Notes on Heat Kernel Estimates and Parabolic Harnack Inequality for Jump Processes on Metric Measure Spaces

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Abstract

In this paper, we discuss necessary and sufficient conditions on jumping kernels for a class of jump-type Markov processes on metric measure spaces to have scale-invariant finite range parabolic Harnack inequality.

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Running title: **Notes on Heat Kernel Estimates and Parabolic Harnack Inequality**

1 Introduction and setup

The purpose of this paper is to investigate the relationships among the scale-invariant finite range parabolic Harnack inequality, short time heat kernel estimates and conditions on jumping kernels for a class of symmetric jump-type Markov process. It can be regarded as a complement to two recent papers [CK2, CKK].

A now well-known fundamental result on diffusions is that a scale-invariant parabolic Harnack inequality is equivalent to Gaussian upper and lower bounds on the heat kernel. But unlike the case of diffusion processes, such equivalent relation is not true for general jump-type Markov process. Very recently in [BBK], the above types of relations have been discussed for the random walk case. This paper is motivated by [BBK] and [CKK]. In [CKK], we established sharp two-sided heat kernel estimate for finite range α-stable-like processes and derived a scale-invariant parabolic Harnack principle for them. Unlike in [BBK] and [CK1], a finite range (i.e. truncated) α-stable-like process does not have the polynomial bounds on the heat kernel for large t. It has the polynomial bounds on the heat kernel only for small t and small $|x-y|$. Therefore a natural question arises.

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Question 1.1 *Suppose Y is a pure jump symmetric Markov process. What is the necessary and sufficient conditions on jumping kernel $J(x,y)$ for Y to have a scale-invariant parabolic Harnack inequality in finite range (that is, for x, y with bounded distance)?*

In this paper, we would like to answer the above question for a class of jump-type Markov processes on metric measure spaces.

The approach of this paper is adapted from those of [CK1, CK2]. However, due to the mild assumption on jumping kernel $J(x,y)$ for x, y with large distance, a quite amount of refinement is required and there are also new challenges to overcome. We hope our paper will motivate more research in this direction.

In the following, if f and g are two functions defined on a set D, $f \asymp g$ means that there exists $c > 0$ such that $c^{-1} f(x) \le g(x) \le c\, f(x)$ for all $x \in D$.

Let (F, ρ, μ) be a locally compact separable metric space with metric ρ and a Radon measure μ having full support on F. We assume $\mu(F) = \infty$ throughout this paper. Assume further that there is a metric space $G \supset F$, and $\rho(\cdot, \cdot)$ can be extended to be a metric on G with dilation for F, i.e. there is a constant $c_1 \ge 1$ such that for every $x, y \in F$ and $\delta > 0$, $\delta^{-1}x, \delta^{-1}y \in G$ with

$$c_1^{-1} \delta^{-1} \rho(x,y) \le \rho(\delta^{-1}x, \delta^{-1}y) \le c_1\, \delta^{-1} \rho(x,y). \tag{1.1}$$

Clearly the above condition is satisfied if $F \subset \mathbb{R}^n$ as we can take G to be $\mathbb{R}^n$. See [HK] for a non-Euclidean example of F satisfying the above condition.

We assume that there exists a strictly increasing function $V : \mathbb{R}_+ \to \mathbb{R}_+$ such that $V(0) = 0$ and there exist constants $c_2 > c_1 > 0$ and $d \ge d_0 > 0$ such that

$$c_1 \Big(\frac{R}{r}\Big)^{d_0} \le \frac{V(R)}{V(r)} \ \le\ c_2 \Big(\frac{R}{r}\Big)^{d} \qquad \text{for every} \ \ 0 < r < R < \infty \tag{1.2}$$

and

$$c_1\, V(r) \le \mu(B(x,r)) \le c_2\, V(r) \qquad \text{for every } x \in F \text{ and } r > 0. \tag{1.3}$$

Let ϕ be a strictly increasing continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$, $\phi(1) = 1$ and satisfies the following: there exist constants $c_2 > c_1 > 0$, $c_3 > 0$, and $\beta_2 \ge \beta_1 > 0$ such that

$$c_1 \Big(\frac{R}{r}\Big)^{\beta_1} \ \le\ \frac{\phi(R)}{\phi(r)} \ \le\ c_2 \Big(\frac{R}{r}\Big)^{\beta_2} \qquad \text{for every } 0 < r < R < \infty, \tag{1.4}$$

$$\int_0^r \frac{s}{\phi(s)}\, ds \ \le\ c_3\, \frac{r^2}{\phi(r)} \qquad \text{for every } r > 0. \tag{1.5}$$

Remark 1.1 Note that conditions (1.2)-(1.4) are equivalent to the existence of constants $c_4, c_5 > 1$ and $L_0 > 1$ such that for every $r > 0$,

$$c_4 \phi(r) \le \phi(L_0 r) \le c_5\, \phi(r) \quad \text{and} \quad c_4 V(r) \le V(L_0 r) \le c_5\, V(r).$$

Denote the diagonal set $\{(x,x);\, x\in F\}$ by d. Let $\kappa>0$ and J be a symmetric measurable function on $F\times F\setminus d$ such that $J(x,y)\mathbf{1}_{\{(x,y):\rho(x,y)>\kappa\}}$ is bounded and for all $(x,y)\in F\times F\setminus d$

$$J(x,y)\asymp \frac{1}{V(\rho(x,y))\phi(\rho(x,y))} \quad \text{for } \rho(x,y)\le\kappa \qquad \text{and} \qquad \sup_{x\in F}\int_{\{y\in F:\rho(y,x)>\kappa\}} J(x,y)\mu(dy)<\infty, \tag{1.6}$$

The constant κ in (1.6) plays no special role, so for convenience we will simply take $\kappa=1$ for the rest of this paper.

For $u\in L^2(F,\mu)$, define

$$\mathcal{E}(u):=\mathcal{E}(u,u):=\int_{F\times F}(u(x)-u(y))^2J(x,y)\mu(dx)\mu(dy) \tag{1.7}$$

and for $\beta>0$,

$$\mathcal{E}_\beta(u):=\mathcal{E}_\beta(u,u):=\mathcal{E}(u,u)+\beta\int_F u(x)^2\mu(dx).$$

Let $C_c(F)$ denote the space of continuous functions with compact support in F, equipped with the uniform topology. Define

$$\mathcal{D}(\mathcal{E}):=\{f\in C_c(F):\mathcal{E}(f)<\infty\}. \tag{1.8}$$

Using [CK2, Lemma 2.1 (1)] and our (1.6), one can follow the proof of [CK2, Proposition 2.2] and check that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(F,\mu)$, where $\mathcal{F}:=\overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$. So there is a Hunt process Y associated with it on F, starting from quasi-every point in F (see [FOT]). Function $J(x,y)$ is the jumping intensity kernel for Y and it determines a Lévy system of Y, which describes the jumps of the process Y: for any non-negative measurable function f on $\mathbb{R}_+\times F\times F$, $x\in F$ and stopping time T (with respect to the filtration of Y),

$$\mathbb{E}_x\left[\sum_{s\le T} f(s,Y_{s-},Y_s)\right]=\mathbb{E}_x\left[\int_0^T\left(\int_F f(s,Y_s,y)J(Y_s,y)\mu(dy)\right)ds\right]. \tag{1.9}$$

When $J(x,y)\equiv 0$ for $\rho(x,y)>1$, we call Y a finite range jump process.

Theorem 1.2 *The process Y is conservative; that is, Y has infinite lifetime $\mathbb{P}_x$-a.s. for q.e. $x\in F$.*

Proof. Note that under condition (1.6), $\sup_{x\in F}\int_F(1\wedge\rho(x,y)^2)J(x,y)\mu(dy)<\infty$. For every $\lambda>0$, using Fubini's theorem and assumptions (1.2)-(1.3), we have

$$\begin{aligned}\int_F e^{-\lambda\rho(x,x_0)}\mu(dx) &= \int_0^\infty e^{-\lambda r}d(\mu(B(x_0,r))=\lambda\int_0^\infty \mu(B(x_0,r))\,e^{-\lambda r}\,dr\\ &\le c\lambda\left(1+\int_1^\infty r^d\,e^{-\lambda r}dr\right)<\infty.\end{aligned}$$

It now follows from [MU, Theorem 3.1] that Y is conservative. □

Remark 1.3 (i) In [CK2], the following condition is assumed as condition (1.1) there. For $x \in F$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x with radius r. We assume that there is a point $x_0 \in F$, a constant $\kappa \in (0,1]$, and an increasing sequence $r_n \to \infty$ so that for every $n \geq 1$, $0 < r < 1$, and $x \in \overline{B(x_0, r_n)}$,

$$\text{there is some ball } B(y, \kappa r) \subset B(x,r) \cap \overline{B(x_0, r_n)}. \tag{1.10}$$

This condition is only used in [CK2] for the proof of conservativeness of Y in [CK2, Theorem 4.7]. So in view of above Theorem 1.2, condition [CK2, (1.1)] can be dropped from [CK2].

(ii) There is a gap in the proof of [CK2, Lemma 6.1] for the case of $\gamma_2 \geq \gamma_1 > 0$. An extra condition on J is needed to ensure the comparability on line 17 of [CK2, p.312] to be valid. Clearly, in the notion of [CK2] where

$$\frac{1}{c\, V(\rho(x,y))\phi_1(\rho(x,y))\psi(C_1\rho(x,y))} \leq J(x,y) \leq \frac{c}{V(\rho(x,y))\phi_1(\rho(x,y))\psi(C_2\rho(x,y))}$$

for $\rho(x,y) > 1$ and some positive constants c, C_1, C_2 with ϕ_1 satisfying the conditions (1.2)-(1.4) for ϕ, if we assume

$$C_1 = C_2 \quad \text{ and } \quad \psi(r+1) \leq c\psi(r) \quad \text{for every } r > 1, \tag{1.11}$$

then comparability and hence Lemma 6.1 of [CK2] hold true. Obviously the second condition in (1.11) is satisfied if $\psi \equiv 1$ or $\psi(r) = e^{\gamma r}$ for some constant $\gamma > 0$. A condition such as (1.11) should have been imposed on the jumping kernel for the main results in [CK2] for the case of $\gamma_2 \geq \gamma_1 > 0$ such as heat kernel estimates [CK2, Theorem 1.2] and the scale-invariant parabolic Harnack inequality [CK2, Theorem 4.12]. The jumping kernel of any relativistic α-stable process on $\mathbb{R}^d$ satisfies the condition (1.11) (see [CK2, Example 2.4]) so all the results in [CK2] are applicable to relativistic α-stable processes on $\mathbb{R}^d$. This paper is motivated by finding the most general condition of $J(x,y)$ for $\rho(x,y) > 1$ (that is, on ψ in setting of [CK2]) under which the scale-invariant parabolic Harnack inequality in finite range and the short time heat kernel estimates established in [CK2] remain true.

In section 3 we will show the Hölder continuity of transition density function of Y so that the process can be refined to start from every point in F. We say **UJS** holds (see [BBK]) if for a.e. $x, y \in F$,

$$J(x,y) \leq \frac{c}{V(r)} \int_{B(x,r)} J(z,y)\mu(dz) \qquad \text{whenever } r \leq \tfrac{1}{2}\rho(x,y). \tag{\textbf{UJS}}$$

For $R > 0$, we say $\mathbf{UJS}_{\leq R}$ holds if the above holds for a.e. $x, y \in F$ and $r \leq \frac{\rho(x,y)}{2} \wedge R$, where the constant c may depend on R. It is easy to check that finite range jump process satisfies $\mathbf{UJS}_{\leq 1}$. Clearly, under condition (1.11), the jumping kernel $J(x,y)$ of [CK2, (1.9) and (1.12)] satisfies **UJS**. We would like to emphasize that **UJS** is a quite weak condition, especially in $\mathbb{R}^n$. For example, if a jumping kernel $J(x,y) \asymp j(|x-y|)$ for some decreasing function j, then by an elementary geometry, one can see easily that for $r \leq \frac{1}{2}|x-y|$,

$$\int_{B(x,r)} j(|z-y|)dz \geq \int_{B(x,r)\cap\{|z-y|\leq|x-y|\}} j(|z-y|)dz \geq c r^n j(|x-y|),$$

for some positive constant c independent of r. Thus **UJS** hold for such case. Under some mild geometric condition on the state space, the above argument works for metric measure spaces.

Let $Z_s := (V_s, Y_s)$ be a space-time process where $V_s = V_0 - s$. The filtration generated by Z satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s;\, s \geq 0\}$. The law of the space-time process $s \mapsto Z_s$ starting from (t, x) will be denoted as $\mathbb{P}^{(t,x)}$. For every open subset D of $[0, \infty) \times F$, define $\tau_D = \inf\{s > 0 : \, Z_s \notin D\}$.

We say that a non-negative Borel measurable function $u(t, x)$ on $[0, \infty) \times F$ is *parabolic* (or *caloric*) on $D = (a, b) \times B(x_0, r)$ for Y if there is a properly exceptional set $\mathcal{N}_u$ of Y so that for every relatively compact open subset D_1 of D, $u(t, x) = \mathbb{E}^{(t,x)}[u(Z_{\tau_{D_1}})]$ for every $(t, x) \in D_1 \cap ([0, \infty) \times F \setminus \mathcal{N}_u)$. Here a set $\mathcal{N} \subset F$ is called properly exceptional with respect to the process Y if $\mu(\mathcal{N}) = 0$ and

$$\mathbb{P}^x \left(\{Y_t, Y_{t-}\} \subset F \setminus \mathcal{N} \text{ for every } t > 0\right) = 1 \qquad \text{for } x \in F \setminus \mathcal{N}.$$

It is well-known (see [FOT]) that every exceptional set is $\mathcal{E}$-polar and every $\mathcal{E}$-polar set is contained in a properly exceptional set. Later we will show in Theorem 3.1 below that every bounded parabolic function is (locally) Hölder continuous and so the properly exceptional set $\mathcal{N}$ can be dropped.

We say the (scale-invariant) parabolic Harnack inequality $\mathbf{PHI}(\phi)_{\leq R_1}$ holds for Y if the following is true; For every $\delta \in (0, 1)$, there exists $c = c(\phi, \delta, R_1) > 0$ such that for every $x_0 \in F$, $t_0 \geq 0$, $R \leq R_1$ and every non-negative function u on $[0, \infty) \times F$ that is parabolic on $(t_0, t_0 + 6\delta\phi(R)) \times B(x_0, 4R)$,

$$\sup_{(t_1,y_1)\in Q_-} u(t_1, y_1) \leq c \inf_{(t_2,y_2)\in Q_+} u(t_2, y_2), \qquad (\mathbf{PHI}(\phi)_{\leq R_1})$$

where $Q_- = (t_0 + \delta\phi(R), t_0 + 2\delta\phi(R)) \times B(x_0, R)$ and $Q_+ = (t_0 + 3\delta\phi(R), t_0 + 4\delta\phi(R)) \times B(x_0, R)$.

Note that we do not assume a priori that the parabolic function $u \geq 0$ is bounded on $(t_0, t_0 + 6\delta\phi(r)) \times B(z, 4R)$.

The purpose of this paper is to give the necessary and sufficient condition for $\mathbf{PHI}(\phi)_{\leq R_1}$ to hold.

Theorem 1.4 *Suppose that the jumping kernel is continuous and satisfies* (1.6). *Then* $\mathbf{PHI}(\phi)_{\leq R_1}$ *is equivalent to* $\mathbf{UJS}_{\leq R_1}$.

Remark 1.5 If $J(x, y) \asymp \frac{1}{V(\rho(x,y))\phi(\rho(x,y))}$ on $F \times F$, it is shown in [CK2, Theorem 4.12] (the case of $\gamma_1 = \gamma_2 = 0$) that a global version of scale-invariant parabolic Harnack inequality $\mathbf{PHI}(\phi)$ holds in the sense that $\mathbf{PHI}(\phi)_{\leq R_1}$ holds for every $R_1 > 0$ with constant $c = c(\phi, \delta) > 0$ independent of $R_1 > 0$.

In Section 4, we show that if $J(x, y)$ is continuous on $F \times F \setminus d$, then condition $\mathbf{UJS}_{\leq R}$ is necessary for $\mathbf{PHI}(\phi)_{\leq R}$ to hold. In Section 5, we show $\mathbf{PHI}(\phi)_{\leq R}$ holds for every $R > 0$ if $\mathbf{UJS}_{\leq R_1}$ is satisfied for some $R_1 > 0$. We also show that if the jumping kernel satisfies (1.6) then bounded parabolic functions are (locally) Hölder continuous. An interesting fact is that $\mathbf{UJS}_{\leq R_1}$ is not needed for the Hölder continuity of bounded parabolic functions of X (see Theorem 3.1).

Notations. Throughout this paper, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. A statement that is said to be hold quasi-everywhere (q.e. in abbreviation) on a set $A \subset F$ if there is an $\mathcal{E}$-polar set $\mathcal{N}$ such that the statement holds for every point in $A \setminus \mathcal{N}$. For a process $\{Y_t\}_{t\geq 0}$ on F and $A \subset F$ (resp. on $[0,\infty) \times F$ and $A \subset [0,\infty) \times F$), let

$$\tau_A = \tau_A(Y) := \inf\{t > 0 : \, Y_t \notin A\}, \quad \sigma_A = \sigma_A(Y) := \inf\{t > 0 : Y_t \in A\}.$$

Given a jumping intensity kernel $J(x, y)$, define $\mathcal{J}_J(x) := \int_F J(x, y)\mathbf{1}_{\{\rho(x,y)>1\}}\mu(dy)$.

2 A priori estimates on heat kernels and exit times

We first recall the following inequalities from [CK2, Lemma 2.1].

Lemma 2.1 *There exists positive constant c such that for all $r > 0$,*

$$\sup_{\eta\in F} \int_{B(\eta,r)^c} \frac{1}{V(\rho(\eta,\xi))\phi(\rho(\eta,\xi))}\mu(d\xi) \, < \, \frac{c}{\phi(r)}.$$

We need two auxiliary symmetric jump-type processes X and Z; their associated Dirichlet forms are all of the type (1.7) but with jumping kernels J_X and J_Z, respectively. These jumping kernels are defined as follows:

$$J_X(x, y) := J(x, y)\mathbf{1}_{\{\rho(x,y)\leq 1\}} \tag{2.1}$$

and

$$J_Z(x, y) := \begin{cases} J(x, y) & \text{for } \rho(x, y) \leq 1; \\ \frac{1}{V(\rho(x,y))\phi(\rho(x,y))} & \text{for } \rho(x, y) > 1. \end{cases} \tag{2.2}$$

Let $\mathcal{E}^X$ and $\mathcal{E}^Z$ denote the Dirichlet form of X and Z on $L^2(F;\mu)$, respectively. Process Z is studied in [CK2]. By Theorem 1.2 and [CK2, Theorem 1.2] (for the case of $\gamma_1 = \gamma_2 = 0$), Z is a conservative process, has jointly continuous transition density function $p_0(t, x, y)$ and so it can be refined to start from every point in X. Since X can be constructed from Z by removing jumps of size larger than 1, X can be refined to start from every point in F and has infinite lifetime (see Remark 3.5 of [BBCK]). On the other hand, Y can be constructed from X by adding jumps of size larger than 1 according to $J(x, y)\mathbf{1}_{\{\rho(x,y)>1\}}$, which can occur at most finite many times in any finite time interval in view of the second condition in (1.6) (see Remark 3.4 of [BBCK]). Consequently, Y can be refined to start from every point in F and has infinite lifetime.

It is easy to see that there is a constant $c > 1$ such that for every $u \in \mathcal{F}$,

$$c^{-1}\mathcal{E}^Z_1(u, u) \leq \mathcal{E}_1(u, u) \leq c\,\mathcal{E}^Z_1(u, u) \quad \text{and} \quad c^{-1}\mathcal{E}^Z_1(u, u) \leq \mathcal{E}^X_1(u, u) \leq c\,\mathcal{E}^Z_1(u, u).$$

This implies in particular that they share the same class of capacity zero sets and the same quasi notions such as quasi-continuity (cf. [FOT]). Moreover, by Theorem 3.1 of [CK2], there is a constant $c_1 > 0$ such that

$$\theta(\|u\|_2^2) \leq c_1 \min\left\{\mathcal{E}^X_1(u, u), \, \mathcal{E}_1(u, u)\right\} \quad \text{for every } u \in \mathcal{F},$$

where $\theta(r) = \frac{r}{\phi(V^{-1}(1/r))}$ and V^{-1} is the inverse function of $r \mapsto V(r)$. Observe that $(\mathcal{E}_1, \mathcal{F})$ and $(\mathcal{E}_1^X, \mathcal{F})$ are the Dirichlet forms of the 1-subprocesses of Y and X, respectively. We have by [BBCK, Theorem 3.1] in the same way as that for [CK2, Theorem 3.2] that there are properly exceptional sets $\mathcal{N}_X$ of X and $\mathcal{N}$ of Y with $\mathcal{N}_X \subset \mathcal{N}$, positive symmetric kernels $p_X(t,x,y)$ defined on $(0,\infty) \times (F \setminus \mathcal{N}_X) \times (F \setminus \mathcal{N}_X)$ and $p(t,x,y)$ on $(0,\infty) \times (F \setminus \mathcal{N}) \times (F \setminus \mathcal{N})$ such that $p_X(t,x,y)$ and $p(t,x,y)$ are the transition density functions of X (starting from $x \in F \setminus \mathcal{N}_X$) and Y (starting from $x \in F \setminus \mathcal{N}$), respectively, with respect to the measure μ on F, for each $y \in F \setminus \mathcal{N}_X$ and $t > 0$, $x \mapsto p_X(t,x,y)$ is $\mathcal{E}^X$-quasi-continuous, and for each $y \in F \setminus \mathcal{N}$ and $t > 0$, $x \mapsto p(t,x,y)$ is $\mathcal{E}$-quasi-continuous. Moreover,

$$p_X(t,x,y) \le \frac{c_2\, e^t}{V(\phi^{-1}(t))} \quad \text{for every } x, y \in F \setminus \mathcal{N}_X \text{ and } t > 0$$

and

$$p(t,x,y) \le \frac{c_2\, e^t}{V(\phi^{-1}(t))} \quad \text{for every } x, y \in F \setminus \mathcal{N} \text{ and } t > 0.$$

Here $\phi^{-1}(r)$ denotes the inverse function of $r \mapsto \phi(r)$. It is easy to see by the Chapman-Kolmogrov equation and the above upper bound estimate on $p(t,x,y)$ that for each $y \in F \setminus \mathcal{N}$, $(t,x) \mapsto p(t,x,y)$ is a parabolic function of Y on $(0,\infty) \times F$.

Proposition 2.2 (i) *For each $T > 0$, there exists $c_1 = c_1(T) > 0$ such that*

$$p_X(t,x,y) \;\le\; c_1 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right)$$

for all $t \in (0,T]$ and $x, y \in F \setminus \mathcal{N}_X$.

(ii) *There exist $0 < R_1 < 1$ and $c_2 > 0$ such that*

$$c_2 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right) \;\le\; p_X(t,x,y)$$

for all $t \in (0,T_1]$ and $x, y \in F \setminus \mathcal{N}_X$ with $\rho(x,y) \in [0,R_1]$ where $T_1 := \phi(R_1)$.

Proof. Let $p_0(t,x,y)$ be the transition density function of the symmetric jump-type Markov process Z on F. Since X can be constructed from Z by removing jumps of size larger than 1 via Meyer's construction, by [BBCK, Lemma 3.6] and [BGK, Lemma 3.1(c)] we have for every $t > 0$ and $x, y \in F \setminus \mathcal{N}_X$,

$$p_X(t,x,y) \;\le\; e^{t\|\mathcal{J}_J\|_\infty} p_0(t,x,y) \quad \text{and} \quad p_0(t,x,y) \;\le\; p_X(t,x,y) + t\|J_1\|_\infty$$

where

$$J_1(x,y) := \frac{1}{V(\rho(x,y))\phi(\rho(x,y))} \mathbf{1}_{\{\rho(x,y)>1\}} \quad \text{and} \quad \mathcal{J}_{J_1}(x) := \int_F J_1(x,y)\mu(dy).$$

Applying the estimates on $p_0(t,x,y)$ in [CK2, Theorem 1.2] (for the case $\gamma_1=\gamma_2=0$) to the above two inequalities, we have

$$p_X(t,x,y) \;\le\; c_1 e^{t\|\mathcal{J}_{J_1}\|_\infty}\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\right) \tag{2.3}$$

and

$$\frac{1}{c_1}\left(\frac{1}{V(\phi^{-1}(t))}\wedge\frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\right)-t\|J_1\|_\infty \;\le\; p_X(t,x,y). \tag{2.4}$$

Now (i) follows immediately from (2.3). Since V and ϕ are strictly increasing functions, there exists $0<R_1<1$ such that

$$\frac{1}{2c_1}\frac{1}{V(\phi^{-1}(t))}\;\le\;\frac{1}{c_1}\frac{1}{V(\phi^{-1}(t))}-t\|J_1\|_\infty \qquad \text{if } t\le\phi(R_1)$$

and

$$\frac{1}{2c_1}\frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\le\frac{1}{c_1}\frac{t}{V(\rho(x,y))\phi(\rho(x,y))}-t\|J_1\|_\infty \qquad \text{if } \rho(x,y)\le R_1.$$

Thus we get (ii) from (2.4) and the above two inequalities. □

For every open subset $D\subset F$, we denote by X^D the subprocess of X killed upon leaving D. Recall that $\mathcal{N}_X$ is the properly exceptional set of X in the paragraph proceeding Proposition 2.2. Then X^D has a transition density function $p^D_X(t,x,y)$ defined on $(0,\infty)\times(F\setminus\mathcal{N}_X)\times(F\setminus\mathcal{N}_X)$; in fact by Dynkin-Hunt formula, which is a consequence of the strong Markov property of X,

$$p^D_X(t,x,y)=p_X(t,x,y)-\mathbb{E}^x[1_{\{\tau_D(X)\le t\}}p(t-\tau_D(X),X_{\tau_D(X)},y)]. \tag{2.5}$$

Proposition 2.3 *For every $c_1\in(0,1)$, $0<c_2<c_3<\infty$, there is a constant $c_4>0$ such that for every $x_0\in F$ and $r\le R_1$,*

$$p_X^{B(x_0,r)}(t,x,y)\ge c_4\frac{1}{V(\phi^{-1}(t))} \qquad \text{for } x,y\in B(x_0,c_1r)\setminus\mathcal{N}_X \text{ and } t\in[c_2\phi(r),\,c_3\phi(r)]. \tag{2.6}$$

Proof. Let $\kappa:=c_2/(2c_3)$ and $B_r:=B(x_0,r)$. We first show that there is a constant $c_5\in(0,1)$ so that (2.6) holds for every $r\le R_*$, $x,y\in B(x_0,c_1r)\setminus\mathcal{N}_X$ and $t\in[\kappa\, c_5\phi(r),\,c_5\phi(r)]$. For $r\le R_*$ and $t\in[\kappa\, c_5\phi(r),\,c_5\phi(r)]$, and $x,y\in B(x_0,c_1r)$, we have $\rho(x,y)\le 2c_1r\le 2c_1\phi^{-1}((\kappa c_5)^{-1}t)$. So, by (1.2)-(1.4),

$$V(2c_1\phi^{-1}((\kappa c_5)^{-1}t))\phi(2c_1\phi^{-1}((\kappa c_5)^{-1}t))\le C(\kappa c_5)^{-1}tV(\phi^{-1}((\kappa c_5)^{-1}t))$$

where $C=C(c_1)$ is independent of c_5. Thus by (2.5) and Proposition 2.2, we have for $x,y\in B(x_0,c_1r)\setminus\mathcal{N}_X$

$$\begin{aligned} p_X^{B_r}(t,x,y) \;\ge\;& \frac{\kappa c_5c_6}{V(\phi^{-1}((\kappa c_5)^{-1}t))}\\ &-c_7\mathbb{E}^x\left[1_{\{\tau_{B_r}(X)\le t\}}\left(\frac{1}{V(\phi^{-1}(t-\tau_{B_r}(X)))}\wedge\frac{t-\tau_{B_r}(X)}{V(\rho(X_{\tau_{B_r}(X)},y))\phi(\rho(X_{\tau_{B_r}(X)},y))}\right)\right], \end{aligned} \tag{2.7}$$

where constants c_6, c_7 are independent of $c_5 \in (0,1]$. Observe that

$$\rho(X_{\tau_{B_r}(X)}, y) \geq (1-c_1)r, \qquad t - \tau_{B_r}(X) \leq t \leq c_5\phi(r)$$

and so

$$\begin{aligned} \frac{t-\tau_{B_r}(X)}{V(\rho(X_{\tau_{B_r}(X)},y))\phi(\rho(X_{\tau_{B_r}(X)},y))} &\leq \frac{t}{V((1-c_1)r)\phi((1-c_1)r)} \\ &\leq \frac{t}{V((1-c_1)\phi^{-1}((c_5)^{-1}t))\phi((1-c_1)\phi^{-1}((c_5)^{-1}t))}. \end{aligned}$$

By (1.2)-(1.4), from the above inequality we get

$$\frac{t-\tau_{B_r}(X)}{V(\rho(X_{\tau_{B_r}(X)},y))\phi(\rho(X_{\tau_{B_r}(X)},y))} \leq C_0 \frac{c_5}{V(\phi^{-1}((c_5)^{-1}t))}. \tag{2.8}$$

where $C_0 = C_0(c_1)$ is independent of c_5. Note that by Proposition 2.2 (i) and Lemma 2.1,

$$\begin{aligned} &\mathbb{P}_x\left(X_t \notin B(x,(1-c_1)r/2)\right) = \int_{B(x,(1-c_1)r/2)^c} p(t,x,y)\mu(dy) \\ &\leq c_7 \int_{B(x,(1-c_1)r/2)^c} \frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\mu(dz) \leq c_8\frac{t}{\phi(r)}. \end{aligned}$$

Here the conservativeness of X is used in the first equality. Thus, for $t \leq c_5\phi(r)$ we have

$$\mathbb{P}_x\left(X_t \notin B(x,(1-c_1)r/2)\right) \leq c_8 c_5$$

where c_8 is independent of c_5.

Now applying [BBCK, Lemma 3.8], we have

$$\mathbb{P}_x\left(\tau_{B(x,(1-c_1)r)}(X) \leq t\right) \leq 2c_8c_5. \tag{2.9}$$

Consequently, we have from (2.7), (2.8) and (2.9)

$$\begin{aligned} p_X^{B_r}(t,x,y) &\geq \frac{\kappa c_5 c_6}{V(\phi^{-1}((\kappa c_5)^{-1}t))} - \frac{c_5 c_7 C_0}{V(\phi^{-1}((c_5)^{-1}t))}\mathbb{P}_x\left(\tau_{B_r}(X) \leq t\right) \\ &\geq c_5\left(\kappa c_6 - c_5 c_7 c_8 C_0 \frac{V(\phi^{-1}((\kappa c_5)^{-1}t))}{V(\phi^{-1}((c_5)^{-1}t))}\right)\frac{1}{V(\phi^{-1}((\kappa c_5)^{-1}t))}. \end{aligned}$$

By (1.4), there exists $c_* > 0$ independent of c_5 such that

$$\frac{V(\phi^{-1}((\kappa c_5)^{-1}t))}{V(\phi^{-1}((c_5)^{-1}t))} \leq c_*.$$

Now we choose c_5 small and apply (1.2)-(1.4) so that $p_X^{B_r}(t,x,y) \geq c_9\frac{1}{V(\phi^{-1}(t))}$. This establishes (2.6) for any $x_0 \in F$, $r \leq R_1$ and $t \in [\kappa\, c_5\phi(r),\, c_5\phi(r)]$.

Now for $r \leq R_1$ and $t \in [c_2\phi(r),\, c_3\phi(r)]$, define $k_0 = [c_3/c_5]+1$. Here for $a \geq 1$, $[a]$ denotes the largest integer that does not exceed a. As the previous argument works when c_5 is small, without loss of generality we may assume that $c_5 \le c_3$. So $t/k_0 \in [\kappa\, c_5\phi(r),\, c_5\phi(r)]$. Using semigroup k_0 times, we conclude that for $x, y \in B(x_0, c_1 r) \setminus \mathcal{N}_X$ and $t \in [c_2\phi(r),\, c_3\phi(r)]$,

$$
\begin{aligned}
& p_X^{B(x_0,r)}(t,x,y) \\
= & \int_{B(x_0,r)} \cdots \int_{B(x_0,r)} p_X^{B(x_0,r)}(t/k_0, x, w_1) \dots p_X^{B(x_0,r)}(t/k_0, w_{k_0-1}, y)\mu(dw_1)\dots\mu(dw_{k_0-1}) \\
\ge & \int_{B(x_0,\phi^{-1}(t/(c_5k_0)))} \cdots \int_{B(x_0,\phi^{-1}(t/(c_5k_0)))} p_X^{B(x_0,r)}(t/k_0, x, w_1) \dots \\
& \qquad \dots p_X^{B(x_0,r)}(t/k_0, w_{k_0-1}, y)\mu(dw_1)\dots\mu(dw_{k_0-1}) \\
\ge & \; c_9^{k_0} \left(\frac{V(\phi^{-1}(t/(c_5k_0)))}{V(\phi^{-1}(t))} \right)^{k_0-1} \frac{1}{V(\phi^{-1}(t))} \ge c_{10} \frac{1}{V(\phi^{-1}(t))},
\end{aligned}
$$

where the last inequality we have used (1.2)-(1.4). The proof of (2.6) is now completed. □

For every open subset $D \subset F$, we denote by Y^D the subprocess of Y killed upon leaving D. Recall that $\mathcal{N}$ is the properly exceptional set of Y in the paragraph proceeding Proposition 2.2. Similarly to the case of X^D, Y^D has a transition density function $p^D(t, x, y)$ defined on $(0, \infty) \times (F \setminus \mathcal{N}) \times (F \setminus \mathcal{N})$.

Theorem 2.4 (i) *There are constants $R_* \in (0, 1)$ and $c > 1$ such that for every $t \le \phi(R_*)$ and $x, y \in F \setminus \mathcal{N}$ with $\rho(x, y) \le R_*$*

$$
p(t,x,y) \le c \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right) \tag{2.10}
$$

and

$$
p(t,x,y) \ge c^{-1} \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right) \tag{2.11}
$$

(ii) *Moreover, for every $c_1 \in (0, 1)$, $c_2, c_3 > 0$, there is a constant $c_4 > 0$ such that for every $x_0 \in F$ and $r \le R_*$,*

$$
p^{B(x_0,r)}(t,x,y) \ge c_4 \frac{1}{V(\phi^{-1}(t))} \qquad \text{for } x, y \in B(x_0, c_1 r) \setminus \mathcal{N} \text{ and } t \in [c_2\phi(r),\, c_3\phi(r)]. \tag{2.12}
$$

Proof. Recall that $p_X(t, x, y)$ is the transition density function of finite range process X on F. Since X can be constructed from Y by removing jumps of size larger than 1 via Meyer's construction, as in the proof of Proposition 2.2, we have for $x, y \in F \setminus \mathcal{N}$,

$$
e^{-t\|\mathcal{J}_{J_2}\|_\infty} p_X(t,x,y) \;\le\; p(t,x,y) \quad \text{and} \quad p(t,x,y) \;\le\; p_X(t,x,y) + t\|J_2\|_\infty
$$

where

$$
J_2(x,y) := J(x,y)\mathbf{1}_{\{\rho(x,y)>1\}} \quad \text{and} \quad \mathcal{J}_{J_2}(x) := \int_F J_2(x,y)\mu(dy).
$$

Applying Proposition 2.2 to the above two inequalities, we have

$$p(t,x,y) \le c_1 \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right) + t\|J_2\|_\infty \qquad \text{for all } t \le 1,\ x,y \in F\backslash\mathcal{N} \tag{2.13}$$

and

$$p(t,x,y) \ge \frac{1}{c_1} e^{-t\|\mathcal{J}_{J_2}\|_\infty} \left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \right) \tag{2.14}$$

for every $t \le \phi(R_1)$ and $x,y \in F \setminus \mathcal{N}$ with $\rho(x,y) \le R_1$. Now (2.11) follows immediately from (2.14). Since V and ϕ are strictly increasing functions, there exists $0 < R_* < 1$ such that

$$c_1 \frac{1}{V(\phi^{-1}(t))} + t\|J_2\|_\infty \le 2c_1 \frac{1}{V(\phi^{-1}(t))} \qquad \text{if } t \le \phi(R_*)$$

and

$$c_1 \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} + t\|J_2\|_\infty \le 2c_1 \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \qquad \text{if } \rho(x,y) \le R_*,$$

we get (2.10) from (2.13).

(ii) follows from our Proposition 2.3 and [BBCK, (3.18) in Lemma 3.6]. In fact, for $x,y \in B(x_0,c_1r) \setminus \mathcal{N}$ and $t \in [c_2\phi(r),\, c_3\phi(r)]$,

$$p^{B(x_0,r)}(t,x,y) \ge e^{-t\|\mathcal{J}_{J_2}\|_\infty} p_X^{B(x_0,r)}(t,x,y) \ge e^{-c_2\phi(R_*)\|\mathcal{J}_{J_2}\|_\infty} p_X^{B(x_0,r)}(t,x,y) \ge c_3 \frac{1}{V(\phi^{-1}(t))}.$$

□

Using Proposition 2.2(i) and [BBCK, (3.18) in Lemma 3.6], we have the following exit time estimate.

Lemma 2.5 *There exists $\gamma \in (0,1)$ such that for every $x \in F \setminus \mathcal{N}$ and $0 < r \le 1$,*

$$\mathbb{P}^x(\tau_{B(x,r/2)}(Y) < \gamma\phi(r)) \le 1 - \frac{1}{2} e^{-\|\mathcal{J}_J\|_\infty}. \tag{2.15}$$

Proof. Recall that X is finite range process on F and that $p_X(t,x,y)$ is its transition density function. As in (2.9), there exists $\gamma \in (0,1)$ such that for every $x \in F \setminus \mathcal{N}$ and $0 < r \le 1$. $\mathbb{P}^x(\tau_{B(x,r/2)}(X) \ge \gamma\phi(r)) > 1/2$. Applying [BBCK, (3.18) in Lemma 3.6], we have that for every $x \in F \setminus \mathcal{N}$ and $0 < r \le 1$.

$$\mathbb{P}^x(\tau_{B(x,r/2)}(Y) \ge \gamma\phi(r)) \ge e^{-\gamma\phi(r)\|\mathcal{J}_J\|_\infty} \mathbb{P}^x(\tau_{B(x,r/2)}(X) \ge \gamma\phi(r)) > \frac{1}{2} e^{-\|\mathcal{J}_J\|_\infty}.$$

□

Recall that $Z_s = (V_s, Y_s)$ is a space-time process where $V_s = V_0 - s$. Let

$$Q^{\downarrow}(t,\delta,x,r) := [t - \delta\phi(r), t] \times B(x,r).$$

Also, for each $A \subset [0,\infty) \times F$, denote $A_s := \{y \in F :\ (s,y) \in A\}$.

Define $U(t,x,r) = \{t\} \times B(x,r)$. We can prove the following lemma in a similar way as those for [CK2, Lemmas 6.2 and Corollary 6.3].

Lemma 2.6 (i) *There exists $C_2 > 0$ such that for every $x \in F \setminus \mathcal{N}$, $r \in (0,1]$, $\delta \in (0,\gamma]$, $t \geq \delta\phi(r)$, and any compact subset $A \subset Q^{\downarrow}(t,\delta,x,r)$,*

$$\mathbb{P}^{(t,x)}(\sigma_A < \tau_r) \geq C_2 \frac{m \otimes \mu(A)}{V(r)\phi(r)},$$

where $\tau_r = \tau_{Q^{\downarrow}(t,\delta,x,r)}$ and $m \otimes \mu$ is a product of the Lebesgue measure m on $\mathbb{R}_+$ and μ on F.

(ii) *For every $0 < \delta \leq \gamma$, there exists $C_3 > 0$ such that for every $R \in (0, R_*/2]$, $r \in (0, R/2]$, $x_0 \in F$ and $(t',x), (t,z) \in [t - 3\delta\phi(R), t] \times B(x_0, R)$ with $t' \leq t - \delta\phi(R)/2$ and $z \notin \mathcal{N}$,*

$$\mathbb{P}^{(t,z)}\left(\sigma_{U(t',x,r)} < \tau_{[t-3\delta\phi(R),t]\times B(x_0,2R)}\right) \geq C_3 \frac{V(r)\phi(r)}{V(R)\phi(R)}. \tag{2.16}$$

Proof. Using Lemma 2.5, the proof of (i) is the same as that of [CK2, Lemmas 6.2].

For (ii), we can prove in the same way as that of [CK2, Corollary 6.3], but here we give an alternative proof. Note that $B(x,r) \subset B(z, 3R/2)$. The left hand side of (2.16) is equal to

$$\mathbb{P}^z\left(X^{B(x_0,2R)}_{t-t'} \in B(x,r)\right) = \int_{B(x,r)} p^{B(x_0,2R)}(t-t',z,y)dy,$$

which, by (2.12) and Remark 1.1, is greater than or equal to

$$c_1\mu(B(x,r))\frac{1}{V(\phi^{-1}(t-t'))} \geq c_2 \frac{V(r)}{V(\phi^{-1}(\delta\phi(R)/2))} \geq c_3 \frac{V(r)}{V(R)} \geq c_3 \frac{V(r)\phi(r)}{V(R)\phi(R)}$$

where c_3 depend on δ. □

3 Hölder continuity of parabolic functions

In this section, using heat kernel estimates established in the previous section, we show that bounded parabolic functions are Hölder continuous. Recall that R_* is the constant in Theorem 2.4.

Theorem 3.1 *Suppose the jumping kernel J satisfies* (1.6). *Then there are constants $c = c(R_*) > 0$ and $\kappa = \kappa(R_*) > 0$ such that for every $0 < R \leq R_*$ and for every function h that is bounded and parabolic in $(0, \phi(2R)) \times B(x_0, 2R)$,*

$$|h(s,x) - h(t,y)| \leq c\, \|h\|_{\infty,R}\, R^{-\kappa} \left(\phi^{-1}(|t-s|) + \rho(x,y)\right)^{\kappa} \tag{3.1}$$

holds for (s,x), $(t,y) \in (0,\phi(2R)) \times (B(x_0,2R) \setminus \mathcal{N}_h)$, where $\|h\|_{\infty,R} := \sup_{(t,y)\in[0,\,\phi(2R)]\times F} |h(t,y)|$ and $\mathcal{N}_h$ is the properly exceptional set of Y in the definition of the parabolic function h. In particular, for the transition density function $p(t,x,y)$ of Y and any $t_0 \in (0, \phi(R_*))$, there are constants $c = c(t_0) > 0$ and $\kappa > 0$ such that for any $t,\ s \in [t_0,\ 1]$ and $(x_i, y_i) \in (F\setminus\mathcal{N}) \times (F\setminus\mathcal{N})$ with $i = 1,2$,

$$|p(s,x_1,y_1) - p(t,x_2,y_2)| \leq \frac{c}{V(\phi^{-1}(t_0))\phi^{-1}(t_0)^{\kappa}} \left(\phi^{-1}(|t-s|) + \rho(x_1,x_2) + \rho(y_1,y_2)\right)^{\kappa}. \tag{3.2}$$

Proof. By enlarge $\mathcal{N}_u$ if needed, we may and do assume that $\mathcal{N}_u \supset \mathcal{N}$, where $\mathcal{N}$ is the properly exceptional set of Y in the paragraph proceeding Proposition 2.2. As before, let $Z_s = (V_s, Y_s)$ be the space-time process of Y, where $V_s = V_0 - s$. Without loss of generality, assume that $0 \le h(z) \le \|h\|_{\infty,R} = 1$ for $z \in [0,\, \phi(2R)] \times F$. Let

$$Q(t,x,r) := [t - \gamma\phi(r), t] \times B(x,r).$$

By Lemma 2.6(i) there is a constant $0 < c_1 < 1$ such that if $x \in F \setminus \mathcal{N}$, $0 < r < 1$, $t \ge \gamma\phi(r)$ and $A \subset Q(t,x,r)$ with $\frac{m\otimes\mu(A)}{m\otimes\mu(Q(t,x,r))} \ge 1/3$, then

$$\mathbb{P}^{(t,x)}(\sigma_A < \tau_r) \ge c_1, \tag{3.3}$$

where $\tau_r := \tau_{Q(t,x,r)}$. By Lévy system formula (1.9) with $f(s,y,z) = 1_{B(x,r)}(y)\, \mathbf{1}_{F\setminus B(x,s)}(z)$ and $T = \tau_r$, there is a constant $c_2 > 0$ such that if $s \ge 2r$,

$$\mathbb{P}^{(t,x)}(Y_{\tau_r} \notin B(x,s)) = \mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \int_{F\setminus \overline{B(x,s)}} J(Y_v, y)\, \mu(dy)\, dv\right]$$

Observe that for $0 < 2r < s \le 1/2$, by condition (1.6) and Lemma 2.1,

$$\begin{aligned}
& \mathbb{E}^x\left[\int_0^{\tau_{B(x,r)}\wedge(\gamma\phi(r))} \int_{F\setminus\overline{B(x,s)}} J(Y_u,y)\mu(dy)du\right] \\
\le\ & \mathbb{E}^x\left[\int_0^{\tau_{B(x,r)}\wedge(\gamma\phi(r))} \left(\int_{B(x,1)\setminus\overline{B(x,s)}} J(Y_u,y)\mu(dy) + \int_{F\setminus\overline{B(Y_u,1/2)}} J(Y_u,y)\mu(dy)\right) du\right] \\
\le\ & c\mathbb{E}^x\left[\int_0^{\tau_{B(x,r)}\wedge(\gamma\phi(r))} \left(\int_{B(x,1)\setminus\overline{B(x,s)}} \frac{1}{V(\rho(x,y))\phi(\rho(x,y))}\mu(dy) \,+\, c\right) du\right] \\
\le\ & c\mathbb{E}^x\left[\int_0^{\tau_{B(x,r)}\wedge(\gamma\phi(r))} \left(\frac{1}{\phi(s)} \,+\, c\right) du\right] \\
\le\ & c_1\mathbb{E}^x\left[\tau_{B(x,r)} \wedge (\gamma\phi(r))\right] \frac{1}{\phi(s)} \le c_2\frac{\phi(r)}{\phi(s)}.
\end{aligned}$$

Thus we have

$$\mathbb{P}^{(t,x)}(Y_{\tau_r} \notin B(x,s)) \le c_2\frac{\phi(r)}{\phi(s)}. \tag{3.4}$$

The remainder part of the proof is an easy modification of the corresponding one in [CK1, Theorem 4.14]. So we omit the details. □

The above theorem in particular implies that Y admits a jointly continuous transition density function $p(t,x,y)$ on $(0,\infty) \times F \times F$. So from now on, we can take the properly exceptional set $\mathcal{N}$ of Y for $p(t,x,y)$ in Section 2 to be empty.

4 Necessary condition for $\mathbf{PHI}(\phi)$

The following fact and its proof are adaptations of [BBK, Proposition 4.7] and its proof.

Proposition 4.1 *Suppose Y satisfies $\mathbf{PHI}(\phi)_{\le R}$. Suppose also $J(\cdot,\cdot)$ is continuous off the diagonal on $F \times F$. Then $\mathbf{UJS}_{\le R}$ holds.*

Proof. Let $r \le \frac{\rho(x,y)}{2} \wedge R$ and $B := B(x,r)$. Recall that R_* is the constant in Theorem 2.4. Denote

$$a := \frac{\phi(R)}{\phi(R_*)} \vee 2 \qquad \text{and} \qquad b := a/(a-1) > 1.$$

For $t \in (0,1)$, $h, \varepsilon \in (0, r/4)$ and $T = \phi(r)$, define

$$u_h(t,x) = \mathbb{P}^x(Y_{\tau_B} \in B(y,\varepsilon), t - T/b < \tau_B < t + h - T/b)$$

Then, by the Lévy system formula in (1.9), for every $z \in B$

$$\begin{aligned} u_h(t,z) &= \mathbb{E}^z\left[\int_0^{\tau_B}\int_{B(y,\varepsilon)} 1_{(t-T/b,t+h-T/b)}(s)J(Y_s,u)\mu(du)ds\right] \\ &= \int_{t-T/b}^{t+h-T/b}\int_{B(y,\varepsilon)} \mathbb{E}^z\left[1_{[0,\tau_B]}(s)J(Y_s,u)\right]\mu(du)ds. \end{aligned}$$

By the continuity of $J(\cdot,\cdot)$, for every $z \in B$

$$\frac{1}{h}u_h(T/b,z) = \frac{1}{h}\int_0^h\int_{B(y,\varepsilon)} \mathbb{E}^z[1_{[0,\tau_B]}(s)J(Y_s,u)]\mu(du)ds \to \int_{B(y,\varepsilon)} J(z,u)\mu(du), \tag{4.1}$$

as $h \to 0$. Next, since u_h is non-negative bounded and parabolic on $(0,\infty)\times B$, applying $\mathbf{PHI}(\phi)_{\le R}$ to u_h in $(0,T)\times B$, we obtain

$$u_h(T/b,x) \le C_1 u_h(T,x). \tag{4.2}$$

Now by (2.10),

$$u_h(T,x) = \int_B p^B(T/a,x,z)u_h(T/b,z)\mu(dz) \le \frac{c}{V(r)}\int_B u_h(T/b,z)\mu(dz).$$

Thus, by (4.1) and (4.2), we have

$$\int_{B(y,\varepsilon)} J(x,u)\mu(du) \le \frac{c}{V(r)}\int_B\int_{B(y,\varepsilon)} J(z,u)\mu(du)\mu(dz).$$

Dividing both sides by $\mu(B(y,\varepsilon))$ and using the continuity of $J(\cdot,\cdot)$, we obtain $\mathbf{UJS}_{\le R}$. □

5 Proof of PHI(ϕ)

Recall that $Z_s = (V_s, Y_s)$ is the space-time process of Y, with $V_s = V_0 - s$. Note that for a space-time ball $Q := [a, b] \times B$,

$$\tau_Q = \tau_Q(Z) = \inf\{s \geq 0 : Z_s \notin Q\} = \tau_B(Y) \wedge (V_0 - a)_+ 1_{\{V_0 \leq b\}}.$$

The following is a standard fact that is proved in [CK1, Lemma 4.13].

Lemma 5.1 (Lemma 4.13 in [CK1]) *For any bounded Borel measurable function $q(t, x)$ that is parabolic in an open subset D of $\mathbb{R}_+ \times F$, $s \mapsto q(Z_{s\wedge\tau_D}) = q(Z_{s\wedge\tau_{D(Y)}})$ is right continuous $\mathbb{P}^{(t,x)}$-a.s. for every $(t, x) \in D$.*

Our aim in this section is to prove the following.

Theorem 5.2 *Assume* (1.6) *and* $\mathbf{UJS}_{\leq R_1}$ *for some $R_1 > 0$. Then* $\mathbf{PHI}(\phi)_{\leq R_2}$ *holds for every $R_2 > 0$.*

In order to prove Theorem 5.2, we will need to prove the following lemma corresponds to [CK2, Lemmas 6.1] (also [CK1, Lemma 4.9]). The statement is changed (in the sense that the size of two space-time balls are different and the initial points are also different) and, due to the very general nature of $J(x, y)$ for $\rho(x, y) > \kappa$ in (1.6), the proof requires major changes from the original one for [CK2, Lemmas 6.1]. A simpler version of Lemma 5.3 with $\phi(r) = r^\alpha$ and $F = \mathbb{R}^d$ appears in [CKK] as Lemma 4.2.

Recall that R_* is the constant in Theorem 2.4.

Lemma 5.3 *Let $R \leq R_*$, $\delta > 0$ and $0 < a < 1/3$. $Q_1 = [t_0, t_0 + 5\delta\phi(R)] \times B(x_0, 3aR/2)$, $Q_2 = [t_0, t_0 + 6\delta\phi(R)] \times B(x_0, 2aR)$ and define*

$$Q_- = [t_0 + \delta\phi(R), t_0 + 2\delta\phi(R)] \times B(x_0, aR), \qquad Q_+ = [t_0 + 3\delta\phi(R), t_0 + 4\delta\phi(R)] \times B(x_0, aR).$$

Let $h : [t_0, \infty) \times F \to \mathbb{R}_+$ be bounded and supported in $[t_0, \infty) \times B(x_0, 3aR)^c$. Then there exists $C_1 = C_1(\delta, a) > 0$ such that the following holds:

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] \leq C_1 \mathbb{E}^{(t_2,y_2)}[h(Z_{\tau_{Q_2}})] \qquad \textit{for } (t_1, y_1) \in Q_- \textit{ and } (t_2, y_2) \in Q_+.$$

Proof. Without loss of generality, assume that $t_0 = 0$. Denote $B_{cR} = B(x_0, cR)$. Using the Lévy system formula in (1.9),

$$
\begin{aligned}
& \mathbb{E}^{(t_2,y_2)}[h(Z_{\tau_{Q_2}})] && (5.1)\\
= \; & \mathbb{E}^{(t_2,y_2)}[h(t_2-(\tau_{B_{2aR}}\wedge t_2), Y_{\tau_{B_{2aR}}\wedge t_2})] \\
= \; & \mathbb{E}^{(t_2,y_2)}\left[\int_0^{t_2} \mathbf{1}_{\{t\le \tau_{B_{2aR}}\}} \int_{B^c_{3aR}} h(t_2-t,v)J(Y_t,v)dt\mu(dv)\right] \\
= \; & \int_0^{t_2} h(t_2-t,v)\int_{B^c_{3aR}} \mathbb{E}^{(t_2,y_2)}[\mathbf{1}_{\{t\le \tau_{B_{2aR}}\}}J(Y_t,v)]\mu(dv)dt \\
= \; & \int_0^{t_2} h(s,v)\int_{B^c_{3aR}} \mathbb{E}^{(t_2,y_2)}\left[\mathbf{1}_{\{t_2-s\le \tau_{B_{2aR}}\}}J(Y_{t_2-s},v)\right]\mu(dv)ds \\
= \; & \int_0^{t_2}\int_{B^c_{3aR}} h(s,v)\int_{B_{2aR}} p^{B_{2aR}}(t_2-s,y_2,z)J(z,v)\mu(dz)\mu(dv)ds && (5.2)\\
\ge \; & \int_0^{t_1}\int_{B^c_{3aR}} h(s,v)\int_{B_{2aR}} p^{B_{2aR}}(t_2-s,y_2,z)J(z,v)\mu(dz)\mu(dv)ds. \\
\ge \; & \int_0^{t_1}\int_{B^c_{3aR}} h(s,v)\int_{B_{3aR/2}} p^{B_{2aR}}(t_2-s,y_2,z)J(z,v)\mu(dz)\mu(dv)ds. && (5.3)
\end{aligned}
$$

Since

$$6\delta\phi(R)\ge t_2-s\ge t_2-t_1\ge \delta\phi(R) \qquad \text{for } s\in[0,\,t_1],$$

by (2.12), we have that the right hand side of (5.3) is greater than or equal to

$$\frac{c_1}{V(R)}\int_0^{t_1}\int_{B^c_{3aR}} h(s,v)\int_{B_{3aR/2}} J(z,v)\mu(dz)\mu(dv)ds.$$

So, the proof is complete once we obtain

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})]\le \frac{c_2}{V(R)}\int_0^{t_1}\int_{B^c_{3aR}} h(s,v)\int_{B_{3aR/2}} J(z,v)\mu(dz)\mu(dv)ds. \tag{5.4}$$

Analogous to (5.2), we have by using the Lévy system in (1.9),

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] = \int_0^{t_1}\int_{B^c_{3aR}} h(s,v)\mu(dv)\int_{B_{3aR/2}} p^{B_{3aR/2}}(t_1-s,y_1,z)J(z,v)\mu(dz)ds.$$

Since

$$
\begin{aligned}
& \int_{B_{3aR/2}} p^{B_{3aR/2}}(t_1-s,y_1,z)\int_{B^c_{3aR}} J(z,v)h(s,v)\mu(dv)\mu(dz) \\
= \; & \int_{B_{5aR/4}} p^{B_{3aR/2}}(t_1-s,y_1,z)\int_{B^c_{3aR}} J(z,v)h(s,v)\mu(dv)\mu(dz) \\
& + \int_{B_{3aR/2}\setminus B_{5aR/4}} p^{B_{3aR/2}}(t_1-s,y_1,z)\int_{B^c_{3aR}} J(z,v)h(s,v)\mu(dv)\mu(dz) \\
=: \; & I_1+I_2.
\end{aligned}
$$

When $z \in B_{3aR/2} \setminus B_{5aR/4}$, we have $aR/4 \le \rho(y_1, z) \le 5aR/2$, so by (2.10),

$$p^{B_{3aR/2}}(t_1 - s, y_1, z) \le c_3 \frac{t_1}{V(\rho(y_1, z))\phi(\rho(y_1, z))} \le c_3 \frac{2\delta\phi(R)}{V(aR/4)\phi(aR/4)} \le c_4 \frac{1}{V(R)}$$

for some constants $c_3, c_4 > 0$ and $\int_0^{t_1} I_2\, ds$ is less than or equal to the right hand side of (5.4). For $z \in B_{5aR/4}$ by $\mathbf{UJS}_{\le R_1}$,

$$\begin{aligned} \int_{B_{3aR}^c} J(z, v) h(s, v) \mu(dv) &\le \frac{c_5}{V(R)} \int_{B(z, aR/6)} \int_{B_{3aR}^c} J(w, v) h(s, v) \mu(dv) \mu(dw) \\ &\le \frac{c_5}{V(R)} \int_{B_{3aR/2}} \int_{B_{3aR}^c} J(w, v) h(s, v) \mu(dv) \mu(dw) \end{aligned}$$

since $B(z, aR/6) \subset B_{3aR/2}$. Note that the right hand side of the above inequality does not depend on z anymore. Multifplying both sides by $p^{B_{3aR/2}}(t_1 - s, y_1, z)$ and integrating over $z \in B(5R/4)$ (and further integrating over $\int_0^{t_1} ds$), we obtain $\int_0^{t_1} I_1 ds$ is less than or equal to the right hand side of (5.4). This proves the lemma. □

Proof of Theorem 5.2. Recall that R_* is the constant in Theorem 2.4 and without loss of generality we assume that $R_* \le R_1$. We first consider the case that u is non-negative and bounded on $[0, \infty) \times F$. Recall that $\gamma > 0$ is the constant in (2.15). It suffices to establish the result for $\delta \le \gamma$. Once this is done, we can extend it to all $\delta < 1$ by simply using the results for $\delta \le \gamma$ at most $[1/\gamma] + 1$ times. We divide the proof into three steps.

Step 1. $\mathbf{PHI}(\phi)_{\le (R_*/2) \wedge 10^{-1}}$ *holds for every non-negative and bounded parabolic function h on $[0, \infty) \times F$.*

Let $R \in (0, R_*/2 \wedge 1/10]$ and for simplicity assume that $t_0 = 0, x_0 = 0$, and denote $B(0, r)$ by B_r. Let $Q_2 := [t_0 + \delta\phi(R)/3,\ t_0 + 6\delta\phi(R)] \times B(x_0, 2R)$. Note that for every $\varepsilon \in (0, 1)$, by the Lévy system formula in (1.9) and (2.12)

$$\begin{aligned}
&\inf_{(t,y) \in Q_+} h(t, y) = \inf_{(t,y) \in Q_+} \mathbb{E}^{(t,y)}[h(Z_{\tau_{Q_2}})] \\
= &\inf_{(t,y) \in Q_+} \mathbb{E}^{(t,y)}[h(t - (\tau_{B_{2R}} \wedge (t - \delta\phi(R)/3)), X_{\tau_{B_{2R}} \wedge (t - \delta\phi(R)/3)})] \\
\ge &\inf_{(t,y) \in Q_+} \mathbb{E}^{(t,y)} \left[\int_0^{t - \delta\phi(R)/3} \mathbf{1}_{\{s \le \tau_{B_{2R}}\}} \int_{B_{3R}^c} h(t - s, v) J(X_s, v) \mu(dv) dt \right] \\
= &\inf_{(t,y) \in Q_+} \int_0^{t - \delta\phi(R)/3} h(t - s, v) \int_{B_{3R}^c} \mathbb{E}^{(t,y)}[\mathbf{1}_{\{s \le \tau_{B_{2R}}\}} J(X_s, v)] \mu(dv) ds \\
= &\inf_{(t,y) \in Q_+} \int_{\delta\phi(R)/3}^{t} h(s, v) \int_{B_{3R}^c} \mathbb{E}^{(t,y)} \left[\mathbf{1}_{\{t - s \le \tau_{B_{2R}}\}} J(X_{t-s}, v) \right] \mu(dv) ds \\
\ge &\inf_{y \in B_R} \int_{\delta\phi(R)/3}^{(3-\varepsilon)\delta\phi(R)} \int_{B_{3R}^c} h(s, v) \inf_{3\delta\phi(R) \le t \le 4\delta\phi(R)} \int_{B_{2R}} p^{B_{2R}}(t - s, y, z) J(z, v) \mu(dz) \mu(dv) ds \\
\ge &\int_{\delta\phi(R)/3}^{(3-\varepsilon)\delta\phi(R)} \int_{B_1 \setminus B_{3R}} h(s, v) \int_{B_R} c_{\varepsilon,\delta} V(R)^{-1} J(z, v) \mu(dz) \mu(dv) ds. \qquad (5.5)
\end{aligned}$$

Since $J(z,v)>0$ when $\rho(z,v)\le 1$ (due to (1.6) with $\kappa=1$), if $\inf_{(t,y)\in Q_+}h(t,y)=0$, then for every $\varepsilon\in(0,1)$, $h(s,v)=0$ for a.e. on $[\delta\phi(R)/3,(3-\varepsilon)\delta\phi(R)]\times(B_1\setminus B_{3R})$ and so

$$h(s,v)=0 \qquad \text{for a.e. } (s,v)\in[\delta\phi(R)/3,3\delta\phi(R)]\times(B_1\setminus B_{3R}).$$

Let $(t,v)\in[8\delta\phi(R)/3,3\delta\phi(R)]\times(B_{1-R}\setminus B_{4R})$ so that $h(t,v)=0$. Define $Q=[\delta\phi(R)/3,3\delta\phi(R)]\times B(v,2R)$. Then by a similar argument as that led to (5.5),

$$\begin{aligned}
0 &= h(t,v) = \mathbb{E}^{(t,v)}\left[h(Z_{\tau_Q})\right] \\
&\ge \mathbb{E}^{(t,v)}\left[h(Z_{\tau_Q});\, Y_{\tau_Q}\in B(v,1)\setminus B(v,3R)\right] \\
&= \int_{\delta\phi(R)/3}^{t}\int_{B(v,1)\setminus B(v,3R)} h(s,w)\int_{B(v,2R)} p^{B(v,2R)}(t-s,v,z)J(z,w)\mu(dz)\mu(dw)ds.
\end{aligned}$$

This implies that $h(s,w)=0$ a.e. on $(\delta\phi(R)/3,t)\times(B(v,1)\setminus B(v,3R))$. In particular, we have $h(s,w)=0$ a.e. on $Q_-=[\delta\phi(R),2\delta\phi(R)]\times B(0,R)$ and so Theorem 5.2 holds. Hence without loss of generality, assume that $\inf_{(t,y)\in Q_+}h(t,y)>0$.

The rest of the proof in this *Step 1* is the same as that for Theorem 4.12 in [CK2]. For reader's convenience, we spell out the details. Taking a constant multiple of h if needed, we may assume that

$$\inf_{(t,y)\in Q_+} h(t,y)=1/2.$$

Let $(t_*,y_*)\in Q_+$ be such that $h(t_*,y_*)\le 1$. It is enough to show that $h(t,x)$ is bounded from above in Q_- by a constant that is independent of the function h.

Recall that $Q^{\downarrow}(t,\delta,x,r)=[t-\delta\phi(r),t]\times B(x,r)$. By Lemma 2.6 (i), there exists $c_1\in(0,1/2)$ such that if $r\le R/2$ and $D\subset Q^{\downarrow}(t,\delta,x,r)$ with $m\otimes\mu(D)/m\otimes\mu(Q^{\downarrow}(t,\delta,x,r))\ge 2/3$, then

$$\mathbb{P}^{(t,x)}(\sigma_D<\tau_{Q^{\downarrow}(t,\delta,x,r)})\ge c_1. \tag{5.6}$$

Let $C_1=C_1(\delta)$ be the constant C_1 in Lemma 5.3 with $a=1/2$. Define

$$\eta=\frac{c_1}{3} \quad\text{and}\quad \xi=\frac{1}{3}\wedge(C_1^{-1}\eta). \tag{5.7}$$

We claim that there is a universal constant $K=K(\delta)$ to be determined later, which is independent of R and function h, such that $h\le K$ on Q_-. We are going to prove this by contradiction. Suppose this is not true. Then there is some point $(t_1,x_1)\in Q_-$ such that $h(t_1,x_1)>K$. We will show that there is a constant $\beta>0$ and there is a sequence of points $\{(t_k,x_k)\}$ in $\widehat{Q}(0,0,R):=[0,4\gamma\phi(R)]\times B(0,2R)$ so that $h(t_k,x_k)\ge(1+\beta)^{k-1}K$, which contradicts to the assumption that h is bounded on $\widehat{Q}(0,0,R)$.

Let $c'>0$ be a constant satisfying $\mu(B(x,r))\ge c'V(r)$ for every $x\in F, r>0$. so that

$$m\otimes\mu(Q^{\downarrow}(t_1,\delta,x_1,r))\ge c'\delta\phi(r)V(r). \tag{5.8}$$

Let c^* a constant satisfying

$$\frac{V(r)\phi(r)}{V(u)\phi(u)}\ge c^*\left(\frac{r}{u}\right)^{d+\beta_2}, \quad \text{for every } 0<r<u<\infty \tag{5.9}$$

and choose the smallest $c_* > 0$ satisfying

$$V(c_*)\phi(c_*) \geq \max\left(\frac{3}{c^*c'\delta C_2\xi}, \frac{2}{c^*C_3\xi}\right). \tag{5.10}$$

where C_2 and C_3 are the constants in Lemma 2.6 (i) and (ii) respectively. Choose K being sufficiently large, so that

$$r := R\left(\frac{V(c_*)\phi(c_*)}{K}\right)^{1/(d+\beta_2)} < \frac{R}{4}. \tag{5.11}$$

With such r, by (5.8)-(5.10), we have

$$\frac{m\otimes\mu(Q^{\downarrow}(t_1,\delta,x_1,r))}{V(R)\phi(R)} \geq \frac{3}{C_2\,\xi K} \qquad \text{and} \qquad \frac{V(r)\phi(r)}{V(R)\phi(R)} \geq \frac{2}{C_3\,\xi K}. \tag{5.12}$$

Take $\tilde{t} = t_1 + 2\delta\phi(r)$ and define $U = \{\tilde{t}\}\times B(x_1,r)$. Were function $h \geq \xi K$ on U, we would have by Lemma 2.6 (ii) that

$$1 \geq h(t_*,y_*) = \mathbb{E}^{(t_*,y_*)}\left[h(Z_{\sigma_U\wedge\tau_Q})\right] \geq \xi K\,\mathbb{P}^{(t_*,y_*)}(\sigma_U < \tau_Q) \geq \xi K\,\frac{C_3V(r)\phi(r)}{V(R)\phi(R)} \geq 2,$$

where $Q := [t-3\delta\phi(R),t]\times B(0,2R)$. This contradiction yields that

$$\text{there is some } y_1 \in B(x_1,r) \text{ such that } h(\tilde{t},y_1) < \xi K. \tag{5.13}$$

We next claim that

$$\mathbb{E}^{(t_1,x_1)}\left[h(Z_{\tau_r}) : Y_{\tau_r} \notin B(x_1,3r/2)\right] \leq \eta K, \tag{5.14}$$

where $\tau_r := \tau_{Q^{\downarrow}(t_1,\delta,x_1,r)}$. If not, then by Lemma 5.3, we would have

$$\begin{aligned}
h(\tilde{t},y_1) &\geq \mathbb{E}^{(\tilde{t},y_1)}\left[h(Z_{\tau_{[t_1-\delta\phi(r),t_1+3\delta\phi(r)]\times B(x_1,5r/4)}}) : Y_{\tau_{[t_1-\delta\phi(r),t_1+3\delta\phi(r)]\times B(x_1,5r/4)}} \notin B(x_1,3r/2)\right] \\
&\geq C_1^{-1}\mathbb{E}^{(t_1,x_1)}\left[h(Z_{\tau_r}) : Y_{\tau_r} \notin B(x_1,3r/2)\right] \\
&\geq C_1^{-1}\eta K \geq \xi K,
\end{aligned}$$

a contradiction to (5.13). So (5.14) holds.

Let A be any compact subset of

$$\widetilde{A} := \{(s,y) \in Q^{\downarrow}(t_1,\delta,x_1,r) : h(s,y) \geq \xi K\}.$$

By Lemma 2.6 (i)

$$1 \geq h(t_*,y_*) \geq \mathbb{E}^{(t_*,y_*)}[h(Z_{\sigma_A}) : \sigma_A < \tau_Q] \geq \xi K\mathbb{P}^{(t_*,y_*)}(\sigma_A < \tau_Q) \geq \xi K\,\frac{C_2\,m\otimes\mu(A)}{V(R)\phi(R)}.$$

So by (5.12),

$$\frac{m\otimes\mu(A)}{m\otimes\mu(Q^{\downarrow}(t_1,\delta,x_1,r))} \leq \frac{V(R)\phi(R)}{C_2\xi K\cdot m\otimes\mu(Q^{\downarrow}(t_1,\delta,x_1,r))} \leq \frac{1}{3}. \tag{5.15}$$

Since (5.15) holds for every compact subset A of $\widetilde{A}$, it holds for $\widetilde{A}$ in place of A.

Let $D = Q^{\downarrow}(t_1,\delta,x_1,r)\setminus \widetilde{A}$ and $M=\sup_{(s,y)\in Q^{\downarrow}(t_1,\delta,x_1,3r/2)} h(s,y)$. We write

$$\begin{aligned} h(t_1,x_1) &= \mathbb{E}^{(t_1,x_1)}\left[h(Z_{\sigma_D\wedge\tau_r})\right] \\ &= \mathbb{E}^{(t_1,x_1)}[h(Z_{\sigma_D}) : \sigma_D<\tau_r] + \mathbb{E}^{(t_1,x_1)}[h(Z_{\tau_r}) : \tau_r\le \sigma_D, Y_{\tau_r}\notin B(x_1,3r/2)] \\ &\quad + \mathbb{E}^{(t_1,x_1)}[h(Z_{\tau_r}) : \tau_r\le\sigma_D, Y_{\tau_r}\in B(x_1,3r/2)]. \end{aligned}$$

The first term on the right is bounded by $\xi K\,\mathbb{P}^{(t_1,x_1)}(\sigma_D<\tau_r)$ in view of Lemma 5.1 the second term is bounded by ηK according to (5.14), and the third term is clearly bounded by $M\mathbb{P}^{(t_1,x_1)}(\tau_r\le\sigma_D)$. Recall that $h(t_1,x_1)>K$. Therefore,

$$K \le \xi K\,\mathbb{P}^{(t_1,x_1)}(\sigma_D<\tau_r) + \eta K + M\,\mathbb{P}^{(t_1,x_1)}(\sigma_D\ge\tau_r).$$

It follows from (5.15) and (5.6)-(5.7)

$$M/K \ge \frac{1-\eta-\xi\,\mathbb{P}^{(t_1,x_1)}(\sigma_D<\tau_r)}{\mathbb{P}^{(t_1,x_1)}(\sigma_D\ge\tau_r)} \ge \frac{1-\eta-\xi}{1-c_1}+\xi = \frac{1-\eta-\xi\,c_1}{1-c_1} \ge \frac{1-(2c_1)/3}{1-c_1} := 1+\beta,$$

where $\beta=\frac{c_1}{3(1-c_1)}$. In other words, $M\ge(1+\beta)K$. As $M=\sup_{(s,y)\in Q^{\downarrow}(t_1,\delta,x_1,3r/2)} h(s,y)$, there exists a point $(t_2,x_2)\in Q^{\downarrow}(t_1,\delta,x_1,2r)\subset \widehat{Q}(0,0,R)$ such that $h(t_2,x_2)\ge(1+\beta)K=:K_2$.

We now iterate the above procedure to obtain a sequence of points $\{(t_k,x_k)\}$ in the following way. Using the above argument (with (t_2,x_2) and K_2 in place of (t_1,x_1) and K), there exists $(t_3,x_3)\in Q^{\downarrow}(t_2,\delta,x_2,2r_2)$ such that

$$r_2 = C'RK_2^{-1/(d+\beta_2)} = C'(1+\beta)^{-1/(d+\beta_2)}K^{-1/(d+\beta_2)}\,R$$

and $h(t_3,x_3)\ge(1+\beta)K_2=(1+\beta)^2K=:K_3$. We continue this procedure to obtain a sequence of points $\{(t_k,x_k)\}$ such that with

$$r_k := C'RK_k^{-1/(d+\beta_2)} = C'(1+\beta)^{-(k-1)/(d+\beta_2)}K^{-1/(d+\beta_2)}\,R,$$

$(t_{k+1},x_{k+1})\in Q^{\downarrow}(t_k,\delta,x_k,2r_k)$ and $h(t_{k+1},x_{k+1})\ge(1+\beta)^kK=:K_{k+1}$. As $0\le t_k-t_{k+1}\le 4\gamma r_k$ and $\rho(x_{k+1},x_k)\le 2r_k$, by (5.12), we can take K large enough (independent of R and h) so that $(t_k,x_k)\in\widehat{Q}(0,0,R)$ for all k. This is a contradiction because $h(t_k,x_k)\ge(1+\beta)^{k-1}K$ goes to infinity as $k\to\infty$ while h is bounded on $\widehat{Q}(0,0,R)$. We conclude that h is bounded by K in Q_-, which completes the proof of $\mathbf{PHI}(\phi)_{\le (R_*/2)\wedge 10^{-1}}$ for the case that u is non-negative and bounded on $[0,\infty)\times F$.

Step 2. $\mathbf{PHI}(\phi)_{\le R_2}$ *holds for every non-negative and bounded parabolic function* u *on* $[0,\infty)\times F$.

For notational convenience, denote $(R_*/2)\wedge 10^{-1}$ by r_*. Suppose $R\in(r_*,R_2]$ and let $(t_1,x_1)\in Q_-$ and $(t_2,x_2)\in Q_+$. Without loss of generality, we also assume $x_0=0$ and $t_0=0$. We further assume that $\rho(x_1,x_2)\le r_*/4$. If not, we just repeat the argument below at most $8[R_2/r_*]$ times

Set $B^1:=B(x_1,r_*)$ and $B^2:=B(x_1,r_*/2)$. Define

$$Q_1=\left(t_1+\tfrac{\delta}{2}\phi(r_*),\, t_1+\tfrac{3\delta}{4}\phi(r_*)\right)\times\left(B^1\setminus B^2\right) \quad\text{and}\quad Q_2=[0,t_2]\times B^2.$$

Since u is parabolic, by $\mathbf{PHI}(\phi)_{\le (R_*/2)\wedge 10^{-1}}$ with

$$\big(t_1 - \tfrac{\delta}{4}\phi(r_*),\, t_1 + \tfrac{\delta}{4}\phi(r_*)\big) \times B^1 \qquad \text{and} \qquad \big(t_1 + \tfrac{\delta}{2}\phi(r_*),\, t_1 + \tfrac{3\delta}{4}\phi(r_*)\big) \times B^1$$

in place of Q_- and Q_+ respective, we have

$$\begin{aligned} u(t_2,x_2) &= \mathbb{E}^{(t_2,x_2)}\Big[u(Z_{\tau_{Q_2}})\Big] \\ &\ge \mathbb{E}^{(t_2,x_2)}\Big[u(Z_{\tau_{Q_2}}) : Z_{\tau_{Q_2}} \in Q_1\Big] \ge c_1\, u(t_1,x_1)\, \mathbb{P}^{(t_2,x_2)}\left(Z\tau_{Q_2} \in Q_1\right). \end{aligned}$$

Since $\rho(y,z) < 2r_* \le R_*$ for every $(y,z) \in B^2 \times B^1$, we have by the Lévy system formula for X (see (1.9)) that

$$\begin{aligned} \mathbb{P}^{(t_2,x_2)}\Big(Z_{\tau_{Q_2}} \in Q_1\Big) &= \mathbb{P}^{x_2}\left(X_{\tau_{B^2}} \in B^1 \setminus B^2,\, t_2 - t_1 - \tfrac{3\delta}{4}\phi(r_*) < \tau_{B^2} < t_2 - t_1 - \tfrac{\delta}{2}\phi(r_*)\right) \\ &\ge c_2 \int_{t_2-t_1-\frac{3\delta}{4}\phi(r_*)}^{t_2-t_1-\frac{\delta}{2}\phi(r_*)} \int_{B^2} \left(\int_{B^1\setminus \overline{B^2}} p^{B^2}(s,x_2,y) J(y,z)\mu(dz) \right) \mu(dy)ds \\ &\ge c_3 \int_{t_2-t_1-\frac{3\delta}{4}\phi(r_*)}^{t_2-t_1-\frac{\delta}{2}\phi(r_*)} \int_{B^2} p^{B^2}(s,x_2,y)\mu(dy)ds \end{aligned}$$

for some positive constants c_2 and $c_3 = c_3(R_*)$. Note that

$$\tfrac{\delta}{4}\phi(r_*) \le t_2 - t_1 - \tfrac{3\delta}{4}\phi(r_*) \le t_2 - t_1 - \tfrac{\delta}{2}\phi(r_*) \le 3\delta\phi(R_2) \le 3\delta c_*\phi(r_*)$$

where c_* depends on R_* and R_2. Applying (2.12) to $p^{B^2}(s,x_2,y)$, we have

$$\begin{aligned} \mathbb{P}^{(t_2,x_2)}\Big(Z_{\tau_{Q_2}} \in Q_1\Big) &\ge c_4 \int_{t_2-t_1-\frac{3\delta}{4}\phi(r_*)}^{t_2-t_1-\frac{\delta}{2}\phi(r_*)} \int_{B(x_1,r_*/8)} \frac{1}{V(\phi^{-1}(s))}\mu(dy)ds \\ &\ge c_5\, \frac{\delta\phi(r_*)V(r_*)}{4V(\phi^{-1}(3\delta\phi(R_2)))} > 0. \end{aligned}$$

This proves that $u(t_2,x_2) \ge c_6 u(t_1,x_1)$ for some positive constant $c_6 = c_6(R_*,R_2,\delta)$; that is, $\mathbf{PHI}(\phi)_{\le R_2}$ holds for every non-negative and bounded parabolic function u on $[0,\infty)\times F$.

Step 3. $\mathbf{PHI}(\phi)_{\le R_2}$ *holds for any non-negative parabolic function u (not necessarily bounded) on $[0,\infty)\times F$*

Let U be a bounded domain such that $\overline{Q_- \cup Q_+} \subset U \subset \overline{U} \subset (t_0, t_0 + 6\delta\phi(R)) \times B(x_0,4R)$. For any $n \in \mathbb{N}$, define $u_n(t,x) = \mathbb{E}^{(t,x)}[(u\wedge n)(Z_{\tau_U})]$. Then u_n is non-negative and bounded on $[0,\infty)\times F$, parabolic on U and $\lim_{n\to\infty} u_n(t,x) = u(t,x)$ for $x \in [0,\infty)\times F$. From the above arguments, we see that $\mathbf{PHI}(\phi)_{\le R_2}$ holds for u_n with the constant c independent of n. Letting $n\to\infty$, we obtain $\mathbf{PHI}(\phi)_{\le R_2}$ for u. □

Let ψ be an increasing function on $[0,\infty)$ with $\psi(r) = 1$ for $0 < r \le 1$ and

$$c_1 e^{\gamma_1 r} \le \psi(r) \le c_2 e^{\gamma_2 r} \qquad \text{for every } 1 < r < \infty$$

for some $c_1, c_2 > 0$ and $\gamma_2 \geq \gamma_1 > 0$. We assume that there exists $c_3, c_4, c_5 > 0$ such that, for every $(x,y) \in F \times F \setminus d$,

$$\frac{1}{c_3 V(\rho(x,y))\phi(\rho(x,y))\psi(c_4\rho(x,y))} \leq J(x,y) \leq \frac{c_3}{V(\rho(x,y))\phi(\rho(x,y))\psi(c_5\rho(x,y))}. \tag{5.16}$$

As mentioned in Remark 1.3(ii), some additional condition on ψ is needed for the validity of Theorem 1.2 in [CK2] for the case of $\gamma_2 \geq \gamma_1 > 0$. Using Theorem 5.2 (instead of [CK2, Theorem 4.12]) and Section 4.9 of [CK2], we get the following corrected form of [CK2, Theorem 1.2] with an additional condition $\mathbf{UJS}_{\leq R}$ imposed.

Theorem 5.4 *Suppose that the measure μ and the jumping kernel J satisfy conditions* (1.1)-(1.5), (5.16) *and* $\mathbf{UJS}_{\leq R}$ *for some $R > 0$. Then Y has joint continuous transition density function $p(t,x,y)$ on $(0,\infty) \times F \times F$ with respect to μ and there are positive constants c_6, c_7 and $C \geq 1$ such that*

$$\begin{aligned} &C^{-1}\left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))\psi(c_6\rho(x,y))}\right) \\ &\leq p(t,x,y) \leq C\left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))\psi(c_7\rho(x,y))}\right), \end{aligned}$$

for every $t \in (0,1]$ and $x, y \in F$

Note that [CK2, Theorem 4.12] (or our Theorem 5.2) is not needed in [CK2] to get the upper bound of the heat kernel.

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