Anomalous random walks and diffusions: From fractals to random media

Takashi Kumagai

(RIMS, Kyoto University, Japan)

http://www.kurims.kyoto-u.ac.jp/~kumagai/

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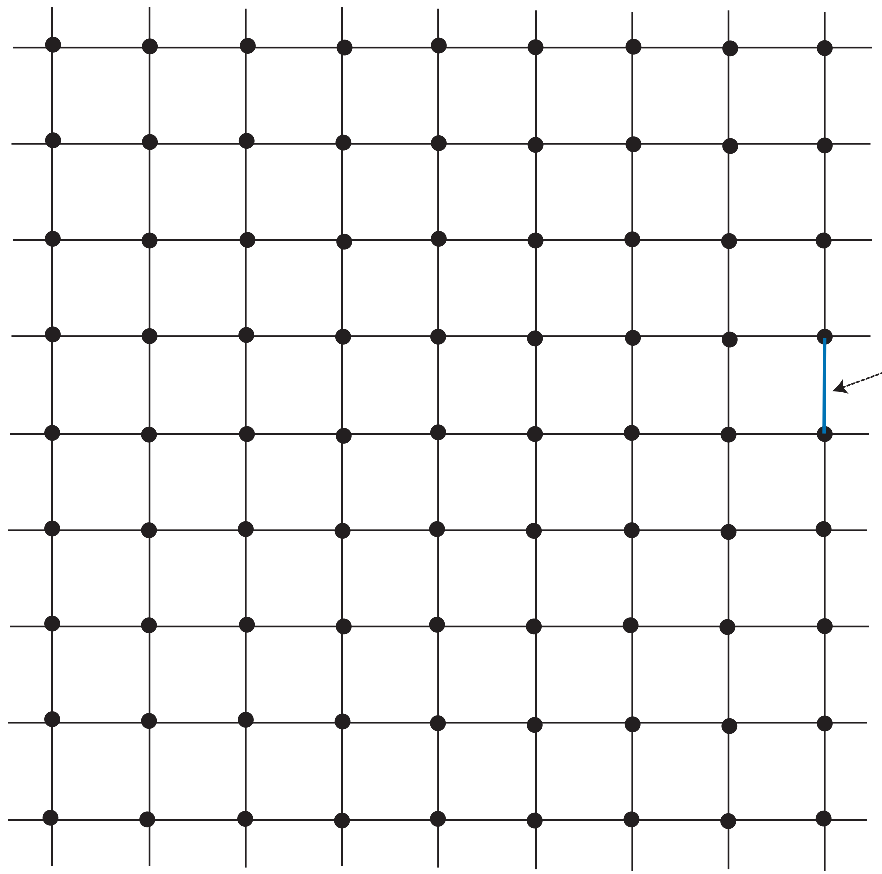
1 Introduction

Bond percolation on $\mathbb{Z}^d \, (d \geq 2)$



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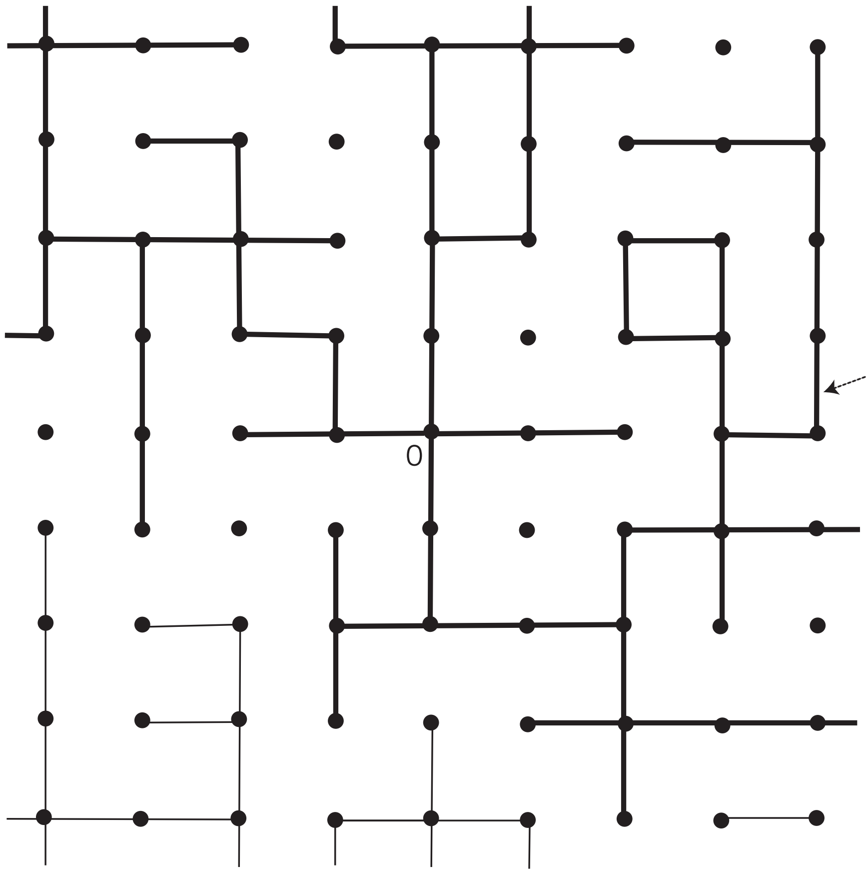

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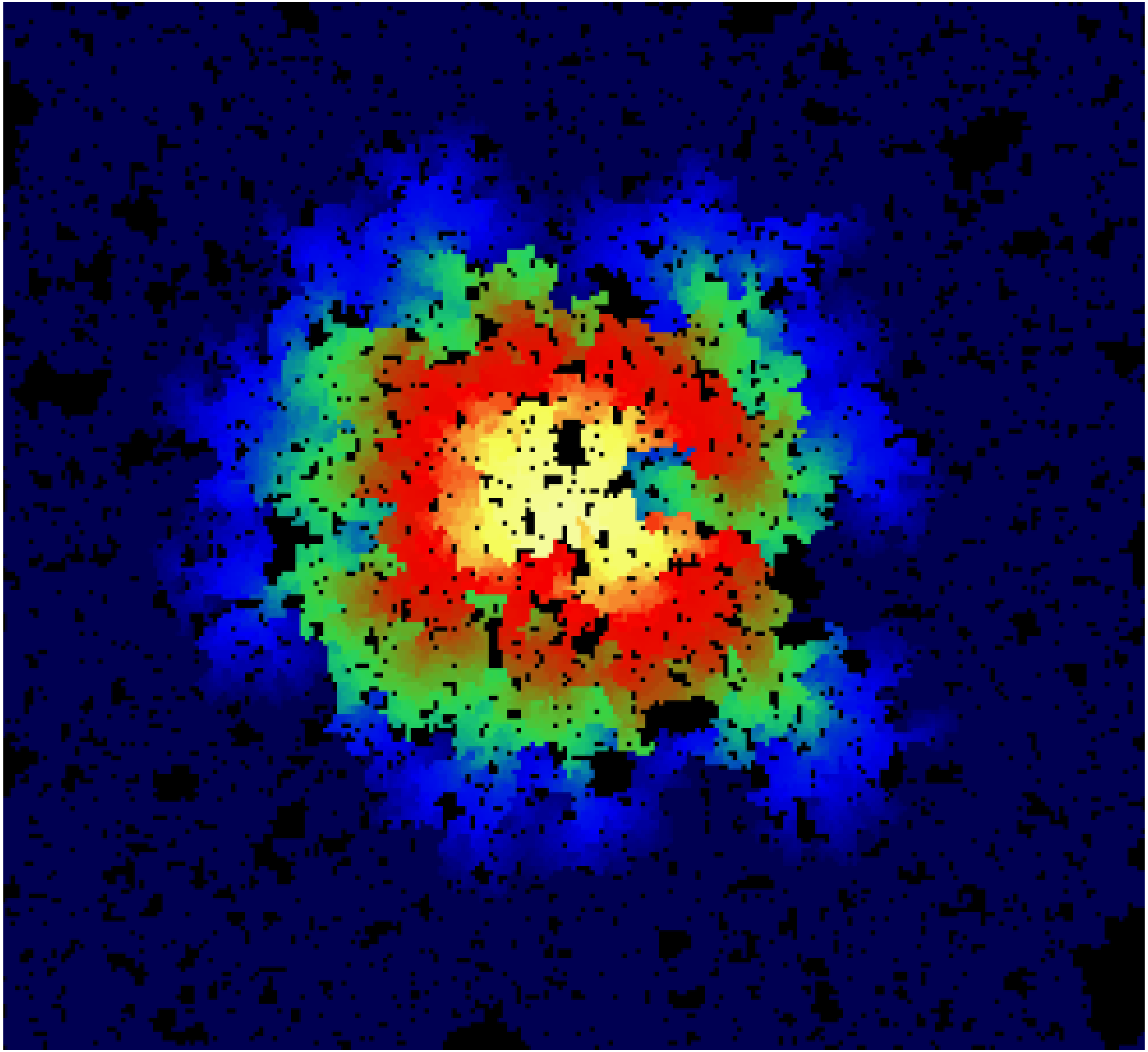

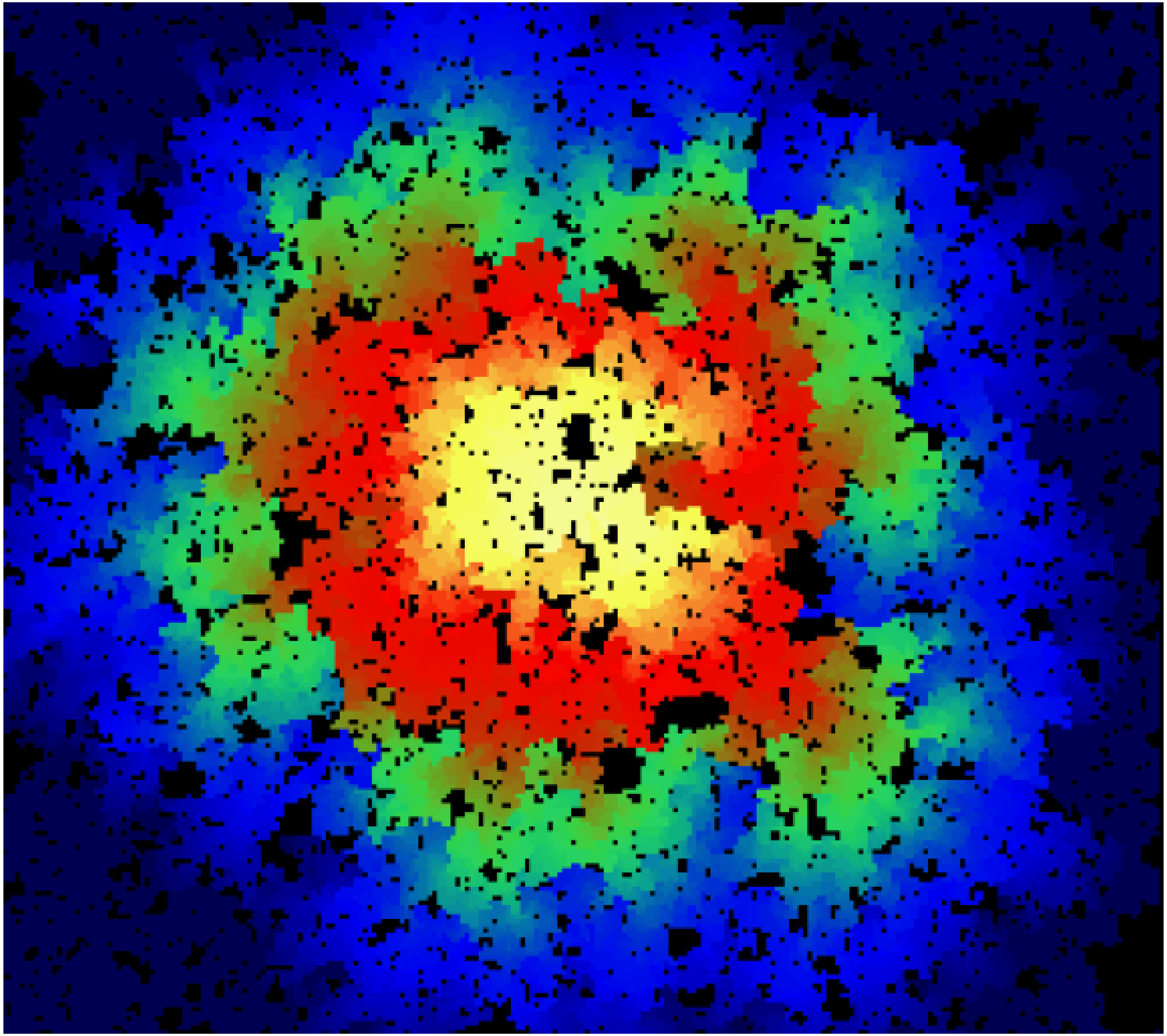

‘Anomalous’ behavior of the random walk at critical probability.

Let $p_n^\omega(x,y) := P_\omega^x(Y_n = y)/\mu_y$ and

$d_s = -2\lim_{n\to\infty} \log p_{2n}^\omega(x,x)/\log n$: Spectral dimension.

Alexander-Orbach conjecture (J. Phys. Lett., ’82)

$d \geq 2 \Rightarrow$ $d_s = 4/3$ (NOT d).

(It is now believed that this is false for small d.)

Motivations and Historical Remark

Analyze "anomalous" random walks or diffusions on disordered media

Math. Physicists' work since late 60's

Survey: Ben-Avraham and S. Havlin ('00)

Detailed study of heat conduction and wave transmission on

- Complicated network $\Rightarrow$ Random walk on fractals Rammal-Toulose ('83) etc.
- Random models at critical probability (Percolation cluster etc.)

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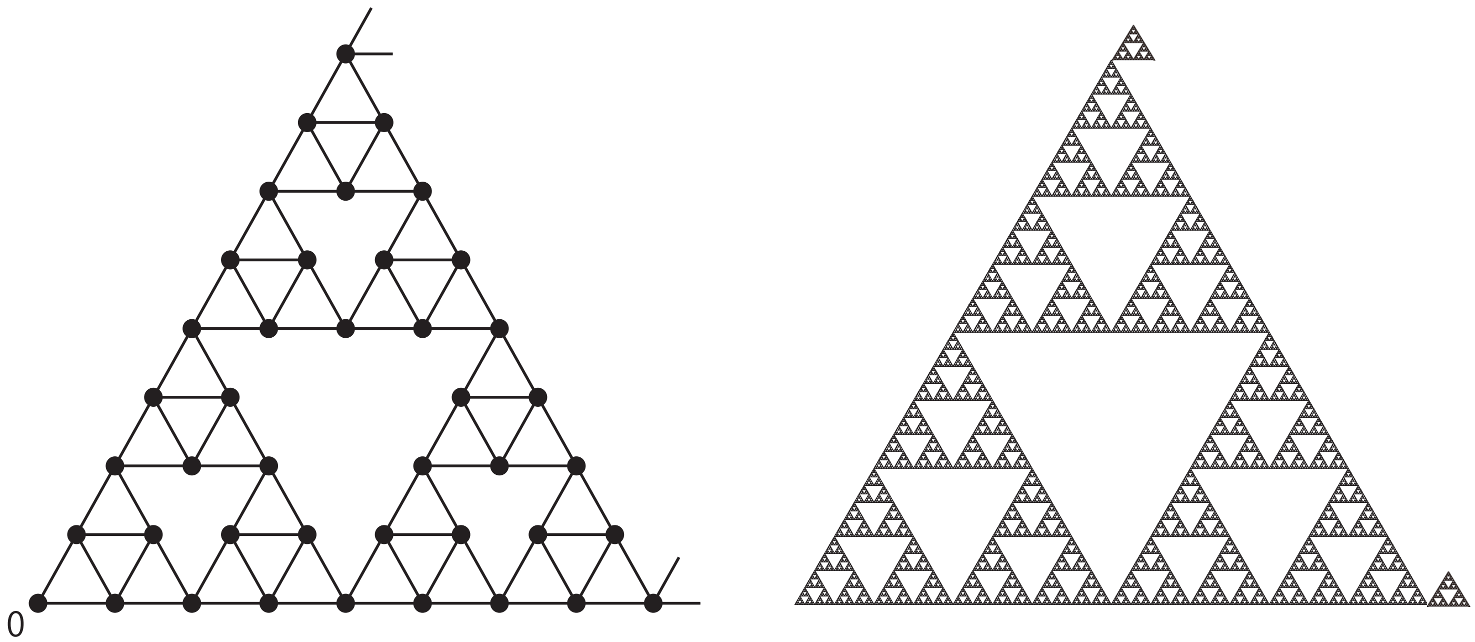
$\Rightarrow$ Late 80's∼: Kesten ('86) anomalous behavior of RW on the critical perco. cluster

$\Rightarrow$ Diffusions / analysis on fractals (Fractals are "ideal" disordered media)

$\Rightarrow$ Stability theory, global analysis $\Rightarrow$ Applications to random media

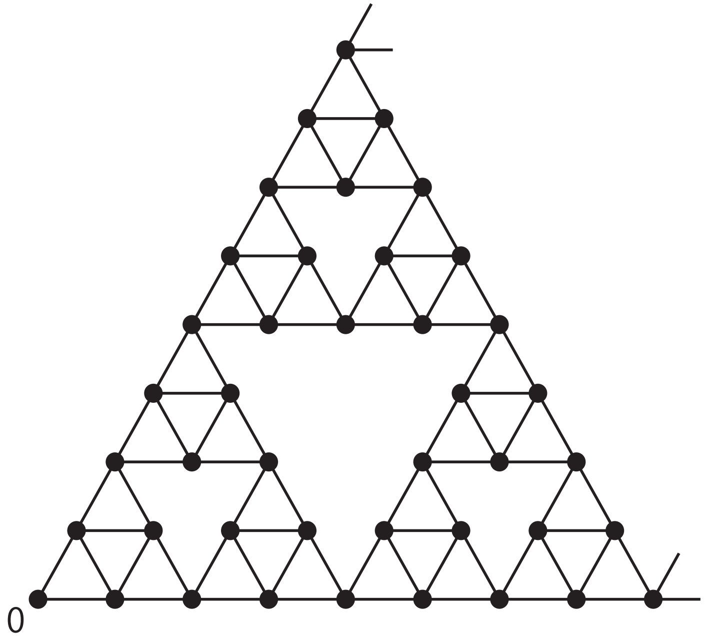
Percolation clusters , Erdős-Rényi random graphs , Uniform spanning trees

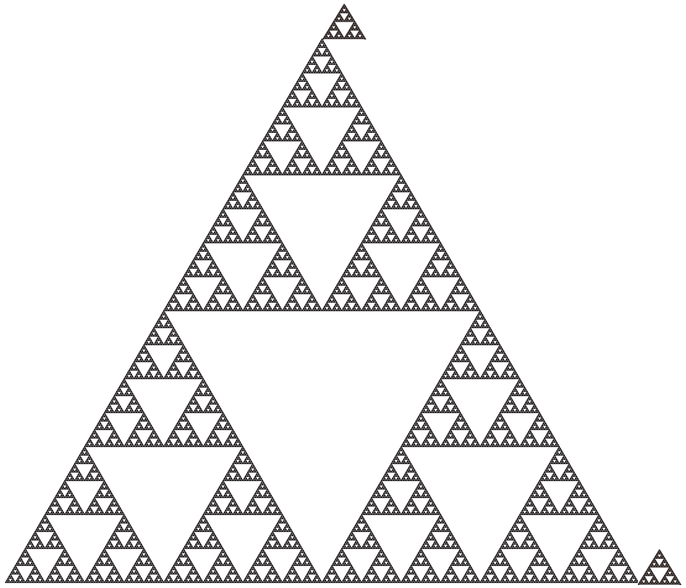


2 Anomalous heat transfer on fractals

G: pre-Sierpinski gasket (left figure), M: Sierpinski gasket (right figure)

$\{Y(n) : n = 0, 1, 2, \cdots\}$: simple random walk (SRW) on G





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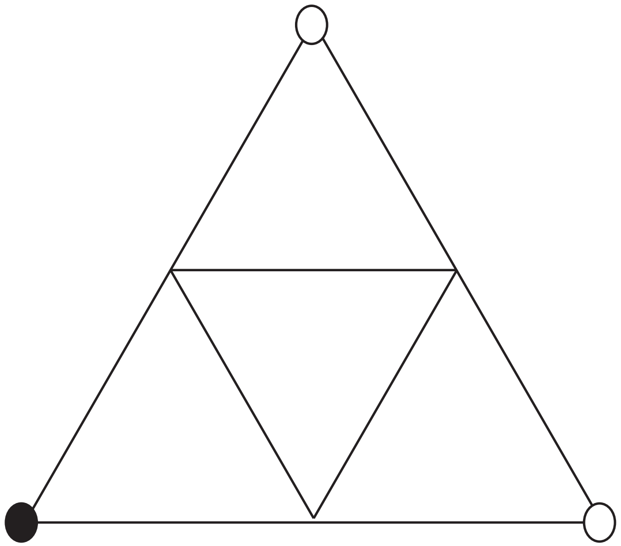
$$2^{-n}Y([5^n t]) \xrightarrow{n\to\infty} B_t : \text{ Brownian motion on } M \text{ [Goldstein '87, Kusuoka '87]}$$

$$\Delta f(x) := \lim_{n\to\infty} 5^n\Big(\frac{1}{4}\sum_{x_i \overset{n}{\sim} x} f(x_i) - f(x)\Big) : \text{ Laplacian on } M \text{ [Kigami '89]}$$



$E^{\bullet}[\sigma_{\circ}] = 5$

Cf.

$E^{\bullet}[\sigma_{\circ}] = 4$

$$2^{-n} Y([5^n t]) \xrightarrow{n\to\infty} B_t : \text{ Brownian motion on } M$$

Cf. Invariance principle on $\mathbb{R}_+$ $\{\tilde{Y}(i)\}$: SRW on $\mathbb{Z}_+$

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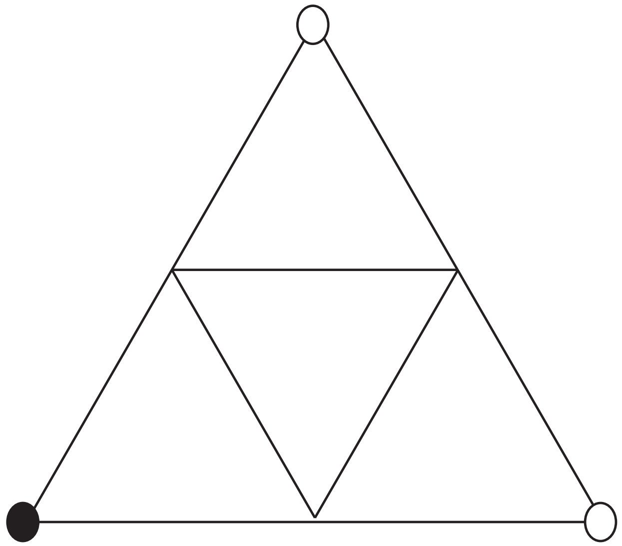

$E^{\bullet}[\sigma_\circ] = 5$

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$$2^{-n}Y([5^n t]) = 2^{-n}Y([2^{d_w n}t]) \xrightarrow{n\to\infty} B_t : \text{ Brownian motion on } M$$

$d_w = \log 5/\log 2 > 2$ is called a walk dimension.

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Theorem 2.1 [Barlow-Perkins '88] Heat kernel estimates (HK(d_w))

$\exists p_t(x,y)$: jointly continuous heat kernel (HK) w.r.t. μ (Hausdorff meas.)

$(P_t f(x) := E^x[f(B(t))] = \int_M p_t(x,y)f(y)\mu(dy)\ \forall x \in M,\ \frac{\partial}{\partial t}p_t(x_0,x) = \Delta p_t(x_0,x)$) s.t.

$$c_1 t^{-d_s/2}\exp(-c_2(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}) \le p_t(x,y) \le c_3 t^{-d_s/2}\exp(-c_4(\frac{d(x,y)^{d_w}}{t})^{\frac{1}{d_w-1}}).$$

$d_f := \log 3/\log 2$: Hausdorff dim., $d_s = 2\log 3/\log 5 < 2$: spectral dim.

Note $d_s/2 = d_f/d_w$: called the Einstein relation. (Cf. BM on $\mathbb{R}^d$: $d_s = d_f = d, d_w = 2$.)

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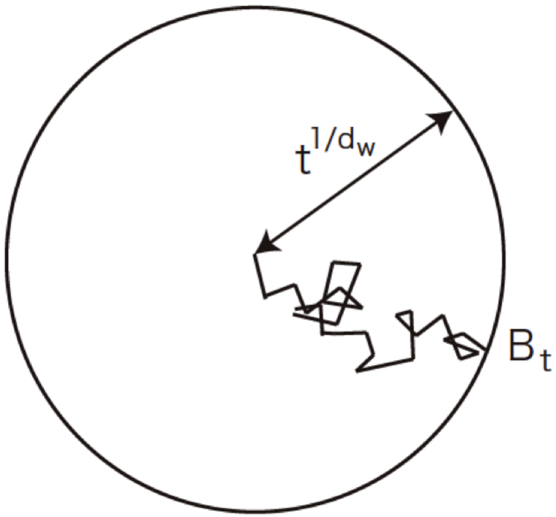
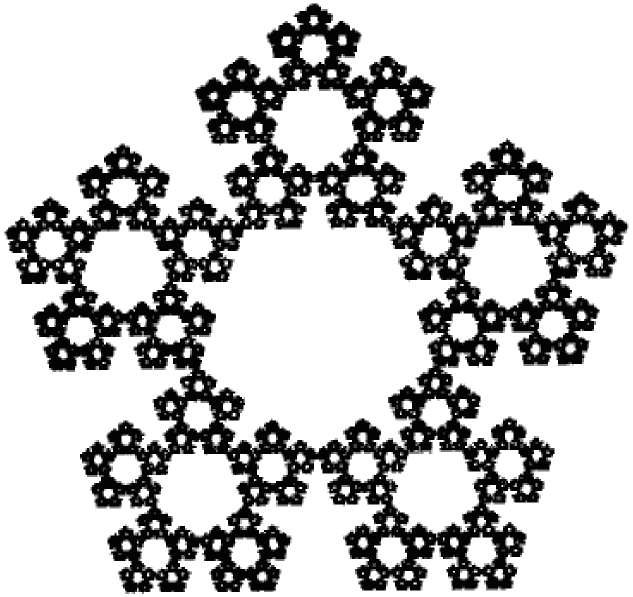
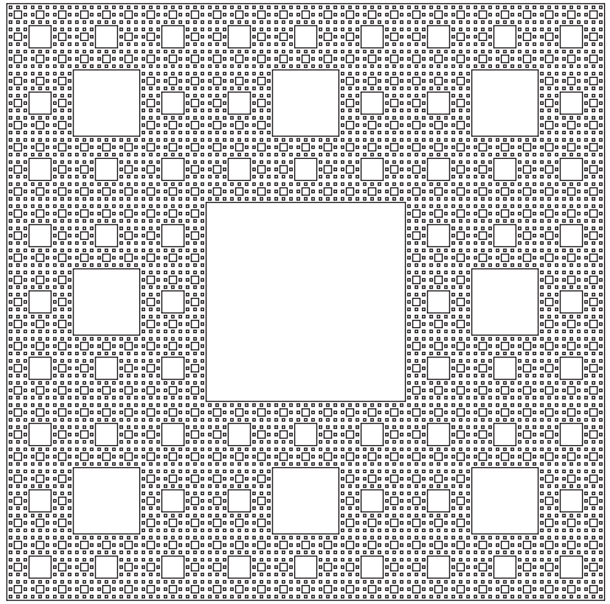
From (HK(d_w)), many properties can be deduced!



- $c_1 t^{1/d_w} \le E^0[d(0,B_t)] \le c_2 t^{1/d_w}$ ($d_w > 2$, sub-diffusive)
- Hölder continuity of harmonic and caloric functions.
- Estimates of Green functions • Laws of iterated laws etc.

Construction of BM and estimates such as (HK (d_w)): Done on various fractals.

(d_f, d_w and d_s depend on fractals.)

Open Prob. Existing construction of BM on the carpet (e.g. [Barlow-Bass '99]) requires detailed uniform control of harmonic functions on the approximating proc.

Construct BM on the carpet without such detailed information.

3 Stability of parabolic Harnack inequalities and sub-Gaussian heat kernel estimates

Sierpinski gasket is "Too ideal"

(Q) Is the heat kernel estimate "stable" under some perturbation?

Back to the classical case

[Aronson '67] $\mathcal{L} = \sum_{i.j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^d$: sym. and uniform elliptic (i.e. $c_1 I \le A(x) = (a_{ij}(x))_{i,j} \le c_2 I$), then (HK(2)) holds.

$$c_1 t^{-d/2} \exp(-c_2 \frac{|x-y|^2}{t}) \le p_t(x,y) \le c_3 t^{-d/2} \exp(-c_4 \frac{|x-y|^2}{t}). \quad (HK(2))$$

[Li-Yau '86] M: Non-cpt R-mfd, Ricci ≥ 0, Δ: Laplace-Beltrami $\Rightarrow$ (HK(2)) holds.

(Q): Stability of (HK(2))?

Assume that the HK for a Dirichlet form $\mathcal{E}$, $\mathcal{E}(f,f) = -\int_M f(x)\mathcal{L}f(x)dx$, satisfies (HK(2)) and $\mathcal{E}'(f,f) \asymp \mathcal{E}(f,f)$ for all f. Does the HK of $\mathcal{E}'$ satisfy (HK(2))?

$\Rightarrow$ YES! By the following characterization of (HK(2)).

(M, d, μ): metric measure space, $\mathcal{E}$: 'nice' Dirichlet form on $L^2(M, \mu)$

[Grigor'yan '92, Saloff-Coste '92, Sturm '96, Delmotte '99]

$$(VD) + (PI(2)) \Leftrightarrow (PHI(2)) \Leftrightarrow (HK(2)).$$

- (VD): volume doubling condition

$$\mu(B(x, 2R)) \leq c_1 \mu(B(x, R)) \qquad \forall \ x \in M, R > 0.$$

- (PI(2)): scaled Poincaré inequality $\forall B_R = B(x_0, R)$, $R > 0$

$$\int_{B_R} (f(x) - \bar{f}_{B_R})^2 \mu(dx) \leq c_1 R^2 \, \mathcal{E}_{B_R}(f, f), \quad \forall f$$

where $\bar{f}_B = \mu(B)^{-1} \int_B f(x) \mu(dx)$.

- (PHI(2)): parabolic Harnack inequality of order 2. 'Regularity' of caloric functions

Theorem 3.4 [Barlow-Bass '03, Barlow-Bass-K '06, Andres-Barlow '13]

$$(VD) + (PI(\beta)) + (CS(\beta)) \Leftrightarrow (PHI(\beta)) \Leftrightarrow (HK(\beta)).$$

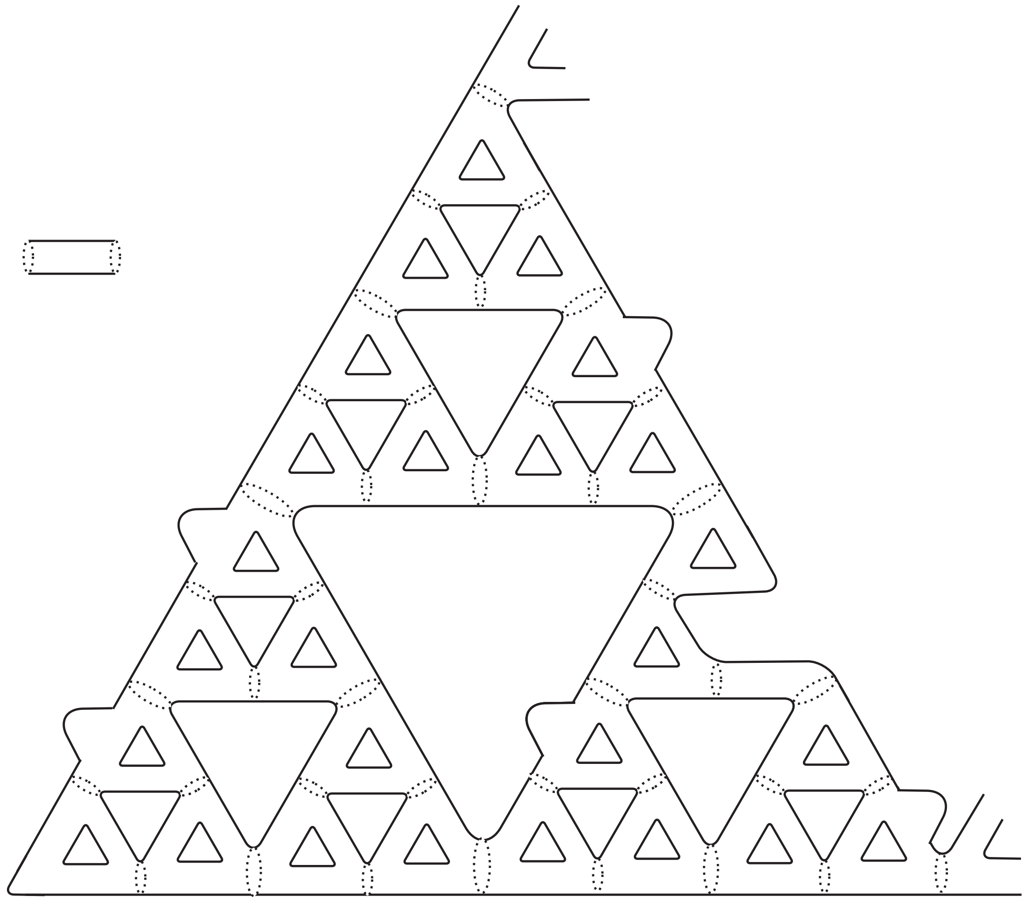
(CS (β)): cut-off Sobolev inequality **Remark.** (CS(2)) always holds.

$$\frac{c_1}{\mu(B(x,t^{1/\beta}))} \exp\big(-c_2(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big) \le p_t(x,y) \le \frac{c_3}{\mu(B(x,t^{1/\beta}))} \exp\big(-c_4(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big). \quad (HK(\beta))$$

Remark. Gasket case: $\beta = d_w = \log 5/\log 2$, $\mu(B(x,t^{1/\beta})) = t^{d_f/d_w} = t^{d_s/2}$.

[The theorem still holds if s^β is replaced by $1_{\{s\le 1\}}s^{\beta_1} + 1_{\{s>1\}}s^{\beta_2}$.]

$\Rightarrow$ Stability of (HK(β)) is established.



Fractal-like manifold

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$\Rightarrow$ Stability of (HK(β)) is established.

BUT (CS (β)) is hard to verify! **Open Prob.** Provide a simpler cond.

Strongly recurrent case: simpler equiv. condition [Barlow-Coulhon-K '05]

$\Rightarrow$ Applicable for random media.

4 Random walk on percolation clusters

4.1 Supercritical case

$(\Omega, \mathcal{F}, \mathbb{P})$: prob. space for the random media, $\mathcal{G} = \mathcal{G}(\omega)$: unique ∞-cluster

$\{Y_n^\omega\}_{n\geq 0}$: SRW on $\mathcal{G}(\omega)$ $p_n^\omega(x,y) := P_\omega^x(Y_n = y)/\mu_y$. ($\mu_y$: $\sharp$ of bonds con. to y.)

Although the media is not 'uniform elliptic', long time behavior is NOT anomalous.

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Theorem 4.1 [Barlow '04] (Gaussian heat kernel estimates)

(HK(2)) holds $\mathbb{P}$-a.s. ω for $t \geq d(x,y) \vee \exists U_x$, $x, y \in \mathcal{G}(\omega)$.

Theorem 4.2 [Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07]

(Quenched invariance principle) $n^{-1}Y_{n^2t}^\omega \to B_{\sigma t}$ $\mathbb{P}$-a.s. ω for some $\sigma > 0$

– Cf. "Annealed" invariance principle: known since 80's

[Kipnis-Varadhan '86, De Masi-Ferrari-Goldstein-Wick '89 ($\sigma > 0$)]

$\Rightarrow$ Extensions to random conductance models. (Skip.)

4.2 Critical case

Percolation on $\mathbb{Z}^d$ with $d > 6$ (rigorously proved for $d \geq 15$)

Let $\mathcal{C}(0)$ be the set of vertices connected to 0 by open bonds (random media!)

At $p = p_c$, $\mathcal{C}(0)$ is a finite cluster with prob. 1!

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(But, in any box of side n, $\exists$ open clusters of diam. $\asymp n$ w.h.p.)

$\Rightarrow$ Consider incipient infinite cluster (IIC). $\mathbb{P}_{\mathrm{IIC}}(\cdot) := \lim_{n\to\infty} \mathbb{P}_{p_c}(\cdot|0 \leftrightarrow \partial B(0,n))$

(I.e. at the critical prob., conditioned on $|\mathcal{C}(0)| = \infty$.)

Belief: Local prop. of the large finite clusters can be captured

by regarding them as subsets the IIC.

Existence of the IIC known for this model. [van der Hofstad-Járai '04]

$(\mathcal{G}(\omega), \omega \in \Omega)$: IIC, $d \geq 15$, $\{Y_n^\omega\}_{n\geq 0}$: SRW on $\mathcal{G}(\omega)$

Theorem 4.4 [Kozma-Nachmias '09] $\exists a_1, a_2 \geq 0$ s.t. the following hold.

(*i*) $(\log n)^{-a_1} n^{-2/3} \leq p_{2n}^\omega(x,x) \leq (\log n)^{a_1} n^{-2/3}$, for large n , $\mathbb{P} - a.s.$

Especially, $d_s(G(\omega)) = \frac{4}{3}$, $\mathbb{P}$–a.s. ω (solves the Alexander-Orbach conjecture).

(*ii*) $(\log R)^{-\alpha_2} R^3 \leq E_\omega^x \tau_{B(0,R)} \leq (\log R)^{\alpha_2} R^3$, for large R, $\mathbb{P} - a.s.$,

where $\tau_A := \inf\{n \geq 0 : Y_n^\omega \notin A\}$.

Why 2/3?

General result: Volume + Resistance $\Rightarrow$ HK estimates

$(\mathcal{G}(\omega), \omega \in \Omega)$: random graph on $(\Omega, \mathcal{F}, \mathbb{P})$, $\exists 0 \in \Omega$ and $D \geq 1$.

For $R, \lambda \geq 1$, we say $B(0, R)$ is λ-good if

$$\frac{R^D}{\lambda} \leq |B(0, R)| \leq \lambda R^D, \quad \frac{R}{\lambda} \leq R_{\text{eff}}(0, B(0, R)^c) \leq R.$$

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Theorem 4.5 [Barlow-Járai-K-Slade '08, K-Misumi '08]

If $\exists q_0$ s.t. $\mathbb{P}(\{\omega : B(0,R) \text{ is } \lambda\text{-good}\}) \geq 1 - \lambda^{-q_0}$, for large R, λ — (*).

$\Rightarrow \exists \alpha_1, \alpha_2 > 0$ s.t. for $\mathbb{P}$-a.s. ω and $x \in \mathcal{G}(\omega)$, $\exists N_x(\omega), R_x(\omega) \in \mathbb{N}$ the following hold

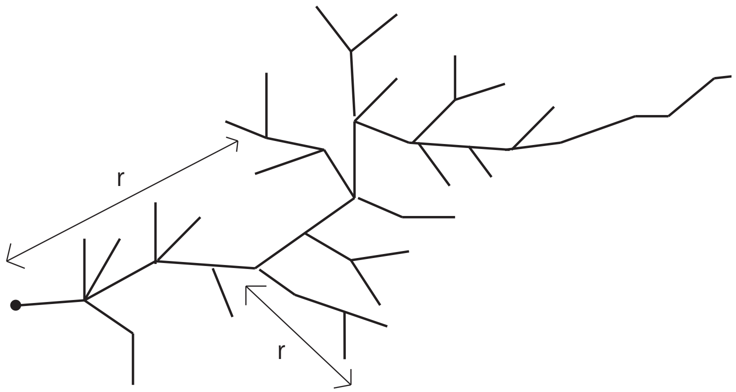
(i) $$(\log n)^{-\alpha_1} n^{-\frac{D}{D+1}} \leq p_{2n}^{\omega}(x,x) \leq (\log n)^{\alpha_1} n^{-\frac{D}{D+1}} \quad \text{for } n \geq N_x(\omega),$$

(ii) $$(\log R)^{-\alpha_2} R^{D+1} \leq E_{\omega}^{x} \tau_{B(0,R)} \leq (\log R)^{\alpha_2} R^{D+1} \quad \text{for } n \geq R_x(\omega).$$

Especially, $d_s(\mathcal{G}(\omega)) = \frac{2D}{D+1} < 2$, $\mathbb{P}$–a.s. ω , and the RW is recurrent.

IIC for high dim. percolation satisfies (*) with $D = 2$.

Open Prob. Provide a simpler sufficient condition for $d_s \geq 2$.



IIC (for Galton-Watson branching tree): $D = 2$

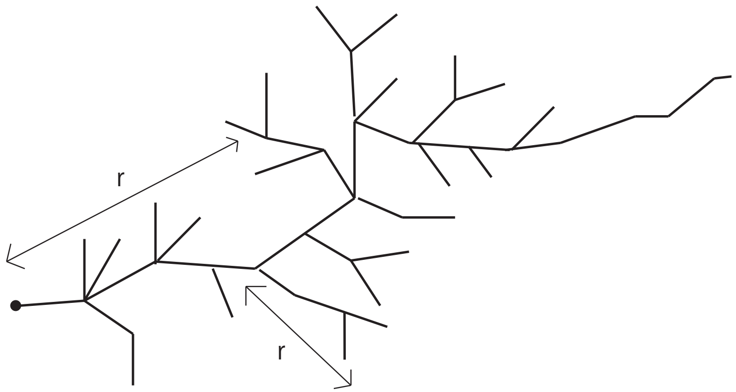
Other examples. (i) Infinite incipient cluster (IIC) for Galton-Watson branching tree

[Barlow-K '06] $D = 2$ and $d_s = 4/3$ — Quenched versions of Kesten's ('86) results.

(ii) IIC for spread out oriented percolation for $d \geq 6$

[Barlow-Jarai-K-Slade '08] ($d \leq 5$ No! for Branching RW [Jarai-Nachmias '13])

(iii) Invasion percolation on a regular tree. [Angel-Goodman-den Hollander-Slade '08]



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(iv) IIC for α-stable GW trees [Croydon-K '08] $D = \alpha/(\alpha - 1), d_s = 2\alpha/(2\alpha - 1)$

(v) 2-dim. uniform spanning trees [Barlow-Masson '11] $D = 8/5 = 2/(5/4)$, $d_s = 13/5$

Below critical dimensions

- RW on the IIC for 2-dimensional critical percolation [Kesten '86]

  (a) $\exists$ of IIC for 2-dimensional crit. perco. cluster is proved.

  (b) Subdiffusive behavior of SRW on IIC is proved in the following sense.

  $\exists \epsilon > 0$ s.t. the $\mathbb{P}$-distribution of $n^{-\frac{1}{2}+\epsilon} d(0, Y_n)$ is tight.

[Damron-Hanson-Sosoe '13] $\tau_{B(0,n)} \geq n^{2+\varepsilon}$ for large n, $\mathbb{P}$-a.s. and a.e. RW path

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Remark. A-O conjecture is believed to holds for $d > 6$ (Critical dimension is $d = 6$)

Numerical simulations suggest that A-O conjecture is false for $d \leq 5$.

$$d = 5 \Rightarrow d_s = 1.34 \pm 0.02, \quad \cdots, \quad d = 2 \Rightarrow d_s = 1.318 \pm 0.001$$

Open Prob. Disprove the Alexander-Orbach conjecture in low dimensions.

Other examples in low dimensional random media

- RW on the uniform infinite planar triangulation ($D = 4$)

  [Benjamini-Curien '13, Gurel-Gurevich and Nachmias '13]

- Liouville BM [Garban-Rhodes-Vargas '13, Berestycki '13,

  Maillard-Rhodes-Vargas-Zeitouni '14, Andres-Kajino '14]

- BM on the critical percolation cluster for the diamond lattice [Hambly-K '10]

- RW on the non-intersecting two-sided random walk trace on $\mathbb{Z}^2$ and $\mathbb{Z}^3$

  [Shiraishi '14]

Open Prob. 1) Lower dimensional models: prove the existence of d_s, d_w.

2) Compute resistance for random media when it is not linear order.

5 Scaling limits of random walks on random media

Ex. 0 T^N: rooted critical Galton-Watson tree (finite var.), cond. to have N vertices.

- Scaling limit of T^N is the cont. random tree $\mathcal{T}$ (Aldous '91). Y^N: SRW on T^N.

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5 Scaling limits of random walks on random media

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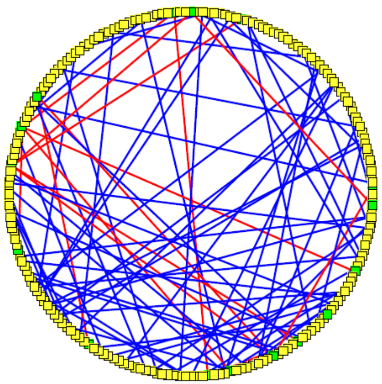
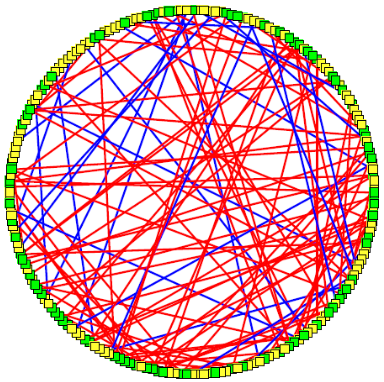
Ex. 1 Erdős-Rényi random graph in critical window

$G(N,p)$: Erdős-Rényi random graph I.e. $V_N := \{1, 2, \cdots, N\}$ vertices

Percolation on the complete graph: each bond open w.p. $p \sim c/N$.

$\mathcal{C}^N$: largest con. comp. E.g. $N = 200, c = 0.8$ $N = 200, c = 1.2$ Pictures by C. Goldschmidt.





Critical window: $p = 1/N + \lambda N^{-4/3}$ for fixed $\lambda \in \mathbb{R}$ $\Rightarrow |\mathcal{C}^N| \asymp N^{2/3}$. (Aldous '97)

- [Addario-Berry, Broutin, Goldschmidt '12]: $\exists \mathcal{M} = \mathcal{M}_\lambda$ (random compact set) s.t.

$$N^{-1/3}\mathcal{C}^N \xrightarrow{d} \exists \mathcal{M} = \mathcal{M}_\lambda \qquad \text{(Gromov-Hausdorff sense).}$$

Theorem 5.1 [Croydon '12] $\{N^{-1/3} Y^{\mathcal{C}^N}_{[Nt]}\}_{t \geq 0} \xrightarrow{d} \{B^{\mathcal{M}}_t\}_{t \geq 0}$: BM on $\mathcal{M}$

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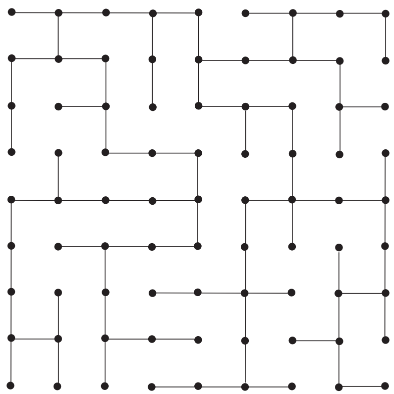
Ex. 2 2-dimensional uniform spanning tree (UST)

$\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$, let $\mathcal{U}^{(n)}$ be a spanning tree on Λ_n (no cycle)

– choose uniformly at random among all spanning trees



$\mathcal{U}$: UST on $\mathbb{Z}^2$ is a local limit of $\mathcal{U}^{(n)}$ (spanning tree of $\mathbb{Z}^2$ a.s.)

- **UST scaling limit**: [Schramm '00] topological properties of any possible scaling lim.

  [Lawler-Schramm-Werner '04] uniqueness of the scaling limit.

Theorem 5.2 [Barlow-Croydon-K '14] $\exists\{\delta_i\}_{i\ge 1} \searrow 0$ s.t. $\{\delta_i Y^{\mathcal{U}}_{\delta_i^{-13/4} t}\}_{t\ge 0} \xrightarrow{d} \{B^{\mathcal{T}}_t\}_{t\ge 0}$.

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Theorem. In all 3 cases, $\exists p^{\mathcal{U}}_t(\cdot,\cdot)$: joint cont. HK of $B^{\mathcal{U}}$, $\exists T_0 > 0$ s.t. for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$p^{\mathcal{U}}_t(x,y) \ \leq \ c_1 t^{-\frac{d_f}{d_w}} \ell(t^{-1}) \exp\left\{ -c_2 \left(\frac{d(x,y)^{d_w}}{t} \right)^{\frac{1}{d_w-1}} \ell\left(\frac{d(x,y)}{t} \right)^{-1} \right\}$$

$$p^{\mathcal{U}}_t(x,y) \ \geq \ c_3 t^{-\frac{d_f}{d_w}} \ell(t^{-1})^{-1} \exp\left\{ -c_4 \left(\frac{d(x,y)^{d_w}}{t} \right)^{\frac{1}{d_w-1}} \ell\left(\frac{d(x,y)}{t} \right) \right\}$$

for all $x, y \in \mathcal{U}, t \leq T_0$ with $\ell(x) := (1 \vee \log x)^{\theta}$, $(\exists \theta > 0)$.

For **Ex 0, 1**, $d_f = 2, d_w = d_f + 1 = 3$, and for **Ex 2**, $d_f = 8/5, d_w = d_f + 1 = 13/5$.

6 Conclusions

Diffusions / analysis on (exactly self-similar) fractals.

⇒ Stability theory, global analysis (generalization of the classical perturbation theory).

New insights to analysis on metric measure spaces.

⇒ Applications to RW/diffusions on random media

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Future challenges • Dynamics on conformal invariant media.

• Dynamics (jump-processes) on random media with long-range correlations.

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Thank you!