$. Hence, by taking L suitably large, it is possible to assume that there is a non-zero probability of 0 being contained in an infinite cluster of white vertices in $\mathbb{L}$. From this result, a standard ergodicity argument with respect to the shift $x \mapsto x + e_1 + \cdots + e_{d_1}$ allows one to check that, $\mathbb{P}$-a.s., there exists at least one infinite connected cluster of white sites in $\mathbb{L}$. In particular, writing $\mathcal{D}(x)$ for the connected cluster of white sites containing a particular vertex $x \in \mathbb{L}$ (taking $\mathcal{D}(x) := \emptyset$ if $G_L(x)$ does not occur), we obtain that the set

$$\mathcal{D}_\infty := \{x \in \mathbb{L} : \mathcal{D}(x) \text{ is infinite}\} \tag{A.3}$$

is non-empty, $\mathbb{P}$-a.s.

Let $\mathcal{C}(x)$ be the connected component of $\mathbb{L}\backslash\mathcal{D}_\infty$ containing x (we set this to be the empty set if $x \in \mathcal{D}_\infty$). The next step of the proof is to check that: for $x \in \mathbb{L}$,

$$\mathbb{P}\left(\operatorname{diam}(\mathcal{C}(x)) \geq n\right) \leq c_3 e^{-c_4 n}. \tag{A.4}$$

To do this, we start by introducing some notions of set boundaries that will be useful. For $x \notin \mathcal{D}_\infty$, the inner boundary of $\mathcal{C}(x)$ is the set

$$\partial^{\mathrm{in}}\mathcal{C}(x) := \{y \in \mathcal{C}(x) : y \text{ is adjacent to a vertex in } \mathbb{L}\backslash\mathcal{C}(x)\}.$$

It is simple to check from its construction that all the vertices in this set are black. Since $\mathbb{L}\backslash\mathcal{C}(x) \supseteq \mathcal{D}_\infty \neq \emptyset$, then $\mathbb{L}\backslash\mathcal{C}(x)$ contains at least one infinite connected component, $\mathcal{D}$ say. The outer boundary of $\mathcal{D}$ is given by

$$\partial^{\mathrm{out}}\mathcal{D} := \{y \in \mathbb{L}\backslash\mathcal{D} : y \text{ is adjacent to a vertex in } \mathcal{D}\}. \tag{A.5}$$

With $\mathcal{D}$ being disjoint from $\mathcal{C}(x)$, we can also define the part of its outer boundary visible from $\mathcal{C}(x)$ by setting

$$\partial^{\mathrm{vis}\mathcal{C}(x)}\mathcal{D} := \left\{y \in \partial^{\mathrm{out}}\mathcal{D} : \text{there exists a path from } y \text{ to } \mathcal{C}(x) \text{ in } \mathbb{L} \text{ that is disjoint from } \mathcal{D}\right\}. \tag{A.6}$$

We claim the following relationship between the various boundary sets:

$$\partial^{\mathrm{vis}\mathcal{C}(x)}\mathcal{D} = \partial^{\mathrm{out}}\mathcal{D} \subseteq \partial^{\mathrm{in}}\mathcal{C}(x). \tag{A.7}$$

To verify the equality, first suppose that there exists a vertex $y \in \partial^{\mathrm{out}}\mathcal{D}\backslash\mathcal{C}(x)$, then $y \in \mathbb{L}\backslash\mathcal{C}(x)$ and we can find a vertex $z \in \mathcal{D} \subseteq \mathbb{L}\backslash\mathcal{C}(x)$ such that y and z are adjacent. This implies that y and z are in the same connected component of $\mathbb{L}\backslash\mathcal{C}(x)$, which is a contradiction because $y \notin \mathcal{D}$ by definition. Hence $\partial^{\mathrm{out}}\mathcal{D} \subseteq \mathcal{C}(x)$, and so (noting that $\mathcal{C}(x)\backslash\mathcal{D} = \mathcal{C}(x)$)

$$\partial^{\mathrm{out}}\mathcal{D} = \{y \in \mathcal{C}(x) : y \text{ is adjacent to a vertex in } \mathcal{D}\} \subseteq \partial^{\mathrm{vis}\mathcal{C}(x)}\mathcal{D}.$$

Since the opposite inclusion is trivial, we obtain the equality at (A.7). From $\partial^{\mathrm{out}}\mathcal{D} \subseteq \mathcal{C}(x)$, the inclusion at (A.7) is also clear.

We proceed by applying the conclusion of the previous paragraph to show that $\mathcal{D} = \mathbb{L}\backslash\mathcal{C}(x)$. First, the boundary connectivity result of [54, Lemma 2] implies that $\partial^{\mathrm{vis}\mathcal{C}(x)}\mathcal{D}$ is $*$-connected. Combining this with (A.7), we obtain that $\partial^{\mathrm{out}}\mathcal{D}$ is a $*$-connected set of black vertices (recall that the vertices of $\partial^{\text{in}}\mathcal{C}(x)$ are black). Secondly, note that if $\mathbb{P}_p$ is the law of a parameter p site percolation process on $\mathbb{Z}_d$ and $\mathcal{C}^*$ is the corresponding $*$-connected component of closed vertices containing 0, then for suitably large p we have that

$$\mathbb{P}_p\left(\mathcal{C}^* \geq n\right) \leq c_5 e^{-c_6 n}, \tag{A.8}$$

(see [1, Theorem 7.3] and [2, Proposition 7.6]). In particular, it is easy to check from this that all $*$-connected components of closed vertices in the site percolation process with this choice of p are finite, $\mathbb{P}_p$-a.s. Hence, because $(\mathbf{1}_{G_L(x)})_{x\in\mathbb{Z}^d}$ dominates a site percolation process whose parameter can be made arbitrarily close to 1 by taking L suitably large, it must be the case that, for large L, $\partial^{\text{out}}\mathcal{D}$ is $\mathbb{P}$-a.s. a finite set. Since $\mathcal{D}$ is infinite, it readily follows that $\mathbb{L}\backslash\mathcal{D}$ is also finite. Now, suppose $\mathcal{D}_1$ is a connected component of $\mathbb{L}\backslash\mathcal{C}(x)$ distinct from $\mathcal{D}$ and such that $\mathcal{D}_1\cap\mathcal{D}_\infty\neq\emptyset$. By the definition of $\mathcal{D}_\infty$, it holds that $\mathcal{D}_1\cap\mathcal{D}(y)\neq\emptyset$ for some y such that $\mathcal{D}(y)$ is an infinite set. Since $\mathcal{D}_1\cup\mathcal{D}(y)$ is an infinite connected component of $\mathbb{L}\backslash\mathcal{C}(x)$, it must be the case that $\mathcal{D}_1$ is infinite. However, this contradicts the finiteness of $\mathbb{L}\backslash\mathcal{D}$, and so no such $\mathcal{D}_1$ can exist. Thus it must be the case that if $\mathcal{D}_1$ is a connected component of $\mathbb{L}\backslash\mathcal{C}(x)$ distinct from $\mathcal{D}$, then $\mathcal{D}_1\cap\mathcal{D}_\infty=\emptyset$. Since, similar to the results of the previous paragraph, we have that $\partial^{\text{out}}\mathcal{D}_1\subseteq\partial^{\text{in}}\mathcal{C}(x)$, the set $\mathcal{D}_1\cup\mathcal{C}(x)$ must be a connected component of $\mathbb{L}\backslash\mathcal{D}_\infty$. By the definition of $\mathcal{C}(x)$, this implies that $\mathcal{D}_1=\emptyset$, which yields $\mathcal{D}=\mathbb{L}\backslash\mathcal{C}(x)$ as required.

An immediate corollary of the equality $\mathcal{D}=\mathbb{L}\backslash\mathcal{C}(x)$ is that $\partial^{\text{in}}\mathcal{C}(x)=\partial^{\text{out}}\mathcal{D}$, which, as we have already established, is a finite $*$-connected component of black vertices. We will use these results to finally prove (A.4). Note first that $\operatorname{diam}(\mathcal{C}(x))=\operatorname{diam}(\partial^{\text{in}}\mathcal{C}(x))$. Hence, writing $\partial B(x,m)$ for the vertices of $\mathbb{L}$ at an ℓ_∞ distance m from x,

$$\begin{aligned}\mathbb{P}\left(\operatorname{diam}(\mathcal{C}(x))\geq n\right) &\leq \sum_{m=0}^{\infty}\mathbb{P}\left(\operatorname{diam}(\partial^{\text{in}}\mathcal{C}(x))\geq n,\ \partial^{\text{in}}\mathcal{C}(x)\cap\partial B(x,m)\neq\emptyset\right)\\ &\leq \sum_{m=0}^{\infty}\sum_{y\in\partial B(x,m)}\mathbb{P}\left(\operatorname{diam}(\mathcal{C}^*(y))\geq n\vee m\right),\end{aligned}$$

where $\mathcal{C}^*(y)$ is the $*$-connected component of black vertices in $\mathbb{Z}^d$ containing y. By again comparing $(\mathbf{1}_{G_L(x)})_{x\in\mathbb{Z}^d}$ to a site percolation process, it is possible to apply (A.8) to deduce that the tail of the probability in the above sum is bounded above by $c_5e^{-c_6(n\vee m)}$. The estimate at (A.4) follows.

We now return to the problem of deriving the estimate at (A.1). For $x\in\mathbb{Z}^d$, define the set $Q'_L(x):=L(x+e_1+\cdots+e_{d_1})+[0,L)^d\cap\mathbb{Z}^d$, so that $(Q'_L(x))_{x\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$. For $x\in\mathbb{L}$, let $a(x)$ be the closest element of $\mathbb{L}$, with respect to ℓ_1 distance, to the $x'\in\mathbb{Z}^d$ such that $x\in Q'_L(x')$. (Only when x is within a distance L of the boundary of $\mathbb{L}$ does $x'\neq a(x)$.) It is then easy to check that if $\operatorname{diam}(\mathcal{G}(x))\geq L$, then

$$\mathcal{G}(x)\subseteq\cup_{x'\in\mathcal{C}(a(x))}\tilde{Q}_{3L}(x'). \tag{A.9}$$

(cf. [17, (3.7)]). In particular, this implies that, if $\operatorname{diam}(\mathcal{G}(x))\geq L$, then it must be the case that $\operatorname{diam}(\mathcal{G}(x))\leq 3L\operatorname{diam}(\mathcal{C}(a(x)))$. Consequently, (A.1) follows from (A.4), and thus the proof of part (i) is complete.

Given part (i), we observe

$$\begin{aligned}
&\mathbb{P}\left(\exists x \in [-n,n]^d \cap \mathbb{L}: \#\mathcal{H}(x) \geq (\log n)^{d+1}\right)\\
\leq\ & (2n+1)^d \sup_{x\in\mathbb{L}} \mathbb{P}\left(x \in \mathcal{C}_1,\ \#\mathcal{H}(x) \geq (\log n)^{d+1}\right)\\
\leq\ & (2n+1)^d \sup_{x\in\mathbb{L}} \mathbb{P}\left(x \in \mathcal{C}_1,\ \mathrm{diam}(\mathcal{H}(x)) \geq (\log n)^{(d+1)/d}\right)\\
\leq\ & (2n+1)^d c_1 e^{-c_2(\log n)^{(d+1)/d}}.
\end{aligned}$$

Since this is summable in n, part (ii) follows by a Borel-Cantelli argument. □

We now work towards the proof of Lemma 3.2.

Proof of Lemma 3.2. To establish the bound in (3.4), let us start by recalling/adapting some definitions from the previous proof. In particular, for $x \in \mathbb{Z}^d$, define $Q_L(x)$ and $\tilde{Q}_{3L}(x)$ as in the proof of Lemma 3.1. Moreover, let $G_L(x)$ be defined similarly, but with $\tilde{\mathcal{O}}_3$ replaced by $\tilde{\mathcal{O}}_1$, and redefine x being 'white' to mean that this version of $G_L(x)$ holds (and say x is 'black' otherwise). Note that the statement (A.2) remains true with this definition of $G_L(x)$, and the dependence between the random variables $(\mathbf{1}_{G_L(x)})_{x\in\mathbb{Z}^d}$ is only finite range, and so we can suppose that it dominates a dominates a site percolation process on $\mathbb{Z}^d$ of density arbitrarily close to 1.

Now, fix $x, y \in \mathbb{L}$, and recall the definition of $a(x)$ from the proof of Lemma 3.1. If n is the ℓ_1 distance between $a(x)$ and $a(y)$, then there exists a nearest neighbor path $a_0, \dots, a_n$ in $\mathbb{L}$ such that $a_0 = a(x)$ and $a_n = a(y)$. We claim that if x and y are both contained in $\mathcal{C}_1$, then there exists a path from x to y along edges of $\mathcal{O}_1$ whose vertices all lie in

$$\cup_{i=0}^n \cup_{b\in\overline{\mathcal{C}}^*(a_i)} \tilde{Q}_{3L}(b), \tag{A.10}$$

where $\overline{\mathcal{C}}^*(a) := \{a\}$ if a is white, otherwise $\overline{\mathcal{C}}^*(a) := \mathcal{C}^*(a) \cup \partial^{\mathrm{out}}\mathcal{C}^*(a)$, where $\mathcal{C}^*(a)$ is the $*$-connected component of black sites in $\mathbb{L}$ containing a ($\partial^{\mathrm{out}}\mathcal{C}^*(a)$ is the outer boundary of $\mathcal{C}^*(a)$, defined similarly to (A.5)). This is essentially [4, Proposition 3.1] rewritten for $\mathbb{L}$ instead of $\mathbb{Z}^d$. The one slight issue with modifying the proof of this result to our situation is that, unlike the $\mathbb{Z}^d$ case, the outer boundary in $\mathbb{L}$ of a finite connected cluster of vertices, $\mathcal{C}$ say, is no longer $*$-connected in general and so it is not possible to run around it in quite the same way. However, this problem is readily overcome by applying [54, Lemma 2], which implies that for each $x \notin \mathcal{C}$, the part of the outer boundary of $\mathcal{C}$ that is visible from x, $\partial^{\mathrm{vis}(x)}\mathcal{C}$ (cf. (A.6)), is $*$-connected. A simple estimate of the number of vertices in the set at (A.10) yields

$$d_1(x,y) \leq (3L+1)^d \sum_{i=0}^n \#\overline{\mathcal{C}}^*(a_i) \leq (3L+1)^d \sum_{i=0}^n \left(1 + 3^d \#\mathcal{C}^*(a_i)\right).$$

(We take $\mathcal{C}^*(a) := \emptyset$ if a is white.) Clearly $\mathcal{C}^*(a) \subseteq \tilde{\mathcal{C}}^*(a)$, where $\tilde{\mathcal{C}}^*(a)$ is the $*$-connected component of black sites in $\mathbb{Z}^d$ containing a. Consequently, we obtain that

$$\mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } d_1(x,y) \geq cR\right) \leq \mathbb{P}\left((3L+1)^d \sum_{i=0}^n \left(1 + 3^d \#\tilde{\mathcal{C}}^*(a_i)\right) \geq cR\right).$$

Moreover, applying [29, Lemma 2.3] as in the proof of [4, Theorem 1.1], we may replace the summands on the right-hand side by independent ones, each with the same distribution as the term they are replacing. Recalling from (A.8) the exponential bound for the size of a $*$-connected vacant cluster in a site percolation process of parameter p close to 1, one readily obtains from this the bound at (3.4). (It is useful to also note that $n \leq c|x-y|/L$.)

We proceed next with the proof of the second bound. In this direction, let us begin by defining a metric d_Z on $\mathcal{C}_2$ related to the process Z introduced in Section 3.2, that is, the time change of Y with time in $\mathcal{H}$ cut out. Assume that K is large enough so that the conclusions of Lemma 3.1 hold. Define a set of edges E'_Z by supposing, for $x, y \in \mathcal{C}_2$, $\{x,y\} \in E'_Z$ if and only if $\{x,y\} \notin \mathcal{O}_2$ and also there exists a path $x = z_0, z_1, \dots, z_k = y$ such that $z_1, \dots, z_{k-1} \in \mathcal{H}$ and $\{z_{i-1}, z_i\} \in \mathcal{O}_1$ for $i = 1, \dots, k$. Thus the jumps of Z will be on edges in either $\mathcal{O}_2$ or E'_Z. Set $E_Z := \mathcal{O}_2 \cup E'_Z$, and let d_Z be the graph distance on $(\mathcal{C}_2, E_Z)$. Our first goal will be to prove that there exist constants c_1, c_2, c_3 such that: for every $x, y \in \mathbb{L}$,

$$\mathbb{P}\left(x, y \in \mathcal{C}_2 \text{ and } d_Z(x,y) \leq c_1^{-1}|x-y|\right) \leq c_2 e^{-c_3|x-y|}, \tag{A.11}$$

where $|x-y|$ is the Euclidean distance between x and y.

For proving (A.11), we suppose that the definition of $G_L(x)$ reverts to that given in the proof of Lemma 3.1, i.e. in terms of $\tilde{\mathcal{O}}_3$. Also, define $G'_L(x)$ to be the event that there are no edges of the set $\tilde{\mathcal{O}}_1 \backslash \tilde{\mathcal{O}}_3$ connecting two vertices of $\tilde{Q}_{3L}(x)$, so that if $G_L(x) \cap G'_L(x)$ holds, then so do the defining properties of $G_L(x)$ when $\tilde{\mathcal{O}}_3$ is replaced by $\tilde{\mathcal{O}}_2$. Clearly, for fixed L, $\mathbb{P}\left(G'_L(x)^c\right) \to 0$ as $p_3 \to p_1$ (i.e. $K \to \infty$). Hence, for any δ, by first choosing L and then K large, we can ensure $\mathbb{P}\left(G_L(x) \cap G'_L(x)\right) \geq 1 - \delta$. For the remainder of this proof, we redefine x being 'white' to mean that $G_L(x) \cap G'_L(x)$ holds, and say x is 'black' otherwise.

Similarly to above, the finite range dependence of the random variables in question means that it is possible to suppose that $(\mathbf{1}_{G_L(x) \cap G'_L(x)})_{x \in \mathbb{Z}^d}$ stochastically dominates a collection $(\eta(x))_{x \in \mathbb{Z}^d}$ of independent and identical Bernoulli random variables whose parameter p is arbitrarily close to 1. Let $\mathcal{C}_\infty$ be the vertices of $\mathbb{L}$ that are contained in an infinite connected component of $\{x \in \mathbb{L} : \eta(x) = 1\}$ (cf. (A.3)). By arguments from the proof of Lemma 3.1, we have that if p is large enough, then this set is non-empty and its complement in $\mathbb{L}$ consists of finite connected components, $\mathbb{P}$-a.s. Now, as in the proof of [17, Lemma 3.1], we 'wire' the holes of $\mathcal{C}_\infty$ by adding edges between every pair of sites that are contained in a connected component of $\mathbb{L} \backslash \mathcal{C}_\infty$ or its outer boundary, and denote the induced graph distance by d'. By proceeding almost exactly as in [17], it is then possible to show that, for suitably large L and K: for $x, y \in \mathbb{L}$,

$$\mathbb{P}\left(d'(x,y) \leq \frac{1}{2}|x-y|\right) \leq e^{-|x-y|}. \tag{A.12}$$

(The one modification needed depends on the observation that, similarly to what was deduced in the proof of Lemma 3.1, the inner boundary of any connected component of $\mathbb{L} \backslash \mathcal{C}_\infty$ is $*$-connected and consists solely of vertices with $\eta(x) = 0$.) Finally, a minor adaptation of (A.9) yields, for $x \in \mathcal{C}_1$

with $\mathrm{diam}(\mathcal{H}(x)) \geq L$,

$$\mathcal{H}(x) \subseteq \cup_{x' \in \mathcal{C}(a(x))} \tilde{Q}_{3L}(x'),$$

where $\mathcal{C}(a(x))$ is now the connected component of $\mathbb{L}\backslash\mathcal{C}_\infty$ containing $a(x)$. It follows that if $x, y \in \mathcal{C}_2$, then $d_Z(x,y) \geq d'(a(x), a(y))$, (cf. [17, (3.10)]). Therefore, since it also holds for $|x-y| \geq 3L$ that $|a(x) - a(y)| \geq c|x-y|/L$, the bound at (A.11) can be obtained from (A.12).

Finally, note that, since $\mu_e \in [K^{-1}, K]$ for every $e \in \mathcal{O}_1$ such that $e \cap e' \neq \emptyset$ for some $e' \in \mathcal{O}_2$, it holds that $t(e) \in [C_A \wedge K^{-1/2}, K^{1/2}]$ for such edges. Moreover, for every other $e \in \mathcal{O}_1$, we have $t(e) \geq 0$. As a consequence, the metric $\overline{d}_1$ is bounded below by a constant multiple of d_Z on $\mathcal{C}_2$. Applying this, as well as setting $\partial^{\mathrm{out}}\mathcal{H}(x) = \{x\}$ for $x \notin \mathcal{H}$, it follows that

$$\begin{aligned}
&\mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } \overline{d}_1(x,y) \leq c_4^{-1}|x-y|\right) \\
&\leq \mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } \inf_{x' \in \partial^{\mathrm{out}}\mathcal{H}(x), y' \in \partial^{\mathrm{out}}\mathcal{H}(y)} d_Z(x', y') \leq c_5^{-1}|x-y|\right) \\
&\leq \sum_{x', y' :\, |x-x'|, |y-y'| \leq |x-y|/4} \mathbb{P}\left(x', y' \in \mathcal{C}_2 \text{ and } d_Z(x', y') \leq c_5^{-1}|x-y|\right) \\
&\qquad + 2\mathbb{P}\left(x \in \mathcal{C}_1 \text{ and } \mathrm{diam}(\mathcal{H}(x)) + 1 \geq |x-y|/4\right).
\end{aligned}$$

From this, the bound at (3.5) is a straightforward consequence of Lemma 3.1(i) and (A.11). □

Given Lemma 3.2, the proof of Lemma 3.3 is straightforward.

Proof of Lemma 3.3. Since $\overline{d}_1 \leq C_A d_1$, the inclusion $B_1(x,R) \subseteq \overline{B}_1(x, C_A R)$ always holds. We will thus concern ourselves with the other two inclusions only. First, by the inequality at (3.5),

$$\begin{aligned}
&\mathbb{P}\left(x \in \mathcal{C}_1 \text{ and } \overline{B}_1(x, C_A R) \not\subseteq B_E(x, c_2 R)\right) \\
&\leq \sum_{y \notin B_E(x, c_2 R)} \mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } \overline{d}_1(x,y) \leq C_A R\right) \\
&\leq c_5 e^{-c_6 R},
\end{aligned}$$

for suitably large c_2. Secondly,

$$\mathbb{P}\left(x \in \mathcal{C}_1 \text{ and } \mathcal{C}_1 \cap B_E(x, c_1 R) \not\subseteq B_1(x,R)\right) \leq \sum_{y \in B_E(x, c_1 R)} \mathbb{P}\left(x, y \in \mathcal{C}_1 \text{ and } d_1(x,y) \geq R\right),$$

and applying (3.4) yields a bound of the form $c_7 e^{-c_8 R}$, thereby completing the proof. □

Next, the comparison of measures stated as Lemma 3.7.

Proof of Lemma 3.7. First note that any point $x \in Q$ that is contained in $\tilde{\mathcal{C}}_1 \backslash \mathcal{C}_1$ must lie in a connected component of $(Q \backslash \mathcal{C}_1, \mathcal{O}_1)$ that meets the inner boundary of Q, which we denote here by $\partial^{\mathrm{in}} Q$. Moreover, we recall that any points $x \in Q$ that are contained in $\mathcal{C}_1$ must also be contained in $\tilde{\mathcal{C}}_1$. It follows that

$$\tilde{\nu}(Q) - \nu(Q) \leq \sum_{x \in \partial^{\mathrm{in}} Q} \#\mathcal{F}(x),$$

where $\mathcal{F}(x)$ is the connected component of $\mathbb{L}\backslash\mathcal{C}_1$ containing x. Now, similarly to (A.1), we have that

$$\mathbb{P}\left(\operatorname{diam}(\mathcal{F}(x)) \geq n\right) \leq c_1 e^{-c_2 n},$$

uniformly in $x \in \mathbb{L}$. Since $\#\partial^{\text{in}} Q$ is bounded above by $c_3 n^{d-1}$, the lemma follows. □

Finally, we prove Lemma 4.1.

Proof of Lemma 4.1. First, observe that

$$\begin{aligned}&\mathbb{P}\left(\mathcal{C}^+(Q_2) \not\subseteq \mathcal{C}_1\right)\\ &\quad\leq \mathbb{P}\left(\mathcal{C}^+(Q_2) \leq \varepsilon|Q_2|\right) + \mathbb{P}\Big(\operatorname{diam}(\mathcal{C}^+(Q_2)) \geq \varepsilon^{1/d}(2n+1),\, \mathcal{C}^+(Q_2) \not\subseteq \mathcal{C}_1\Big).\end{aligned}$$

As in the proof of Proposition 3.5, the first term here is bounded above by $c_1 e^{-c_2 n}$. The second term is bounded above by

$$c_3 n^d \sup_{x \in Q_2} \mathbb{P}\left(\text{the diameter of the connected component of } \mathbb{Z}^d_+\backslash\mathcal{C}_1 \text{ containing } x \text{ is } \geq c_4 n\right).$$

That this admits a bound of the form $c_5 e^{-c_6 n}$ follows from (A.1) (replacing $\mathcal{C}_3$ by $\mathcal{C}_1$).

Consequently, to complete the proof, it will suffice to show that

$$\mathbb{P}\left(\mathcal{C}^+(Q_2) \cap Q_1 \subset \mathcal{C}_1 \cap Q_1\right) \leq c_7 e^{-c_8 n}.$$

For the event on the left-hand side of the above to hold, it must be the case that there exists an open path in $(Q_2, \mathcal{O}_1)$ of diameter at least $(1-\varepsilon)n$ that is not part of $\mathcal{C}^+(Q_2)$. Moreover, as in the previous paragraph, we have that, with probability at least $1-c_1e^{-c_2n}$, $\operatorname{diam}(\mathcal{C}^+(Q_2)) \geq \varepsilon^{1/d}(2n+1)$. However, by (the bond percolation version of) [49, Theorem 5], we have that with probability at least $1-c_9e^{-c_{10}n}$, there is a unique open cluster in $(Q_2, \mathcal{O}_1)$ of diameter at least $\varepsilon^{1/d}(2n+1)$. Hence, by taking ε suitably small, the result follows. □

A.2 Arbitrary starting point quenched invariance principle

This section contains the proof of Theorem 3.13. For it, we note that the full $\mathbb{Z}^d$ model also satisfies the conclusions of Propositions 3.10 and 3.11, as well as Assumption 2.3 (in the same sense as we checked for the $\mathbb{L}$ model in Section 3.3).

Proof of Theorem 3.13. To begin with, we recall the quenched invariance principle of [3, Theorem 1.1(a)] for the VSRW started at the origin: there exists a deterministic constant $c > 0$ such that, for $\mathbb{P}_1$-a.e ω, the laws of the processes $\tilde{Y}^n$ under $\tilde{P}^\omega_0$ converge weakly to the law of $(B_{ct})_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is standard Brownian motion on $\mathbb{R}^d$ started from 0. Here, $\mathbb{P}_1$ is the conditional law $\mathbb{P}(\cdot|\, 0 \in \tilde{\mathcal{C}}_1)$. Moreover, we note that by proceeding as in [3, Remark 5.16], one can check that the result remains true if $\mathbb{P}_1$ is replaced by $\mathbb{P}$, and $\tilde{P}^\omega_0$ is replaced by $\tilde{P}^\omega_{x_0}$, where x_0 is chosen to be the (not necessarily uniquely defined) closest point to the origin in the infinite cluster $\tilde{\mathcal{C}}_1$.

Given the above result, the argument of this paragraph applies for $\mathbb{P}$-a.e. ω. Fix $x \in \mathbb{R}^d$, $\varepsilon > 0$, and define

$$\tau^n := \inf\left\{t > 0 : \tilde{Y}^n_t \in B_E(x,\varepsilon)\right\},$$
$$\tau^B := \inf\left\{t > 0 : B_{ct} \in B_E(x,\varepsilon)\right\}.$$

A standard argument gives us that, jointly with the convergence of the previous paragraph, τ^n converges in distribution to τ^B. Letting $T > 0$ be a deterministic constant, it follows that the laws of the processes $(\tilde{Y}^n_{\tau^n+t})_{t\geq 0}$ under $\tilde{P}^\omega_{x_0}(\cdot|\,\tau^n \leq T)$ converge weakly to the law of $(B_{c(\tau+t)})_{t\geq 0}$, started from 0 and conditional on $\tau \leq T$. Consequently, the Markov property gives us that for every bounded, continuous function $f : C([0,\infty),\mathbb{R}^d) \to \mathbb{R}$,

$$\int_{\overline{B}_E(x,\varepsilon)} \tilde{E}^\omega_{ny} f(\tilde{Y}^n) P^\omega_{x_0}\left(\tilde{Y}^n_{\tau^n} \in dy | \tau^n \leq T\right) \to \int_{\overline{B}_E(x,\varepsilon)} E^B_y f(B_{c\cdot}) P^B_0\left(B_{c\tau^B} \in dy | \tau^B \leq T\right), \quad \text{(A.13)}$$

where $\overline{B}_E(x,\varepsilon)$ is the closure of $B_E(x,\varepsilon)$, P^B_x is the law of the standard Brownian motion B started from x, and E^B_x is the corresponding expectation. Furthermore, it is elementary to check for such f that, as $\varepsilon \to 0$,

$$\int_{\overline{B}_E(x,\varepsilon)} E^B_y f(B_{c\cdot}) P^B_0\left(B_{c\tau^B} \in dy | \tau^B \leq T\right) \to E^B_x f(B_{c\cdot}). \quad \text{(A.14)}$$

Suppose that the following also holds for every sequence of starting points $x_n \in n^{-1}\tilde{\mathcal{C}}_1$ such that $x_n \to x$, every finite collection of times $0 < t_1 < t_2 < \cdots < t_k$ and all bounded, continuous functions $f_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1,\dots,k$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{\overline{B}_E(x,\varepsilon)} \tilde{E}^\omega_{ny} f(\tilde{Y}^n_{\mathbf{t}}) P^\omega_{x_0}\left(\tilde{Y}^n_{\tau^n} \in dy | \tau^n \leq T\right) - \tilde{E}^\omega_{nx_n} f(\tilde{Y}^n_{\mathbf{t}})\right| = 0, \quad \text{(A.15)}$$

where $\tilde{Y}^n_{\mathbf{t}} := (\tilde{Y}^n_{t_1}, \tilde{Y}^n_{t_2}, \dots, \tilde{Y}^n_{t_k})$. Combining (A.13), (A.14) and (A.15) readily yields that the finite-dimensional distributions of $\tilde{Y}^n$ (under $P^\omega_{nx_n}$) converge to those of $B_{c\cdot}$ (under P^B_x). Moreover, from the full plane version of Proposition 3.10, we have that the laws of $\tilde{Y}^n$ under $P^\omega_{nx_n}$ are tight. These two facts yields the desired quenched invariance principle.

To complete the argument of the previous paragraph and the proof of the theorem, it remains to check the limit at (A.15) holds (simultaneously over sequences of starting points $x_n \to x$, times $\mathbf{t} = (t_1,\dots,t_k)$, and functions f) for $\mathbb{P}$-a.e. ω. In fact, using the independent increments property of $\tilde{Y}^n$ and some standard analysis, it will suffice to check the result for $k = 2$ and for functions f of the form $f(\tilde{Y}^n_{t_1}, \tilde{Y}^n_{t_2}) = f_1(\tilde{Y}^n_{t_1}) f_2(\tilde{Y}^n_{t_2})$. Writing the semigroup of $\tilde{Y}^n$ as $\tilde{P}^n$, we have for such a function f that

$$\begin{aligned}
&\left|\int_{\overline{B}_E(x,\varepsilon)} \tilde{E}^\omega_{ny} f(\tilde{Y}^n_{t_1}, \tilde{Y}^n_{t_2}) P^\omega_{x_0}\left(\tilde{Y}^n_{\tau^n} \in dy | \tau^n \leq T\right) - \tilde{E}^\omega_{nx_n} f(\tilde{Y}^n_{t_1}, \tilde{Y}^n_{t_2})\right| \\
&= \left|\int_{\overline{B}_E(x,\varepsilon)} \tilde{P}^n_{t_1} g(y) P^\omega_{x_0}\left(\tilde{Y}^n_{\tau^n} \in dy | \tau^n \leq T\right) - \tilde{P}^n_{t_1} g(x_n)\right| \\
&\leq \sup_{y \in \overline{B}_E(x,\varepsilon)\cap n^{-1}\tilde{\mathcal{C}}_1} \left|\tilde{P}^n_{t_1} g(y) - \tilde{P}^n_{t_1} g(x_n)\right|,
\end{aligned} \quad \text{(A.16)}$$

where $g := f_1 \times (\tilde{P}^n_{t_2-t_1} f_2)$ (which is a bounded, continuous function). Take $R > 2$ with $x \in B_E(0, R/2) \cap n^{-1}\tilde{\mathcal{C}}_1$. For each $\lambda > 1$, it holds that

$$\tilde{P}^n_t g = \tilde{P}^{n,B_{\lambda R}}_t (g 1_{B_{\lambda R}}) + (\tilde{P}^n_t - \tilde{P}^{n,B_{\lambda R}}_t)(g 1_{B_{\lambda R}}) + \tilde{P}^n_t (g 1_{B^c_{\lambda R}}),$$

where $B_s := B_E(0, s) \cap n^{-1}\tilde{\mathcal{C}}_1$. We have

$$\begin{aligned} |(\tilde{P}^n_t - \tilde{P}^{n,B_{\lambda R}}_t)(g 1_{B_{\lambda R}})(z) + \tilde{P}^n_t (g 1_{B^c_{\lambda R}})(z)| &\leq \|g\|_\infty \tilde{P}^z_\omega(\tilde{\tau}^n_{B_{\lambda R}} \leq t) + \|g\|_\infty \tilde{P}^z_\omega(\tilde{Y}^n_t \in B^c_{\lambda R}) \\ &\leq 2\|g\|_\infty \tilde{P}^z_\omega(\tilde{\tau}^n_{B_{\lambda R}} \leq t). \end{aligned}$$

So, setting $\overline{B}_{x,\varepsilon} := \overline{B}_E(x, \varepsilon) \cap n^{-1}\tilde{\mathcal{C}}_1$, for large n we have

$$\begin{aligned} &\sup_{y \in \overline{B}_{x,\varepsilon}} \left| \tilde{P}^n_{t_1} g(y) - \tilde{P}^n_{t_1} g(x_n) \right| \\ \leq\ &\sup_{y \in \overline{B}_{x,\varepsilon}} \left| \tilde{P}^{n,B_{\lambda R}}_{t_1} (g 1_{B_{\lambda R}})(y) - \tilde{P}^{n,B_{\lambda R}}_{t_1} (g 1_{B_{\lambda R}})(x_n) \right| + 4\|g\|_\infty \sup_{z \in \overline{B}_{x,2\varepsilon}} \tilde{P}^z_\omega(\tilde{\tau}^n_{B_{\lambda R}} \leq t) \\ =:\ &I_1 + 4\|g\|_\infty I_2. \end{aligned}$$

Now let us note that Assumption 2.3 holds for X^n killed on exiting $B_{\lambda R}$ when D_n is replaced by $B_{\lambda R}$ (which can be verified similarly to the discussion in Section 3.3 for the $\mathbb{L}$ case; for this, it is useful to note that the killing does not have any effect since points in $B_{\lambda R/2}$ are suitably far away from the boundary of $B_{\lambda R}$). Moreover, applying the scaling relation $\tilde{q}^n_t(x, y) = n^d q^{\tilde{Y}}_{tn^2}(nx, ny)$, by Proposition 3.11 (for the full $\mathbb{Z}^d$ model) we have, $\|P^{n,B_{\lambda R}}_t g\|_{\infty,n,\lambda R} \leq c_1 t^{-d/2} \|g\|_{1,n,\lambda R}$ whenever we also have $tn^2 \geq c_2(\sup_{x \in B_{\lambda R}} R_x \vee 1 \vee \sup_{x,y \in B_{\lambda R}} 2d_1(x, y))^{1/4}$, where R_x is defined as in Proposition 3.11. Hence a simple Borel-Cantelli argument using the tail estimate of that proposition to control $\sup_{x \in B_{\lambda R}} R_x$ and Lemma 3.2 to control $\sup_{x,y \in B_{\lambda R}} 2d_1(x, y)$ yields that the above bound holds true for all large n, $\mathbb{P}$-a.s. Hence, by applying Proposition 2.5, we have $I_1 \leq C_t(2\varepsilon)^{\gamma'} \|g\|_{2,n,\lambda R}$ for all $x \in B_{R/2}$ and all large n, $\mathbb{P}$-a.s. By applying the full $\mathbb{Z}^d$ version of Proposition 3.14, i.e. the vague convergence of m_n to a multiple of Lebesgue, it follows that for $\mathbb{P}$-a.e. ω: for $t > 0$, $R > 2$ and $\lambda > 1$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{x \in \overline{B}_{R/2}} I_1 = 0.$$

For I_2, we will apply the full $\mathbb{Z}^d$ version of the exit time bound of Proposition 3.8. In particular, this result implies that for $\mathbb{P}$-a.e. ω: for $t > 0$, $R > 2$ and $\lambda > 1$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{x \in \overline{B}_{R/2}} I_2 \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} c_2 \Psi(c_3(\lambda R/2 - 2\varepsilon)n, tn^2) = c_2 e^{-c_3 \lambda^2 R^2 / t}.$$

(Note that a Borel-Cantelli argument that depends on the tail estimate for R_x of Proposition 3.8 is hidden in the inequality.) Letting $\lambda \to \infty$, this converges to 0. Thus, we deduce that for fixed $R > 2$ and $\mathbb{P}$-a.e.-ω that: for every sequence of starting points $x_n \in n^{-1}\tilde{\mathcal{C}}_1$ such that $x_n \to x \in B_E(0, R/2)$, for every $0 < t_1 < t_2$, for every bounded, continuous f_1, f_2, the expression at (A.16) converges to 0 as $n \to \infty$ and then $\varepsilon \to 0$. Since there is no problem in extending this result to allow any $x \in \mathbb{R}^d$, we have thus completed the proof of (A.15). $\square$

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