Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs

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Abstract

Sub-Gaussian estimates for random walks are typical of fractal graphs. We characterize them in the strongly recurrent case, in terms of resistance estimates only, without assuming elliptic Harnack inequalities.

1 Introduction

1.1 Statement of the main result

Let Γ be an infinite locally finite connected graph. That is, Γ is a set whose elements are called vertices; some of the vertices are connected by an edge, in which case one says that they are neighbours. If $x, y \in \Gamma$ are neighbours, one writes $x \sim y$. That Γ is locally finite means that every vertex has a finite number of neighbours. That Γ is connected means that for every pair x, y of vertices in Γ, there is at least one path in Γ joining x and y, that is a sequence $x_0 = x, x_1, \ldots, x_\ell = y$ such that

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$x_i \sim x_{i+1}$ for $i = 0, ..., \ell - 1$. The length of such a path is ℓ. The smallest possible length of a path joining $x, y \in \Gamma$ is denoted by $d(x,y)$, which defines a metric on Γ.

Assume that the graph Γ is endowed with a weight (or conductance) μ_{xy}, that is a symmetric nonnegative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if $x \sim y$. We call the pair (Γ, μ) a weighted graph. Such an object may also be viewed as an electric network, in which there is a wire of resistance μ_{xy}^{-1} between each pair x, y with $x \sim y$.

Now, define $\mu_x = \sum_{y \in \Gamma} \mu_{xy}$ for each $x \in \Gamma$, and set $\mu(A) = \sum_{x \in A} \mu_x$ for each $A \subset \Gamma$. μ is then a measure on Γ, and (Γ, d, μ) is a metric measure space.

Denote by $B(x,r)$ the ball in Γ of radius $r \geq 0$ centered at $x \in \Gamma$ with respect to the metric d and by $V(x,r)$ its measure, i.e.

$$B(x,r) := \{y \in \Gamma; d(x,y) \leq r\}, \qquad V(x,r) := \mu(B(x,r)).$$

We will consider graphs satisfying the volume doubling condition, that is we shall assume that there exists $C > 0$ such that

$$V(x,2r) \leq CV(x,r), \quad x \in \Gamma, \quad r \geq 0. \tag{VD}$$

It follows easily from (VD) that there exist $C, \alpha > 0$ such that for all $x, y \in \Gamma$, $r \geq s \geq 0$,

$$V(x,r) \leq C \left(\frac{r}{s}\right)^{\alpha} V(x,s), \tag{1.1}$$

and consequently, for all $x, y \in \Gamma$, $r \geq s \geq 0$,

$$V(x,r) \leq C \left(\frac{d(x,y) + r}{s}\right)^{\alpha} V(y,s). \tag{1.2}$$

We shall say that (Γ, μ) satisfies the condition $(VG(\alpha))$ if (1.1) holds, and we shall say that (Γ, μ) satisfies the stronger condition $(VG(\alpha\_))$ if $(VG(\gamma))$ holds for some $\gamma \in (0, \alpha)$. In particular $(VG(\alpha))$ holds if (Γ, μ) has polynomial volume growth of exponent α:

$$V(x,r) \simeq r^{\alpha}, \ r > 0, \ x \in \Gamma,$$

and $(VG(\alpha\_))$ holds if (Γ, μ) has polynomial growth of exponent γ with $0 < \gamma < \alpha$. (Throughout this article, if f and g depend on a variable ζ ranging in a set I, $f \simeq g$ means that there exists $C > 0$ such that $C^{-1} f(\zeta) \leq g(\zeta) \leq C f(\zeta)$ for all $\zeta \in I$.)

For each $x \sim y$, define

$$p(x,y) = \mu_{xy}/\mu_x.$$

In this paper, we will consider the discrete time Markov chain $\{X_n, n \geq 0, \mathbb{P}^x, x \in \Gamma\}$, with transition probabilities $p(x,y)$. The chain X is reversible with respect to μ since

$$p(x,y)\mu_x = \mu_{xy} = \mu_{yx} = p(y,x)\mu_y.$$

The associated Markov operator P, given by

$$Pf(x) = \sum_{y \in \Gamma} p(x,y) f(y),$$

is self-adjoint on $\ell^2(\Gamma,\mu)$.

For $n \in \mathbb{Z}_+ := \{0,1,\cdots\}$, let p_n denote the n-th convolution power of p, that is

$$p_0(x,y) = \delta_{x,y} := \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases}$$

and

$$p_n(x,y) = \sum_{z\in\Gamma} p_{n-1}(x,z)p(z,y), \qquad n \geq 1.$$

Alternatively, $p_n(x,y)$ is the transition function of the random walk X_n, i.e.

$$p_n(x,y) = \mathbb{P}^x(X_n = y),$$

or the kernel of the operator P^n with respect to the counting measure. Define the heat kernel, that is the kernel of P^n with respect to μ, or the transition density of X_n, by

$$h_n(x,y) := \frac{p_n(x,y)}{\mu_y}.$$

Clearly, h_n is symmetric, that is $h_n(x,y) = h_n(y,x)$. As a consequence of the semigroup law $P^{m+n} = P^m P^n$, the heat kernel satisfies the Chapman-Kolmogorov equation

$$h_{n+m}(x,y) = \sum_{z\in\Gamma} h_n(x,z)h_m(z,y)\mu_z, \tag{1.3}$$

for all $x,y \in \Gamma$, $n,m \in \mathbb{Z}_+$.

Our aim is to give a geometric necessary and sufficient condition for sub-Gaussian heat kernel upper and lower estimates to hold:

$$h_n(x,y) \leq \frac{C}{V(x,n^{1/\beta})} \exp\left(-\left(\frac{d(x,y)^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right), \text{ for all } x,y \in \Gamma,\ n \in \mathbb{N} \tag{$UHK(\beta)$}$$

and

$$h_n(x,y) + h_{n+1}(x,y) \geq \frac{c}{V(x,n^{1/\beta})} \exp\left(-\left(\frac{d(x,y)^\beta}{cn}\right)^{\frac{1}{\beta-1}}\right), \tag{$LHK(\beta)$}$$

for all $x,y \in \Gamma$, $n \in \mathbb{N}$ such that $n \geq d(x,y)$. The reason $(LHK(\beta))$ uses $h_n(x,y) + h_{n+1}(x,y)$ instead of $h_n(x,y)$ is that if Γ is bipartite, then $h_{2n+1}(x,x) = 0$, and so no non-trivial lower bound can hold just for $h_n(x,y)$. The conjunction of $(UHK(\beta))$ and $(LHK(\beta))$ will be denoted by $(HK(\beta))$.

A priori $\beta > 1$, but in fact the estimates $(HK(\beta))$ can hold only if $\beta \geq 2$. One way to see this is to observe that the upper bound $h_n(x,x) \leq CV(x,n^{1/\beta})^{-1}$, which follows from $(UHK(\beta))$, is compatible with the lower bound from [48], $h_n(x,x) \geq cV(x,n^{1/2}\log n)^{-1}$, which always holds under (VD), only if $\beta \geq 2$. Further, if (Γ,μ) has polynomial volume growth of exponent α, the estimates $(HK(\beta))$ can hold only if $\beta \leq \alpha + 1$. This can be seen in several ways: for instance, the lower bound $h_n(x,x) \geq cn^{-\alpha/\beta}$, which follows from $(LHK(\beta))$, must be compatible with the upper bound $h_n(x,x) \leq Cn^{-\alpha/(\alpha+1)}$ from [12]. Conversely, it was proved in [4] that for every couple α, β such that $2 \leq \beta \leq \alpha+1$, there exists a graph (Γ, μ) with polynomial volume growth of exponent α such that $(HK(\beta))$ holds.

The graphs associated with many regular fractals, such as the Sierpinski gaskets, carpets, and the Vicsek sets, do satisfy $(HK(\beta))$. However, the existing proofs all use some kind of Harnack inequality. While this is sometimes very easy to prove (so easy it is often not stated explicitly) for some families of finitely ramified sets, for infinitely ramified sets such as Sierpinski carpets the argument is considerably harder. See [37], [2], [3], [4], [9], [10], and the references therein.

In this paper, we will characterize the estimates $(HK(\beta))$ in the so-called 'strongly recurrent' case, that is the case where the volume growth of the graph (Γ, μ) is limited by the exponent β, which governs the scaling between time and space, in the sense that $(VG(\beta_{-}))$ holds. Note that the Sierpinski gaskets, the Vicsek graphs and the two-dimensional Sierpinski carpet are strongly recurrent, and that our method probably gives the quickest way so far to check $(HK(\beta))$ (see Section 5 below for the treatment of some examples).

In the case $\beta = 2$, it was proved in [22] that $(HK(2))$ is equivalent to (VD) plus the standard Poincaré inequality $(PI(2))$. For $\beta > 2$ the situation is more complicated. Characterizations of $(HK(\beta))$ have been given in [29], [52], [53], [30], but these all involve the elliptic Harnack inequality (EHI) (see Section 1.2 for a definition), whose geometric characterization remains an open question. In particular, it is not known whether or not (EHI) is invariant under rough isometries (see Section 1.3). In [11], a characterization of $(HK(\beta))$ was given in terms of a β-Poincaré inequality $(PI(\beta))$ (see Section 1.2), and a condition, denoted $(CS(\beta))$, requiring the existence of suitable families of cut-off functions; these two conditions are known to be invariant under rough isometry (see [34]). In this paper we give in the strongly recurrent case a more transparent and geometric characterization, in terms of electrical resistance, which is also clearly invariant under rough isometry, as we shall explain in Section 1.3.

For an introduction to the connection between random walks and electrical networks see [24]. For $f \in \mathbb{R}^\Gamma$, define

$$\mathscr{E}(f,f) = \frac{1}{2} \sum_{\substack{x,y \in \Gamma \\ x \sim y}} (f(x) - f(y))^2 \mu_{xy}, \tag{1.4}$$

and for $f, g \in \mathbb{R}^\Gamma$ such that $\mathscr{E}(f,f)$, $\mathscr{E}(g,g) < +\infty$ define

$$\mathscr{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y \in \Gamma \\ x \sim y}} (f(x) - f(y))(g(x) - g(y)) \mu_{xy}. \tag{1.5}$$

A straightforward computation shows that, for $f, g \in \ell^2(\Gamma, \mu)$,

$$\mathscr{E}(f,g) = \langle (I-P)f, g \rangle, \tag{1.6}$$

where $\langle .,. \rangle$ denotes the scalar product in $\ell^2(\Gamma,\mu)$. We abbreviate $\mathscr{E}(f,f)$ as $\mathscr{E}(f)$. In terms of electrical networks, we can regard $\mathscr{E}(f)$ as the energy dissipation in the network Γ associated with the potential f.

We shall use the fact that $\mathscr{E}$ is a Dirichlet form, and in particular the fact, which is easily checked, that for $f \in \mathbb{R}^\Gamma$ and $a \in \mathbb{R}$,

$$\mathscr{E}((f-a)_+) \le \mathscr{E}(f). \tag{1.7}$$

Let A, B be subsets of Γ. We define the effective resistance between A and B as follows.

$$R(A,B)^{-1} = \inf\{\mathscr{E}(f); f \in \mathbb{R}^\Gamma, f|_A = 1, f|_B = 0\}, \tag{1.8}$$

where we take $\inf \emptyset = +\infty$. We write $R(x,y)$ for $R(\{x\},\{y\})$. Taking $f = 1_A$ or $f = 1 - 1_B$ in (1.8) we see that $R(A,B) > 0$ if $A \cap B = \emptyset$ and one of A, B is finite. It is easy to prove (see Lemma 2.1 below) that the infimum in (1.8) is always attained, and that $R(A,B) < \infty$ for any $A,B \subset \Gamma$. Note that if $A \subset A'$ and $B \subset B'$ then $R(A,B) \ge R(A',B')$.

In fact, $R(x,y)$ defines a metric on Γ (this is non-trivial, see [3], Proposition 4.25, or [40]); note that in [14], section 6, the metric considered is $\delta_2(x,y) = \sqrt{R(x,y)}$.

The following easy lemma will play a key role in this paper.

Lemma 1.1. *For all $f \in \mathbb{R}^\Gamma$ and $x,y \in \Gamma$,*

$$|f(x) - f(y)|^2 \le R(x,y)\mathscr{E}(f). \tag{1.9}$$

Furthermore, for each $x,y \in \Gamma$, there exists $f \in \mathbb{R}^\Gamma$ such that equality holds in (1.9).

The inequality (1.9) is an immediate consequence of the definition of R. For the second assertion, see Lemma 2.1 below.

Definition 1.2. Following [4], we say that a graph is very strongly recurrent if there exists $p_1 > 0$ such that, writing $T_A = \min\{n \ge 0 : X_n \in A\}$, $T_y = T_{\{y\}}$, $\tau(x,r) = T_{B(x,r)^c}$,

$$\mathbb{P}^x(T_y < \tau(x,2r)) \ge p_1, \quad \text{for all } \ x \in \Gamma, r \ge 1, y \in B(x,r). \tag{1.10}$$

This property is called 'strongly recurrent' in [23], while in [53] 'strongly recurrent' is used for the property that there exists $c > 0$, $M > 1$ such that

$$R(x,B(x,Mr)^c) \ge (1+c)R(x,B(x,r)^c) \quad \text{for all } \ x \in \Gamma, r \ge 1. \tag{1.11}$$

We prove below that (1.10) implies (1.11) – see Lemma 3.6. Further (see Example 5 in Section 5) there exist graphs which satisfy (1.11) but not (1.10). It is easy to see that (1.11) implies that Γ is recurrent.

We now introduce the following condition:

$$p(x,y) \ge p_0 > 0 \quad \text{for all } x,y \in \Gamma, \ x \sim y. \tag{p_0}$$

For $x,y \in \Gamma$, define $V(x,y) = V(x,d(x,y))$. If (VD) holds then by (1.2) there exists C such that $V(x,y) \le CV(y,x)$ for all $x,y \in \Gamma$.

Our main theorem is the following:

Theorem 1.3. *Let (Γ,μ) be a weighted graph satisfying condition (p_0). Assume $(VG(\beta\_))$ for some $\beta \ge 2$. Then $(HK(\beta))$ holds if and only if*

$$R(x,y) \simeq \frac{d^\beta(x,y)}{V(x,y)},\ x,y \in \Gamma. \tag{$R(\beta)$}$$

In this case (Γ,μ) satisfies (1.10).

The condition $(R(\beta))$ can be decomposed into a lower estimate:

$$\text{there exists } c > 0 \text{ such that } R(x,y) \ge c\frac{d^\beta(x,y)}{V(x,y)} \text{ for all } x,y \in \Gamma, \qquad (RL(\beta)),$$

and an upper estimate:

$$\text{there exists } C > 0 \text{ such that } R(x,y) \le C\frac{d^\beta(x,y)}{V(x,y)} \text{ for all } x,y \in \Gamma. \qquad (RU(\beta))$$

Note that the conditions $(VG(\beta))$, $(VG(\beta\_))$, $(PI(\beta))$ and $(RU(\beta))$ all become weaker as β increases, while $(RL(\beta))$ becomes stronger.

In Section 2.2, we will see that, under some additional assumptions, $(PI(\beta))$ and $(RU(\beta))$ are equivalent. This helps to give a geometric understanding of $(R(\beta))$. Indeed, $(PI(\beta))$ (and therefore $(RU(\beta))$) is a quantitative connectivity property: balls of all radii are sufficiently connected, to an extent governed by β. Conversely, one can see $(RL(\beta))$ as a property saying that there cannot be more connections than the exponent β allows. In other words $(R(\beta))$ contains at the same time a β-Poincaré inequality and the matching anti-Poincaré inequality.

Here is the plan of this paper. In Section 2, we will show that the resistance estimate $(RL(\beta))$ can be strengthened and discuss the equivalence of $(PI(\beta))$ and $(RU(\beta))$ under some additional conditions. In Section 3, we shall show that for any $\beta \ge 2$, $(VG(\beta\_))$ (and, in fact, a weaker volume growth condition, see (2.1) below) and $(R(\beta))$ suffice for $(HK(\beta))$ to hold. The first step, in Section 3.1, is to observe that $(RU(\beta))$ together with $(VG(\beta))$ is enough for on-diagonal upper estimates. In Section 3.2, we estimate the exit time of the random walk from a ball in terms of its radius. One can then conclude that $(HK(\beta))$ holds by using the results in [30], but, taking advantage of our strong recurrence assumption, we will give a more direct proof. In Section 4, we shall show that, together with $(VG(\beta\_))$, $(HK(\beta))$ implies $(R(\beta))$. This will use the implication from $(PI(\beta))$ to $(RU(\beta))$ proved in Section 2.2. In Section 5, we give examples.

We note that, applying the technique in this paper, one of the authors obtained the measure metric space version of our results using resistance forms (see [44]).

Throughout the paper, we will use c, C with or without subscripts, to denote strictly positive constants whose values are not important, and which may change from line to line.

In the remainder of this section we give the comments on Harnack inequalities, and invariance under rough isometry that we already announced.

1.2 Harnack and Poincaré inequalities

Let $\mathscr{L} = P - I$. We say that $u : B(x,r) \to \mathbb{R}$ is harmonic in $B = B(x,r)$ if u is defined on $\overline{B} = \{y : y \sim x, x \in B\}$, and $\mathscr{L}u(x) = 0$, $x \in B$.

We say that (Γ, μ) satisfies the elliptic Harnack inequality (with constant C) if for all $x \in \Gamma, r \geq 0$, and for any non-negative harmonic function u in $B(x,2r)$, the following holds

$$\max_{y \in B(x,r)} u(y) \leq C \min_{y \in B(x,r)} u(y). \tag{EHI}$$

The statement of the parabolic Harnack inequality is a little more complicated, and depends on the index β. We say $(PHI(\beta))$ holds if whenever $u(n,x) \geq 0$ is defined on $[0,4N] \times \bar{B}(y,2R)$ and satisfies

$$u(n+1,x) - u(n,x) = \mathscr{L}u(n,x), \qquad \forall (n,x) \in [0,4N] \times B(y,2R), \tag{1.12}$$

then

$$\max_{\substack{N \leq n \leq 2N \\ x \in B(y,R)}} u(n,x) \leq C \min_{\substack{3N \leq n \leq 4N \\ x \in B(y,R)}} (u(n,x) + u(n+1,x)), \tag{1.13}$$

when $N \geq 2R$ and $N \simeq R^{\beta}$. Clearly, $(PHI(\beta))$ implies (EHI). It is known that $(PHI(\beta))$ is equivalent to $(HK(\beta))$ – see Theorem 3.1 in [30]. The original proof for the case of $\beta = 2$ goes back to [45] in a continuous setting; see [22] for an adaptation to the graph case.

For $B \subset \Gamma$ set

$$\mathscr{E}_B(f,f) = \frac{1}{2} \sum_{\substack{x,y \in B \\ x \sim y}} (f(x) - f(y))^2 \mu_{xy}. \tag{1.14}$$

We say that (Γ, μ) satisfies a scaled Poincaré inequality of order β if there exist $C > 0, C' \geq 1$ such that for every $f \in \mathbb{R}^{\Gamma}$ and every ball $B := B(x_0, r)$, $x_0 \in \Gamma$, $r \geq 0$,

$$\sum_{x \in B} (f(x) - \bar{f}_B)^2 \mu_x \leq C r^{\beta} \mathscr{E}_{C'B}(f), \tag{$PI(\beta)$}$$

where $C'B := B(x_0, C'r)$ and $\bar{f}_B = \frac{1}{\mu(B)} \sum_{x \in B} f(x) \mu_x$.

1.3 Invariance under rough isometry

Definition 1.4. Let $(\Gamma^{(1)},\mu^{(1)})$, $(\Gamma^{(2)},\mu^{(2)})$ be weighted graphs satisfying condition (p_0). A map $T:\Gamma^{(1)}\to\Gamma^{(2)}$ is called a rough isometry if the following holds. There exist positive constants $a,c>1,b>0$ and $M>0$ such that

$$a^{-1}d^{(1)}(x,y)-b\le d^{(2)}(T(x),T(y))\le ad^{(1)}(x,y)+b\ \forall x,y\in\Gamma^{(1)}, \tag{1.15}$$

$$d^{(2)}(T(\Gamma^{(1)}),y')\le M\quad \forall y'\in\Gamma^{(2)}, \tag{1.16}$$

$$c^{-1}\mu_x^{(1)}\le\mu_{T(x)}^{(2)}\le c\mu_x^{(1)}\quad \forall x\in\Gamma^{(1)}, \tag{1.17}$$

where $d^{(i)}(\cdot,\cdot)$ is the graph distance of $(\Gamma^{(i)},\mu^{(i)})$ for $i=1,2$.
If there exists a rough isometry between two spaces, they are said to be roughly isometric. (One can check this is an equivalence relation.)

The concept of rough isometry was introduced (for manifolds) by M. Kanai in [38, 39], but without the condition (1.17). Under the assumption of bounded geometry made in those papers, the analogue of (1.17) could be proved. A general definition similar to the above one was given in [21], see also [34].

In [34], it is proved that $(PHI(\beta))$ is stable under rough isometry. We can verify this fact under the assumption of $(VG(\beta\_))$ or $(VG(\beta))$ in the following way. It is known that conditions $(VG(\beta))$ and $(VG(\beta\_))$ are stable under rough isometry, see [21], Proposition II.2. In [54], it is proved that the effective resistance is preserved under rough isometry up to a multiplicative constant. (In [54], only unweighted graphs (i.e. $\mu_{xy}\equiv 1$ when $x\sim y$) are considered. But a simple modification of the proof gives the same result for weighted graphs satisfying condition (p_0).) Thus, $(R(\beta))$ is stable under rough isometry. This together with Theorem 1.3 gives the desired fact.

2 Resistance estimates and Poincaré inequalities

In this section, we shall only need a weaker form of $(VG(\beta\_))$, namely

$$V(x,r)\le\varphi\left(\frac{r}{s}\right)V(x,s) \tag{2.1}$$

for all $x\in\Gamma$, $r\ge s\ge 0$, where φ satisfies

$$\limsup_{t\to+\infty}\frac{\varphi(t)}{t^\beta}=0. \tag{2.2}$$

In other words, it will be enough to assume that the volume growth of (Γ,μ) is uniformly strictly below $(VG(\beta))$, without necessarily being polynomial of a smaller exponent than β. We shall often use (2.1) in the following form: for any $\varepsilon>0$, there exists $\eta>0$ such that

$$\frac{s^\beta}{V(x,s)}\le\varepsilon\frac{r^\beta}{V(x,r)}, \tag{2.3}$$

for every $x\in\Gamma$ and every $r,s\ge0$ such that $s\le\eta r$.

We start with some basic properties of resistance.

Lemma 2.1. *Let* (Γ,μ) *be a weighted graph, and* $A,B\subset\Gamma$ *with* $A\cap B=\emptyset$.
(a) Then $R(A,B)<\infty$. *Further if*

$$\inf_{x\sim y}\mu_{x,y}=c>0, \tag{2.4}$$

then

$$R(A,B)\le c^{-1}d(A,B). \tag{2.5}$$

(b) There exists a function f *which attains the infimum in (1.8).*

Proof. (a) Let $f\in\mathbb{R}^{\Gamma}$, $\ell=d(A,B)$, and $a=x_0,x_1,\dots,x_\ell=b$ be a shortest path joining $a\in A$, $b\in B$. Let f satisfy the constraints in (1.8), and let $0<\kappa\le\min_i\mu_{x_{i-1},x_i}$. Then

$$\begin{aligned}1=|f(a)-f(b)|^2&\le\ell\left(\sum_{i=0}^{\ell-1}|f(x_i)-f(x_{i+1})|^2\right)\\&\le\kappa^{-1}\ell\left(\sum_{i=0}^{\ell-1}|f(x_i)-f(x_{i+1})|^2\mu_{x_ix_{i+1}}\right)\le\kappa^{-1}\ell\mathscr{E}(f).\end{aligned}$$

Thus $R(A,B)^{-1}\ge\mathscr{E}(f)\ge\kappa\ell^{-1}>0$. If (2.4) holds then we can take $\kappa=c$ and obtain (2.5).
(b) Let f_n be a sequence satisfying the constraints in (1.8) with $\mathscr{E}(f_n)\to R(A,B)^{-1}$. We can assume that $0\le f_n\le1$. A diagonalization argument gives a sequence $g_k=f_{m_k}$ such that g_k converges pointwise to a function g, and using Fatou

$$\mathscr{E}(g)=\mathscr{E}(\liminf g_k)\le\liminf\mathscr{E}(g_k)=R(A,B)^{-1}.$$

Thus a minimiser exists. □

In general, $R(x,y)$ can be substantially smaller than $d(x,y)$, but there are extremal situations, such as trees, where the two quantities are comparable. In particular, on an unweighted tree one has $R(x,y)=d(x,y)$. See Example 2 in Section 5 below.

We now show that the lower resistance estimate $(RL(\beta))$ self-improves in the presence of the other assumptions. Consider the condition:

$$\text{there exists } c>0 \text{ such that } R(x,B^c(x,r))\ge c\frac{r^\beta}{V(x,r)} \text{ for all } x\in\Gamma,\ r>0. \tag{$SRL(\beta)$}$$

It is easy to see that $(SRL(\beta))$ implies $(RL(\beta))$, since if $d(x,y)=r\ge2$ then

$$R(x,y)\ge R(x,B(x,r-1)^c)\ge\frac{(r-1)^\beta}{V(x,r-1)}\ge c2^{-\beta}\frac{r^\beta}{V(x,r)}.$$

(If $r=d(x,y)=1$ then the bound $(RL(\beta))$ always holds, since $R(x,y)\ge R(x,\{x\}^c)=\mu_x^{-1}\ge V(x,1)^{-1}$.)

In fact, in the presence of $(RU(\beta))$ and $(VG(\beta\_))$, the converse is true.

Lemma 2.2. *Let* (Γ,μ) *be a weighted graph. Assume* (2.1) *and* $(R(\beta))$. *Then* $(SRL(\beta))$ *holds.*

Using this, in the presence of $(RL(\beta))$ and $(VG(\beta\_))$, the upper estimate $(RU(\beta))$ is equivalent to $(PI(\beta))$.

Lemma 2.3. *Let* (Γ,μ) *be a weighted graph.*
(a) (2.1) *and* $(R(\beta))$ *imply* $(PI(\beta))$.
(b) $(VG(\beta\_))$[1] *and* $(PI(\beta))$ *imply* $(RU(\beta))$.
In particular, if (Γ,μ) *satisfies* $(VG(\beta\_))$ *and* $(RL(\beta))$*, then* $(PI(\beta))$ *and* $(RU(\beta))$ *are equivalent.*

2.1 Improvement of the lower estimates

In this section, we prove Lemma 2.2, obtaining it as a particular case of a more general result.

Lemma 2.4. *Let* (Γ,μ) *be a weighted graph satisfying* (VD). *Assume that*

$$R(x,y) \simeq \frac{\eta\,(d(x,y))}{V(x,y)},\ x,y\in\Gamma \tag{2.6}$$

holds, with η *increasing and satisfying*

$$\sup_{\substack{x\in\Gamma\\ r>0}} \frac{\eta(\lambda r)V(x,r)}{\eta(r)V(x,\lambda r)} \to 0 \text{ as } \lambda\to 0_+. \tag{2.7}$$

Then there exists $c>0$ *such that*

$$R(x,B^c(x,r)) \ge c\frac{\eta(cr)}{V(x,r)},\ \forall x\in\Gamma,\ r>0. \tag{2.8}$$

Proof. Fix $x_0\in\Gamma$ and $r>0$, and let $A=B(x_0,r)-B(x_0,r/2)$. For $x\in A$ let h_x be the function on Γ given by Lemma 1.1, such that $h_x(x)=0$, $h_x(x_0)=1$, and $\mathscr{E}(h_x)=1/R(x_0,x)$. Let $\lambda<\frac{1}{2}$. As h_x is harmonic on $\Gamma\setminus\{x,y\}$, $h_x(y)$ is maximised over $B(x,\lambda r)$ by a y_1 with $d(x,y_1)=\lambda r$. So, for $y\in B(x,\lambda r)$, by (1.9) and the upper bound in (2.6),

$$|h_x(y)|^2 \le |h_x(y_1)-h_x(x)|^2 \le C\frac{\eta(\lambda r)}{V(x,\lambda r)}\mathscr{E}(h_x) = C\frac{\eta(\lambda r)}{V(x,\lambda r)R(x,x_0)}. \tag{2.9}$$

Using the lower bound in (2.6), and (VD), if $y\in B(x,\lambda r)$ then

$$|h_x(y)|^2 \le C\frac{\eta(\lambda r)V(x,x_0)}{V(x,\lambda r)\eta(d(x_0,x))} \le C'\frac{\eta(\lambda r)V(x,r/2)}{V(x,\lambda r)\eta(r/2)}. \tag{2.10}$$

Thus, by (2.7), there exists a constant $\delta>0$ such that $x\in A$, $d(x,y)\le\delta r$, implies that $h_x(y)\le\frac{1}{2}$. We can assume $\delta<1/6$.

[1] A careful reader will notice in the proof below that, again, one can slightly weaken $(VG(\beta\_))$ here, by assuming (2.1) with $\sum_{i=0}^{\infty} 2^{-i\beta}\varphi(2^i)<+\infty$ instead of (2.2).

Now use (VD) to cover A by balls $B(x_i, \delta r)$, $1 \le i \le j$, with $x_i \in A$. Here j is bounded from above, the bound only depending on the volume doubling constant. Let $g = \min_{\{i=1,..,j\}} h_{x_i}$, $h = 2(g - \frac{1}{2})_+$, and $h' = h 1_{B(x_0,r)}$. Then $h'(x_0) = h(x_0) = 1$, and $h' \equiv 0$ outside $B(x_0, r)$, so that

$$R(x_0, B^c(x_0, r))^{-1} \le \mathscr{E}(h').$$

But it is clear that $\mathscr{E}(h') \le \mathscr{E}(h)$ since $h = 0$ on A. Also, by (1.7),

$$\mathscr{E}(h) \le 4\mathscr{E}(g).$$

Now, for $x, y \in \Gamma$ such that $g(x) \ge g(y)$, if $g(y) = h_{x_i}(y)$, then

$$(g(x) - g(y))^2 = (g(x) - h_{x_i}(y))^2 \le (h_{x_i}(x) - h_{x_i}(y))^2 \le \sum_{i=1}^{j} (h_{x_i}(x) - h_{x_i}(y))^2.$$

Summing over x, y, we obtain

$$\mathscr{E}(g) \le \sum_{i=1}^{j} \mathscr{E}(h_{x_i}). \tag{2.11}$$

Now using the lower bound in (2.6), and (VD),

$$\mathscr{E}(h_{x_i}) = \frac{1}{R(x_0, x)} \le C \frac{V(x_0, x)}{\eta(d(x_0, x))} \le C' \frac{V(x_0, r)}{\eta(r/2)} \le C'' \frac{V(x_0, r/2)}{\eta(r/2)}.$$

Combining this with (2.11) implies that $\mathscr{E}(g) \le CV(x_0, r/2)/\eta(r/2)$, and this yields (2.8). □

Proof of Lemma 2.2. For this it is enough to observe that if $\eta(r) = r^\beta$ then (2.6) follows from $(R(\beta))$, (2.7) from (2.1), and use Lemma 2.4. □

2.2 Upper estimates and Poincaré inequalities

This section, where we prove Lemma 2.3, can be skipped in a first reading. In fact the implication from $(VG(\beta\_))$ and $(R(\beta))$ to $(PI(\beta))$ will not be used in the proof of the main result, and, furthermore is in any case be a consequence of Theorem 1.3 and Proposition 4.2, although this route is rather indirect. As for the implication from $(PI(\beta))$ to $(RU(\beta))$, it is used only in the end to deduce the converse part of Theorem 1.3 from Proposition 4.2.

We first need a version of Lemma 3.5 in [18], to compare resistance in Γ with resistance in a large ball. Recall the definition of $\mathscr{E}_B$ in (1.14), and for $B \subset \Gamma$ set

$$R_B(x, y) = \inf\{\mathscr{E}_B(f) : f(x) = 0, f(y) = 1\}. \tag{2.12}$$

Lemma 2.5. *Assume* (2.1) *and* $(R(\beta))$. *Then there exists* $C' \ge 1$ *such that*

$$R(x, y) \le R_{B(x, C'd(x,y))}(x, y) \le 2R(x, y). \tag{2.13}$$

Proof. Since $\mathcal{E}_B(f) \le \mathcal{E}(f)$ the left side of (2.13) is immediate.

For the right hand side we begin by proving the inequality

$$\frac{1}{R(x,y)} \le \frac{1}{R_{B(x,2C'd(x,y))}(x,y)} + \frac{1}{R(x,B(x,C'd(x,y))^c)}. \tag{2.14}$$

Let $C \ge 1$, and write $B = B(x,Cd(x,y))$, $B' = B(x,2Cd(x,y))$. Let f_1, f_2 be functions which attain the infimum in the variational problems for $R_{B'}(x,y)$ and $R(x,B^c)$. Thus $f_1(x) = f_2(x) = 1$, $f_1(y) = 0$, and $f_2 = 0$ on B^c. We can take $f_1 = 0$ on $(B')^c$. Let $f = \min(f_1, f_2)$. Then, using (2.11) in B', and the fact that $f = 0$ on $(B')^c$,

$$\begin{aligned} R(x,y)^{-1} \le \mathcal{E}(f) &= \mathcal{E}_{B'}(f) \le \mathcal{E}_{B'}(f_1) + \mathcal{E}_{B'}(f_2) = \mathcal{E}_{B'}(f_1) + \mathcal{E}(f_2) \\ &= R_{B'}(x,y)^{-1} + R(x,B^c)^{-1}, \end{aligned}$$

proving (2.14). Using $(RU(\beta))$, $(SRL(\beta))$ it follows that there exists C' (not depending on x,y) such that

$$R(x,B(x,C'd(x,y))^c) \ge 2R(x,y),$$

and (2.14) therefore gives

$$\frac{1}{2R(x,y)} \le \frac{1}{R_{B(x,2C'd(x,y))}(x,y)},$$

completing the proof of (2.13). $\square$

We now return to the proof of Lemma 2.3.

Proof of $(2.1) + (R(\beta)) \Rightarrow (PI(\beta))$. Fix $B = B(x_0,r)$. By Lemma 2.5, there exist $C, C' > 0$ such that

$$|f(x) - f(y)| \le C \frac{[d(x,y)]^{\beta/2}}{\sqrt{V(x,y)}} \sqrt{\mathcal{E}_{C'B}(f)}, \qquad \forall f \in \mathbb{R}^\Gamma,\ x,y \in B.$$

Thus, for $x \in B$, we may write

$$|f(x) - \bar{f}_B| \le \frac{1}{\mu(B)} \sum_{y \in B} |f(x) - f(y)|\mu_y \le \frac{C\sqrt{\mathcal{E}_{C'B}(f)}}{\mu(B)} \sum_{y \in B} \frac{[d(x,y)]^{\beta/2}}{\sqrt{V(x,y)}} \mu_y.$$

Note that, since $x,y \in B$, $d(x,y) \le 2r$, therefore

$$\begin{aligned} |f(x) - \bar{f}_B| &\le \frac{C\sqrt{\mathcal{E}_{C'B}(f)}}{\mu(B)} \sum_{s=1}^{2r} \frac{s^{\beta/2}\mu(\{y : d(x,y) = s\})}{\sqrt{V(x,s)}} \\ &\le \frac{C' r^{\beta/2}\sqrt{\mathcal{E}_{C'B}(f)}}{\mu(B)} \sum_{s=1}^{2r} \frac{V(x,s) - V(x,s-1)}{\sqrt{V(x,s)}}. \end{aligned}$$

Recall that for any sequences $\{a_s\}_{s \ge 1}, \{b_s\}_{s \ge 1}$, the following holds

$$\sum_{s=1}^{2r} a_s b_s = \sum_{s=1}^{2r-1} A_s(b_s - b_{s+1}) + A_{2r} b_{2r},$$

where $A_s := \sum_{n=1}^{s} a_n$. Applying this with

$$a_s = V(x,s) - V(x,s-1),\ b_s = \frac{1}{\sqrt{V(x,s)}},$$

we obtain

$$\begin{aligned}\sum_{s=1}^{2r} \frac{V(x,s)-V(x,s-1)}{\sqrt{V(x,s)}} &= \sum_{s=1}^{2r-1} \frac{\sqrt{V(x,s)}}{\sqrt{V(x,s+1)}}(\sqrt{V(x,s+1)} - \sqrt{V(x,s)}) + \sqrt{V(x,2r)} \\ &\le 2\sqrt{V(x,2r)} \le C\sqrt{\mu(B)}.\end{aligned}$$

In the last inequality we used the fact that $x \in B = B(x_0,r)$ and (VD). Taking squares and summing, we have

$$\sum_{x\in B}(f(x)-\bar{f}_B)^2\mu_x \le \frac{Cr^\beta}{\mu(B)}\mathscr{E}_{C'B}(f)\sum_{x\in B}\mu_x = Cr^\beta\mathscr{E}_{C'B}(f),$$

that is $(PI(\beta))$. □

Proof of $(VG(\beta_-)) + (PI(\beta)) \Rightarrow (RU(\beta))$. This can be proved by modifying Proposition 3.3 of [18]. First, note that by applying Cauchy-Schwarz to $(PI(\beta))$, we obtain for each $B = B(x,r)$,

$$\begin{aligned}\frac{1}{\mu(B)}\sum_{z\in B}|f(z)-\bar{f}_B|\mu_z \le \left(\frac{1}{\mu(B)}\sum_{z\in B}(f(z)-\bar{f}_B)^2\mu_z\right)^{1/2} &\le \left(\frac{C_1 r^\beta}{\mu(B)}\mathscr{E}_{C'B}(f)\right)^{1/2} \\ &\le \left(\frac{C_1 r^\beta}{\mu(B)}\mathscr{E}(f)\right)^{1/2}. \qquad (2.15)\end{aligned}$$

Fix x,y in Γ and $f \in \mathbb{R}^\Gamma$. Write $B_i = B(x, 2^{-i}d(x,y))$, $i \in \mathbb{Z}_+$. We have

$$\begin{aligned}|f(x)-\bar{f}_{B_0}| &\le \sum_{i=0}^{\infty}|\bar{f}_{B_i}-\bar{f}_{B_{i+1}}| \le \sum_{i=0}^{\infty}\frac{1}{\mu(B_{i+1})}\sum_{z\in B_{i+1}}|f(z)-\bar{f}_{B_i}|\mu_z \\ &\le C\sum_{i=0}^{\infty}\frac{1}{\mu(B_i)}\sum_{z\in B_i}|f(z)-\bar{f}_{B_i}|\mu_z \le C'\sum_{i=0}^{\infty}\left(\frac{(2^{-i}d(x,y))^\beta}{\mu(B_i)}\mathscr{E}(f)\right)^{1/2}.\end{aligned}$$

Here we have used (VD) and (2.15). Now $(VG(\beta\_))$ implies that

$$\frac{1}{\mu(B_i)} \le \frac{C2^{i\alpha}}{V(x,y)},$$

with $\alpha < \beta$. This yields

$$|f(x)-\bar{f}_{B(x,d(x,y))}|^2 \le C\frac{d^\beta(x,y)}{V(x,y)}\mathscr{E}(f). \qquad (2.16)$$

Similarly,

$$|f(y)-\bar{f}_{B(y,d(x,y))}|^2 \le C\frac{d^\beta(x,y)}{V(x,y)}\mathscr{E}(f). \tag{2.17}$$

Finally, under (VD),

$$|\bar{f}_{B(x,d(x,y))}-\bar{f}_{B(y,d(x,y))}| \le \frac{C}{V(x,2d(x,y))}\sum_{z\in B(x,2d(x,y))}|f(z)-\bar{f}_{B(x,2d(x,y))}|\mu_z. \tag{2.18}$$

Using (2.15), we have

$$\frac{1}{V(x,2d(x,y))}\sum_{z\in B(x,2d(x,y))}|f(z)-\bar{f}_{B(x,2d(x,y))}|\mu_z \le C\left(\frac{d(x,y)^\beta}{V(x,y)}\mathscr{E}(f)\right)^{1/2}. \tag{2.19}$$

By (2.16), (2.17), (2.18) and (2.19), we obtain

$$|f(x)-f(y)|^2 \le C\frac{d^\beta(x,y)}{V(x,y)}\mathscr{E}(f), \qquad \forall f\in\mathbb{R}^\Gamma, \forall x,y\in\Gamma,$$

which is the claim. □

The proof of Lemma 2.3 is complete.

3 From resistance estimates to heat kernel estimates

In this section, we shall prove that $(R(\beta))$ together with $(VG(\beta\_))$ implies $(HK(\beta))$. In fact, we will only need the weaker form of $(VG(\beta\_))$ given by (2.1).

3.1 On-diagonal upper heat kernel estimate

We obtain the on-diagonal heat kernel upper estimate from the resistance upper estimate and a volume upper bound in a relatively direct way, and this is the first main simplification in our case with respect to [30]. Here, we need only assume that the volume growth exponent does not exceed β.

Theorem 3.1. *Assume $(VG(\beta))$ and $(RU(\beta))$. Then there exists $C>0$ such that*

$$h_n(x,x) \le \frac{C}{V(x,n^{1/\beta})}, \qquad \forall x\in\Gamma, n\in\mathbb{N}. \tag{$DUHK(\beta)$}$$

We obtain this from a more general result. Note that in the following statement, we do not assume (VD).

Proposition 3.2. *Assume that there exists a one-to-one increasing function η from $[0,\infty)$ to itself such that*

$$R(x,y) \le \frac{\eta(d(x,y))}{V(x,y)}, \ \forall x,y\in\Gamma, \tag{3.1}$$

and such that, for some $A > 0$, $\eta(r)/V(x,r)$ satisfies

$$\frac{\eta(s)}{V(x,s)} \leq A\frac{\eta(r)}{V(x,r)}, \forall x \in \Gamma,\ r,s \text{ such that } 0 \leq s \leq r. \tag{3.2}$$

Then there exist $C, c > 0$ such that

$$h_n(x,x) \leq \frac{C}{V(x,\eta^{-1}(cn))}, \qquad \forall x \in \Gamma, n \geq 4. \tag{3.3}$$

The above estimate holds also for small n if one assumes that $\mu_x \simeq \mu_y$ for $x \sim y$.

Proof. Fix $x_0 \in \Gamma$. For $n \in \mathbb{N}$ and $x \in \Gamma$, set $f_n(x) = h_n(x_0,x) + h_{n+1}(x_0,x)$.

Let $r > 0$. Write $B(r) = B(x_0,r)$ and $V(r) = V(x_0,r)$. If $x_- \in B(r)$ is such that $f_n(x_-) = \min_{x\in B(r)} f_n(x)$,

$$f_n(x_-)V(r) \leq \sum_{x\in B(r)} f_n(x)\mu_x \leq \sum_{x\in\Gamma} h_n(x_0,x)\mu_x + \sum_{x\in\Gamma} h_{n+1}(x_0,x)\mu_x \leq 2,$$

so that $f_n(x_-) \leq 2/V(r)$.

Using (1.9), (3.1) and (3.2), we can write

$$\begin{aligned} f_n^2(x_0) &\leq 2\left(f_n^2(x_-) + |f_n(x_0) - f_n(x_-)|^2\right) \leq \frac{8}{V^2(r)} + 2R(x_0,x_-)\mathscr{E}(f_n) \\ &\leq \frac{8}{V^2(r)} + \frac{\eta\left(d(x_0,x_-)\right)}{V(x_0,x_-)}\mathscr{E}(f_n) \leq \frac{8}{V^2(r)} + \frac{C\eta(r)}{V(r)}\mathscr{E}(f_n). \end{aligned}$$

It is easy to check, using (1.3) and (1.6), that

$$0 \leq \mathscr{E}(f_n) = f_{2n}(x_0) - f_{2n+2}(x_0). \tag{3.4}$$

We obtain therefore

$$f_n^2(x_0) \leq 8V(r)^{-2} + C\eta(r)V(r)^{-1}\left(f_{2n}(x_0) - f_{2n+2}(x_0)\right).$$

Fix $N \geq 2$ and sum this from N to $2N-1$:

$$\sum_{n=N}^{2N-1} f_n^2(x_0) \leq 8NV(r)^{-2} + CV(r)^{-1}\eta(r)(f_{2N}(x_0) - f_{4N}(x_0)).$$

Since for each even $n \in \{N,...,2N-1\}$ we have, using (3.4), $f_n(x_0) \geq f_{2N}(x_0)$, it follows that

$$\frac{1}{2}(N-1)f_{2N}^2(x_0) \leq 8NV(r)^{-2} + CV(r)^{-1}\eta(r)f_{2N}(x_0).$$

Then

$$f_{2N}^2(x_0) \leq CV(r)^{-2} + C'N^{-1}V(r)^{-1}\eta(r)f_{2N}(x_0).$$

Take $r = \eta^{-1}(cN)$, with $c > 0$ small enough, to obtain

$$f_{2N}(x_0) \leq CV(r)^{-1} = \frac{C}{V(x_0,\eta^{-1}(cN))},$$

that is

$$h_{2N}(x_0,x_0)+h_{2N+1}(x_0,x_0)\le \frac{C}{V(x_0,\eta^{-1}(cN))},$$

from which (3.3) follows. $\square$

Taking $\eta(r)=Cr^{\beta}$, and using (VD) once more in the end to obtain exactly the desired estimate, we obtain Theorem 3.1.

The above argument can also be used without any volume growth assumption or resistance estimate. The following proposition gives an estimate similar to that in [12], Theorem 2.1, but with much weaker hypotheses.

Proposition 3.3. *Let* (Γ,μ) *be a weighted graph satisfying assumption* (2.4). *Then there exists* $C>0$ *such that*

$$h_n(x,x)\le \frac{C}{V(x,w^{-1}(n))},\qquad \forall x\in\Gamma, n\in\mathbb{N},$$

where $w(r):=rV(x,r)$.

Proof. We use the same notation as in the proof of Proposition 3.2. Proceeding as before and using Lemma 2.1, we obtain

$$\frac{1}{2}f_n^2(x_0)\le 4V(r)^{-2}+Cr\mathscr{E}(f_n),$$

and hence

$$f_n^2(x_0)\le 8V(r)^{-2}+C'r(f_{2n}(x_0)-f_{2n+2}(x_0)).$$

This leads to

$$f_{2N}^2(x_0)\le CV(r)^{-2}+C'rN^{-1}f_{2N}(x_0).$$

So taking $r_N=\sup\{s:sV(s)\le N\}$ we obtain

$$f_{2N}\le C'V(r_N)^{-1}.$$

$\square$

Instead of using $R(x,y)\le C\,d(x,y)$, the universal estimate from Lemma 2.1, one could also derive other upper estimates of $h_n(x,x)$ under assumptions of the form

$$R(x,y)\le\theta\left(d(x,y)\right),$$

where $\theta(t)<<t$. We leave this to the reader.

3.2 Off-diagonal upper heat kernel estimate

If $\beta = 2$, $(DUHK(2))$ and (VD) imply $(UHK(2))$. (See [35] for the case when the volume growth is polynomial, and [16] or [17] for the general doubling volume case). The situation is quite different when $\beta > 2$, and further tools are needed.

Recall the definition of $\tau(x,r)$, and consider the following estimate for the expected exit time from a ball:

$$\mathbb{E}^x[\tau(x,r)] \simeq r^\beta, \qquad r \geq 1. \tag{E_β}$$

Proposition 3.4. *Assume η is increasing and satisfies (VD), (2.6), (3.2) and (2.7). Then there exist $C, c > 0$ such that:*
(3.5)
$$\mathbb{E}^x[\tau(x_0,r)] \leq C\eta(r),\ \mathbb{E}^{x_0}[\tau(x_0,r)] \geq c\eta(cr), \qquad \forall r \geq 1,\ x_0 \in \Gamma, x \in B(x_0,r).$$
In particular, taking $\eta(r) = r^\beta$, (2.1) and $(R(\beta))$ imply (E_β).

Proof. The argument for the upper bound in (3.5) goes back to [51], and is quite general. Let X_n^B be the random walk on (Γ,μ) killed on exiting $B := B(x_0,r)$. The associated sub-Markov kernel is defined by

$$p^B(x,y) := \begin{cases} p(x,y) & x,y \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Define the transition function $p_n^B(x,y)$ as the n-th convolution power of p^B, and let $h_n^B(x,y) := p_n^B(x,y)/\mu_y$. The Green kernel $g_B(x,y)$ of X_n^B is defined by $g_B(x,y) = \sum_{n=0}^\infty h_n^B(x,y)$. It is easy to check that $g_B(x,\cdot)$ is harmonic on $B \setminus \{x\}$ and that $\mathscr{L} g_B(x,\cdot)$ at x is $-(\mu_x)^{-1}$. Thus $g_B(\cdot,\cdot)$ has the following reproducing property:

$$\mathscr{E}(g_B(x,\cdot), f) = f(x) \text{ for all } f \in \mathbb{R}^\Gamma \text{ such that } f|_{B^c} = 0. \tag{3.6}$$

Set

$$p_x(y) := \mathbb{P}^y(T_x < \tau(x,r)) = \frac{g_B(x,y)}{g_B(x,x)}.$$

The second equality is because both functions are 1 at x, 0 outside B and harmonic elsewhere. Using the reproducing property of g_B and the fact that p_x is an equilibrium potential for $R(x,B^c)$, we have

$$R(x,B^c)^{-1} = \mathscr{E}(p_x) = g_B(x,x)^{-1}. \tag{3.7}$$

Since $p_x(y) \leq 1$ for all $y \in \Gamma$,

$$g_B(x,y) \leq g_B(x,x) \qquad \forall x,y \in \Gamma. \tag{3.8}$$

Summarizing, we have

$$R(x,B^c) = g_B(x,x) = \sum_{n=0}^\infty h_n^B(x,x) \qquad \forall x \in \Gamma. \tag{3.9}$$

By the monotonicity of resistance,

$$R(x,B^c) \leq R(x,y) \qquad \forall x \in \Gamma, y \in B^c.$$

Thus, by the upper bound in (2.6),

$$g_B(x,x) = R(x,B^c) \le C\frac{\eta(r)}{V(x,r)}. \tag{3.10}$$

Now, since, for $x \in B$,

$$\mathbb{E}^x[\tau(x_0,r)] = \sum_{y\in B} g_B(x,y)\mu_y, \tag{3.11}$$

we have

$$\mathbb{E}^x[\tau(x_0,r)] \le \frac{C\eta(r)}{V(x,r)}V(x_0,r) \le C'\eta(r),$$

where we use (3.8), (3.10), and (VD). We thus obtain the upper bound in (3.5).

For the lower bound, by (1.9) and (3.6) we have

$$|1-p_{x_0}(y)|^2 \le \frac{R(x_0,y)}{g_B(x_0,x_0)} = \frac{R(x_0,y)}{R(x_0,B^c)}.$$

Thus, if $d(x_0,y) = \lambda r$, using the upper bound in (2.6), and (2.8) (which holds due to Lemma 2.4) we obtain

$$|1-p_{x_0}(y)|^2 \le C\frac{\eta(\lambda r)}{\eta(cr)}\frac{V(x_0,r)}{V(x_0,\lambda r)}.$$

Hence, by (2.7), there exists $\delta > 0$ such that

$$p_{x_0}(y) = \frac{g_B(x_0,y)}{g_B(x_0,x_0)} \ge 1/2 \qquad \forall y \in B(x_0,\delta r). \tag{3.12}$$

On the other hand, by (3.9) and (2.8) , we have

$$g_B(x_0,x_0) = R(x_0,B^c) \ge \frac{c\eta(cr)}{V(x_0,r)}. \tag{3.13}$$

Combining this with (3.12),

$$g_B(x_0,y) \ge \frac{c\eta(cr)}{2V(x_0,r)}, \qquad \forall y \in B(x_0,\delta r).$$

Thus, using (3.11) and (VD),

$$\mathbb{E}^{x_0}[\tau(x_0,r)] = \sum_{x\in B} g_B(x_0,y)\mu_y \ge \frac{c\eta(cr)}{2V(x_0,r)}V(x_0,\delta r) \ge c'\eta(cr),$$

where $c' > 0$ depends on δ. We thus obtain the second estimate in (3.5). $\square$

As a by-product of Proposition 3.4, we obtain:

Proposition 3.5. *Let* (Γ,μ) *be a weighted graph satisfying* (VD)*,* (2.6)*,* (3.2) *and* (2.7)*. Then it is very strongly recurrent and satisfies the elliptic Harnack inequality* (EHI).

Proof. (3.12), which was obtained in the proof of Proposition 3.4, implies immediately that (Γ,μ) satisfies (1.10). This implies (*EHI*) by [4], Lemma 1.6. □

We also can prove that (1.10) implies (1.11).

Lemma 3.6. *If* (Γ,μ) *is very strongly recurrent it is strongly recurrent.*

Proof. Note first that if $d(x,y) \le r$ then $\tau(x,2r) \le \tau(y,3r)$. So (1.10) implies that

$$\mathbb{P}^x(T_y < \tau(y,3r)) \ge p_1, \quad \text{for all } y \in \Gamma, r \ge 1, x \in B(y,r). \tag{3.14}$$

Fix $y \in \Gamma$, and set

$$L_n = \mu_y^{-1} \sum_{r=0}^{n} 1_{\{X_r = y\}}.$$

Then $\mathbb{E}^y L_{\tau_B} = g_B(y,y)$.

Let $B = B(y,r)$, and write $A = B(y,3r)$. Then if $d(x,y) = r+1$ the condition (1.10) implies that

$$g_A(x,y) \ge p_2 g_A(y,y). \tag{3.15}$$

So,

$$\begin{aligned} g_A(y,y) &= \mathbb{E}^x L_{\tau_A} = \mathbb{E}^x(L_{\tau_A} - L_{\tau_B}) + g_B(y,y) \\ &= \mathbb{E}^x \mathbb{E}^{X_{\tau_B}} L_{\tau_A} + g_B(y,y) \ge p_2 g_A(y,y) + g_B(y,y). \end{aligned}$$

Thus $g_A(y,y) \ge (1-p_2)^{-1} g_B(y,y)$, and using (3.7) we deduce that (Γ,μ) satisfies (1.11). □

We now come back to our main goal, which is to prove that $(R(\beta))$+$(VG(\beta_-)) \Rightarrow$ $(HK(\beta))$. Given Proposition 3.4 we could finish its proof by using known results. By [30], Theorem 6.2, (*VD*)+($DUHK(\beta)$)+(E_β) implies ($UHK(\beta)$). We could even have avoided Section 3.1, since Lemma 2.2 and Proposition 3.4 show that $(R(\beta))$+$(VG(\beta_-))$ implies (E_β), while as we just observed (*EHI*) is a by-product of Proposition 3.4. Now one concludes by invoking [30], Theorem 3.1 (i), (iv), i.e. that (*VD*)+(*EHI*)+(E_β) is equivalent to $(HK(\beta))$. We did not choose this way because our Theorem 3.1 is of independent interest, and has a much simpler proof than the general on-diagonal upper bound in [30].

We will also give full details of the final steps of the proof, because again they are much simpler in our strongly recurrent situation than in [30].

We also mention that yet another approach can be found in [27], where it is proved that $(DUHK(\beta))$+ (E_β) implies (and in fact is equivalent to) $(UHK(\beta))$, the intermediate step being a β-version of the so-called relative Faber-Krahn inequality used for instance in [16]. See also [42] for related sufficient conditions for $(UHK(\beta))$.

Finally, under the assumptions of Proposition 3.5, one can probably obtain general heat kernel estimates in the style of [36], Section 5. We will not pursue this here, and we will limit ourselves in this regard to the case where η is a power function. See [44] for this generalization.

Consider the following estimate of the repartition function of the exit time:

$$\Psi_n(x,r) := \mathbb{P}^x(\tau(x,r) \le n) \le C\exp\left(-\left(\frac{r^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right). \qquad (\Psi)$$

The next lemma is known (see for instance, Proposition 7.1 of [29]) but we will give a shorter probabilistic proof based on an argument which dates back to [6], with several subsequent variations.

Lemma 3.7. *On any weighted graph* (Γ,μ), $(E_\beta) \Rightarrow (\Psi)$.

Proof. Assume (E_β). We first prove that there exist $0 < p < 1$ and $A > 0$ such that

$$\text{(3.16)} \qquad \mathbb{P}^x(\tau(x,r) \le n) \le p + An/r^\beta \qquad \forall x \in \Gamma,\, r > 0,\ n \in \mathbb{Z}_+.$$

Indeed, by the Markov property we have

$$\mathbb{E}^x[\tau(x,r)] \le n + \mathbb{E}^x\left[1_{\{\tau(x,r)>n\}}\mathbb{E}^{X_n}[\tau(x,r)]\right] \le n + \mathbb{E}^x\left[1_{\{\tau(x,r)>n\}}\mathbb{E}^{X_n}[\tau(X_n,2r)]\right].$$

Applying (E_β), we have

$$cr^\beta \le n + Cr^\beta \mathbb{P}^x(\tau(x,r) > n) = n + Cr^\beta(1 - \mathbb{P}^x(\tau(x,r) \le n)).$$

Rearranging gives (3.16).

Next, let $l \ge 1$, $b = [r/l]$, and define stopping times σ_i, $i \ge 0$ by

$$\sigma_0 = 0,\ \sigma_{i+1} = \inf\{m \ge \sigma_i : d(X_{\sigma_i}, X_m) \ge b\}.$$

Let $\xi_i = \sigma_i - \sigma_{i-1}$, $i \ge 1$. Let $\mathscr{F}_m$ be the filtration generated by $\{X_i : i \le m\}$ and let $\mathscr{G}_m = \mathscr{F}_{\sigma_m}$. We have by (3.16)

$$\mathbb{P}^x(\xi_{i+1} \le n|\mathscr{G}_i) = \mathbb{P}^{X_{\sigma_i}}(\tau(X_{\sigma_i}, b-1) \le n) \le p + An/(b-1)^\beta \le p + A'n/b^\beta.$$

Since $d(X_{\sigma_i}, X_{\sigma_{i+1}}) = b$, we have $d(X_0, X_{\sigma_l}) \le r$, so that $\sigma_l = \sum_{i=1}^l \xi_i \le \tau(X_0, r)$. So, by Lemma 3.14 in [3],

$$\log \mathbb{P}^x(\tau(x,r) \le n) \le 2\left(\frac{A'ln}{pb^\beta}\right)^{1/2} - l\log\left(\frac{1}{p}\right) = C(r^{-\beta}l^{1+\beta}n)^{1/2} - cl.$$

Now we optimize on l; namely, we consider the case $r^\beta n^{-1} \ge a$ for some $a > 0$ and take l_0 the greatest integer l that satisfies

$$\text{(3.17)} \qquad Cl/2 > c(r^{-\beta}l^{1+\beta}n)^{1/2}.$$

Note that by taking a large enough, (3.17) holds for small $l \in \mathbb{N}$. Then

$$l_0^{\beta-1} < (c^2/4C^2)r^\beta n^{-1} \le (l_0+1)^{\beta-1}, \text{ and } \log \mathbb{P}^x(\tau(x,r) \le n) \le -cl_0/2.$$

We thus obtain (Ψ) when $r^\beta n^{-1} \ge a$. Adjusting the constant if necessary, (Ψ) clearly holds also if $r^\beta n^{-1} < a$. $\square$

Proposition 3.8. $(VD)+(DUHK(\beta))+(\Psi)\Rightarrow(UHK(\beta))$.

Proof. We adapt the proof of Proposition 8.1 of [29], which is written for polynomial growth, to the volume doubling situation – see also [8], Theorem 6.2. This uses the following general inequality for reversible Markov chains:

$$h_{n+m}(x,y)\le \Psi_n(x,r)\sup_z h_m(y,z)+\Psi_m(y,r)\sup_z h_n(x,z), \qquad \forall x,y\in\Gamma, n,m\in\mathbb{N}, \tag{3.18}$$

where $r<d(x,y)/2$.

Let us recall its proof for the sake of completeness. For $x,y\in\Gamma$ distinct, let $r<d(x,y)/2$. Then, since $B(x,r)$ and $B(y,r)$ do not intersect, and for $n,m\in\mathbb{N}$,

$$\begin{aligned} h_{n+m}(x,y) &\le \sum_{z\notin B(x,r)} h_n(x,z)h_m(z,y)\mu_z+\sum_{z\notin B(y,r)} h_n(x,z)h_m(z,y)\mu_z \\ &\le \sup_z h_m(z,y)\sum_{z\notin B(x,r)} p_n(x,z)+\sup_z h_n(x,z)\sum_{z\notin B(y,r)} p_m(y,z) \\ &= \sup_z h_m(y,z)\mathbb{P}^x(X_n\notin B(x,r))+\sup_z h_n(x,z)\mathbb{P}^y(X_m\notin B(y,r)). \end{aligned}$$

Since $\mathbb{P}^x(X_n\notin B(x,r))\le\Psi_n(x,r)$, (3.18) follows. Now, using Chapman-Kolmogorov, Cauchy-Schwarz, and again the symmetry of h_n,

$$\begin{aligned} h_{2n}(x,y)=\sum_{z\in\Gamma} h_n(x,z)h_n(z,y)\mu_z &\le \left(\sum_z h_n^2(x,z)\mu_z\right)^{1/2}\left(\sum_z h_n^2(y,z)\mu_z\right)^{1/2} \\ &= h_{2n}(x,x)^{1/2}h_{2n}(y,y)^{1/2}. \end{aligned}$$

Thus, by $(DUHK(\beta))$,

$$h_{2n}(x,y)\le\frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}}, \qquad \forall x,y\in\Gamma, n\ge 1. \tag{3.19}$$

Taking $m=n$ in (3.18), applying (Ψ) and taking, say, $r=d(x,y)/3$ yields

$$h_{2n}(x,y)\le\frac{C'}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}}\exp\left(-\left(\frac{r^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right), \qquad \forall x,y\in\Gamma, n\ge 1, \tag{3.20}$$

and a similar estimate follows for odd n since

$$h_{2n+1}(x,y)=\sum_{z\in\Gamma} h_{2n}(x,z)p(z,y)\le\max_{z\sim x} h_{2n}(x,z).$$

By (VD), these estimates are equivalent to $(UHK(\beta))$. $\square$

3.3 Lower heat kernel estimates

We now prove the lower bounds. Our approach is much more direct than the one in [30], essentially because we rely fully on the assumption $(VG(\beta_-))$ and incorporate some arguments from [15]. The first step is an argument of Benjamini-Chavel-Feldman [13] in the case $\beta = 2$ (see also Lemma 7.1 of [8]), which can easily be adapted (see [15], proof of Theorem 3.1), to show that the upper estimate $(UHK(\beta))$ together with (VD) always implies an on-diagonal lower heat kernel estimate. We sketch a proof for the sake of completeness.

Proposition 3.9. *Assume (VD) and $(UHK(\beta))$. Then there exists a constant $c > 0$ such that*

$$h_{2n}(x,x) \geq \frac{c}{V(x,n^{1/\beta})},\ \forall x \in \Gamma, n \in \mathbb{N}. \qquad (DLHK(\beta))$$

Proof. Using (VD) and $(UHK(\beta))$, one checks that, for C large enough, every $n \in \mathbb{N}$ and every $x \in \Gamma$,

$$\sum_{y \notin B(x,Cn^{1/\beta})} h_n(x,y)\,\mu_y \leq 1/2,$$

(see the computations in [28], proof of Theorem 3.2). Thus

$$\sum_{y \in B(x,Cn^{1/\beta})} h_n(x,y)\,\mu_y = 1 - \sum_{y \notin B(x,Cn^{1/\beta})} h_n(x,y)\,\mu_y \geq 1/2$$

Write now

$$\begin{aligned} h_{2n}(x,x) &= \sum_{y\in\Gamma} h_n^2(x,y)\,\mu_y \geq \sum_{B(x,Cn^{1/\beta})} h_n^2(x,y)\,\mu_y \\ &\geq \frac{1}{V(x,Cn^{1/\beta})}\left(\sum_{B(x,Cn^{1/\beta})} h_n(x,y)\,\mu_y\right)^2 \geq \frac{1}{4V(x,Cn^{1/\beta})}. \end{aligned}$$

Adjusting for parity and using (VD), $(DLHK(\beta))$ follows. □

We now prove a near-diagonal lower estimate. The next proposition is inspired by [15], pp. 800-801, with the usual additional difficulties due to discrete time. See also [20], Lemma 2.4.

We first need a lemma. For a fixed x_0 in Γ, set $f_n(\cdot) = h_n(x_0,\cdot) + h_{n+1}(x_0,.)$.

Lemma 3.10.

$$\mathscr{E}(f_n) \leq \frac{C}{n} h_{2[n/2]}(x_0,x_0),\ \forall n \geq 1. \qquad (3.21)$$

Proof. Set $g_n(\cdot) = h_n(x_0,\cdot)$. One checks easily that $Pg_n = g_{n+1}$. Thus

$$\begin{aligned} \mathscr{E}(f_n) &= \langle (I-P)(g_n + g_{n+1}), g_n + g_{n+1}\rangle \\ &= \langle (I-P)(P^n + P^{n+1})g_0, (P^n + P^{n+1})g_0\rangle \\ &= \langle (I-P^2)P^{2n}g_0, g_0\rangle + \langle (I-P^2)P^{2n+1}g_0, g_0\rangle. \end{aligned}$$

Using the fact that P is self-adjoint and contractive on $\ell^2(\Gamma,\mu)$, write

$$\begin{aligned}\mathscr{E}(f_n) &= \langle (I-P^2)g_n, g_n\rangle + \langle (I-P^2)g_{n+1}, g_n\rangle \\ &\leq \|(I-P^2)g_n\|_2\|g_n\|_2 + \|(I-P^2)g_{n+1}\|_2\|g_n\|_2 \\ &\leq 2\|(I-P^2)g_n\|_2\|g_n\|_2 \leq 2\|(I-P^2)P^{[(n+1)/2]}g_{[n/2]}\|_2\|g_{[n/2]}\|_2.\end{aligned}$$

Now

$$\|(I-P^2)P^n\|_{2\to 2} \leq \frac{C}{n}, n\geq 1$$

by spectral theory (this is because P^2 is a non-negative operator on $\ell^2(\Gamma,\mu)$); for details and comments, see [19], p. 426). Therefore, for $n \geq 1$,

$$\mathscr{E}(f_n) \leq \frac{C'}{[(n+1)/2]}\|g_{[n/2]}\|_2^2 = \frac{C''}{n}h_{2[n/2]}(x_0,x_0),$$

according to (1.3). □

Set $u_n(x,y) := h_n(x,y) + h_{n+1}(x,y)$.

Proposition 3.11. *Assume* (2.1), $(RU(\beta))$, *and* $(DLHK(\beta))$. *Then, there exist* $c, C > 0$ *such that*

$$u_n(x,y) \geq \frac{c}{V(x,n^{1/\beta})}, \forall x,y\in\Gamma, n\in\mathbb{N} \quad \textit{such that } d(x,y)\leq Cn^{1/\beta}. \quad (NLHK(\beta))$$

Proof. Let $x_0 \in \Gamma$. Putting $f_n(\cdot) = u_n(x_0,\cdot)$ in (1.9) gives

(3.22) $$|f_n(x) - f_n(y)|^2 \leq R(x,y)\mathscr{E}(f_n) \text{ for all } x,y\in\Gamma.$$

Since $(DUHK(\beta))$, $(DLHK(\beta))$ and (VD) hold,

$$h_{2[n/2]}(x_0,x_0) \leq C f_n(x_0,x_0).$$

Thus, using (3.21), one obtains

(3.23) $$|f_n(x) - f_n(y)|^2 \leq \frac{C}{n}R(x,y)f_n(x_0,x_0).$$

So, using $(RU(\beta))$,

$$|f_n(x) - f_n(y)|^2 \leq \frac{C'd^\beta(x,y)}{nV(x,y)}f_n(x_0,x_0),$$

and using $(DLHK(\beta))$ again,

$$|f_n(x) - f_n(y)|^2 \leq C''\frac{d^\beta(x,y)}{n}\frac{V(x_0,n^{1/\beta})}{V(x,y)}f_n^2(x_0,x_0).$$

In particular, choosing $x_0 = x$,

$$|u_n(x,x) - u_n(x,y)|^2 \leq C''\frac{d^\beta(x,y)}{n}\frac{V(x,n^{1/\beta})}{V(x,y)}u_n^2(x,x).$$

By (2.1), there exists $\delta > 0$ such that

$$C'' \frac{d^\beta(x,y)}{n} \frac{V(x,n^{1/\beta})}{V(x,y)} \le \frac{1}{4}$$

as soon as $d(x,y) \le \delta n^{1/\beta}$. For such y,

$$|u_n(x,x) - u_n(x,y)| \le \frac{1}{2} u_n(x,x).$$

Thus

$$u_n(x,y) \ge \frac{1}{2} u_n(x,x),$$

and $(NLHK(\beta))$ follows from $(DLHK(\beta))$. □

The full heat kernel lower bound is now within reach.

Proposition 3.12. $(p_0) + (VD) + (NLHK(\beta)) \Rightarrow (LHK(\beta))$

Proof. This is a classical iteration argument, see for instance [22], Theorem 3.8, for the case $\beta = 2$. We write the proof for the sake of completeness, and also to emphasize the role of condition (p_0). We write $\tilde{p}_n(x,y) = u_n(x,y)\mu_y = p_n(x,y) + p_{n+1}(x,y)$. We consider the following cases:
Case 1: $d(x,y) \le Cn^{1/\beta}$;
Case 2: $Cn^{1/\beta} < d(x,y) \le \varepsilon n$;
Case 3: $\varepsilon n < d(x,y) \le n$,
where C is the constant in Proposition 3.11 and $\varepsilon > 0$ is a small constant chosen later.

In Case 1, $(LHK(\beta))$ follows from Proposition 3.11. In Case 3, $(LHK(\beta))$ becomes

$$\tilde{p}_n(x,y) \ge \frac{c\mu_y}{V(x,n^{1/\beta})} \exp(-c'n), \tag{3.24}$$

which can be deduced directly from (p_0). Indeed, since there is a path from x to y of length either n or $n+1$, the $\mathbb{P}^x$-probability that the random walk will follow the path is at least $p_0^{-(n+1)}$. Thus, $\tilde{p}_n(x,y) \ge \exp(-c'n)$. Clearly $\mu_y/V(y,n^{1/\beta}) \le 1$, so we obtain (3.24).

We now consider Case 2, which is the main case. Denote $d = d(x,y)$, take $k \in \mathbb{N}$ such that

$$k \le d, \tag{3.25}$$

and define m by $m = \lfloor n/k \rfloor - 1$. Since $k \le d \le \varepsilon n$, we see that $n/k \ge \varepsilon^{-1}$ and m is positive. Since $n \ge k(m+1)$, by a simple calculation using (p_0) condition and Chapman-Kolmogorov, (cf. Lemma 13.6 in [29]), we have

$$C^{n-mk} \tilde{p}_n(x,y) \ge (\tilde{p}_m)^k(x,y), \tag{3.26}$$

where $(\tilde{p}_m)^k$ is the k-th convolution power of $\tilde{p}_m = p_m + p_{m+1}$. Note that there exists a sequence $o_1, o_2, \cdots o_k \in \Gamma$ such that $x = o_1, y = o_k$ and

$$d(o_i, o_{i+1}) \leq \lceil \frac{d(x,y)}{k} \rceil := r \qquad \forall i = 1, 2, \cdots, k-1. \tag{3.27}$$

Clearly, we have

$$(\tilde{p}_m)^k(x,y) \geq \sum_{z_1 \in B(o_1, r)} \cdots \sum_{z_{k-1} \in B(o_{k-1}, r)} \tilde{p}_m(x, z_1) \tilde{p}_m(z_1, z_2) \cdots \tilde{p}_m(z_{k-1}, y). \tag{3.28}$$

Assume that we have in addition

$$3r \leq C m^{1/\beta}. \tag{3.29}$$

Since $d(z_{i-1}, z_i) \leq 3r$, by Proposition 3.11, we have $\tilde{p}_m(z_{i-1}, z_i) \geq c\mu_{z_i} V(z_{i-1}, m^{1/\beta})^{-1}$ for all $i = 2, \cdots, k-1$. The same applies to $\tilde{p}_m(x, z_1)$ and $\tilde{p}_m(z_{k-1}, y)$. So, we obtain from (3.26) and (3.28)

$$C^{n-mk} \tilde{p}_n(x,y) \geq \frac{c^k \mu_y}{V(x, m^{1/\beta})} \prod_{i=1}^{k-1} \left(\sum_{z_i \in B(o_i, r)} \frac{\mu_{z_i}}{V(z_i, m^{1/\beta})} \right). \tag{3.30}$$

We now specify the choice of k to ensure that both (3.25) and (3.29) hold. Using the definition of m and r, we see that (3.29) is equivalent to $cd/k \leq C(n/k)^{1/\beta}$ or

$$k \geq c' C^{-\beta/(\beta-1)} \left(\frac{d^\beta}{n} \right)^{1/(\beta-1)}. \tag{3.31}$$

Let k be the minimal possible integer satisfying (3.31). By the hypothesis $d \geq Cn^{1/\beta}$, we have

$$k \geq c', \qquad k \simeq \left(\frac{d^\beta}{n} \right)^{1/(\beta-1)}. \tag{3.32}$$

The condition (3.25) follows from the hypothesis $n \geq \varepsilon^{-1} d$ provided $\varepsilon > 0$ is small enough. By (3.27), (3.29), (3.32) and by the choice of m, we obtain

$$m \simeq \left(\frac{n}{d} \right)^{\beta/(\beta-1)}, \qquad r \simeq \left(\frac{n}{d} \right)^{1/(\beta-1)}. \tag{3.33}$$

By (VD) and (3.33),

$$\sum_{z_i \in B(o_i, r)} \frac{\mu_{z_i}}{V(z_i, m^{1/\beta})} \geq \frac{c \sum_{z_i \in B(o_i, r)} \mu_{z_i}}{V(o_i, m^{1/\beta})} = \frac{c' V(o_i, r)}{V(o_i, m^{1/\beta})} \geq c''.$$

Combining this with (3.30), we obtain

$$\tilde{p}_n(x,y) \geq \frac{c^k C^{mk-m} \mu_y}{V(x, m^{1/\beta})} \geq \frac{c^k C^{-(n-mk)} \mu_y}{V(x, m^{1/\beta})} \geq \frac{\mu_y}{V(x, n^{1/\beta})} \exp(-c' k), \tag{3.34}$$

where we use the facts $m \leq n$, $n - mk \leq 2k$ which follow from the definition of m. Putting (3.32) into (3.34), we obtain $(LHK(\beta))$. □

4 From heat kernel estimates to resistance estimates

In this section, we prove that $(HK(\beta))$ together with $(VG(\beta_-))$ implies $(R(\beta))$. In fact, we shall prove that $(HK(\beta))$ alone implies $(PI(\beta))$ and $(SRL(\beta))$. The above claim then follows from Lemma 2.3. Note that $(HK(\beta)) \Rightarrow (VD)$ holds (see Proposition 7.2 in [53], or Theorem 3.1 in [30]).

We first give a lemma, which is a sub-Gaussian version of Lemma 5.1 in [25] (see also Lemma 3.9 in [22]).

Lemma 4.1. *Let $x_0 \in \Gamma$, $r > 0$, $B := B(x_0, r)$ and let $h_n^B(x,y)$ be the transition density of the random walk X_n killed on exiting B. Suppose $(HK(\beta))$ holds. Given $0 < \varepsilon < 1$, there exist $c, a > 0$ such that*

$$u_n^B(x,y) := h_n^B(x,y) + h_{n+1}^B(x,y) \geq \frac{c}{V(x,n^{1/\beta})} \text{ for } x,y \in B(x_0,(1-\varepsilon)n^{1/\beta}), n \leq (ar)^\beta.$$

Proof. Let $x,y \in \Gamma$, $n \in \mathbb{N}$. By $(LHK(\beta))$, we have

$$u_n(x,y) \geq \frac{c_0}{V(x,n^{1/\beta})}, \text{ for } x,y \in B(x_0,n^{1/\beta}), \tag{4.1}$$

for some $c_0 > 0$. Then, note that the following holds:

$$h_n^B(x,y) = h_n(x,y) - \mathbb{E}^x[1_{\{T_B \leq n\}} h_{n-T_B}(X_{T_B}, y)]. \tag{4.2}$$

As a consequence,

$$h_n(x,y) - h_n^B(x,y) = h_n(y,x) - h_n^B(y,x) = \sum_{\substack{0 \leq s \leq n \\ \xi \in B^c}} \mathbb{P}^y(X_s = \xi, T_{B^c} = s) h_{n-s}(\xi, x). \tag{4.3}$$

Now we estimate (4.3) from above. By $(UHK(\beta))$, for $0 \leq s \leq n$ (in fact we can assume $n - s \geq r > 0$ since otherwise $h_{n-s}(\xi,x) = 0$)

$$h_{n-s}(\xi,x) = h_{n-s}(x,\xi) \leq \frac{C}{V(x,(n-s)^{1/\beta})} \exp\left(-\left(\frac{d(x,\xi)^\beta}{C(n-s)}\right)^{1/(\beta-1)}\right).$$

If $\xi \in B^c$, $x \in B(x_0,(1-\varepsilon)n^{1/\beta})$, and $n \leq r^\beta$, it follows that $d(x,\xi) \geq \varepsilon r$, hence

$$h_{n-s}(\xi,x) \leq \frac{C}{V(x,(n-s)^{1/\beta})} \exp\left(-\left(\frac{\varepsilon^\beta r^\beta}{C(n-s)}\right)^{1/(\beta-1)}\right).$$

Also,

$$\begin{aligned} h_{n-s}(\xi,x) &\leq \frac{C}{V(x,n^{1/\beta})} \left(\frac{V(x,r)}{V(x,(n-s)^{1/\beta})} \exp\left(-\left(\frac{\varepsilon^\beta r^\beta}{C(n-s)}\right)^{1/(\beta-1)}\right)\right) \\ &\leq \frac{C'}{V(x,n^{1/\beta})} \exp\left(-c'\left(\frac{r^\beta}{n-s}\right)^{1/(\beta-1)}\right) \end{aligned}$$

by doubling . Now, if in addition $n \le (ar)^\beta$, $a \le 1$,

$$\frac{r^\beta}{n-s} \ge \frac{r^\beta}{n} \ge a^{-\beta},$$

and

$$h_{n-s}(\xi,x) \le \frac{C'}{V(x,n^{1/\beta})} \exp\left(-c' a^{-\beta/(\beta-1)}\right).$$

Taking a small enough we can ensure that

$$h_{n-s}(\xi,x) \le \frac{c_0}{4V(x,n^{1/\beta})}, \tag{4.4}$$

for all $\xi \in B^c$ and $0 \le s \le n$. Gathering (4.3) and (4.4), we obtain

$$h_n(x,y) - h_n^B(x,y) \le \frac{c_0}{4V(x,n^{1/\beta})} \sum_{\substack{0\le s\le n \\ \xi \in B^c}} \mathbb{P}^y(X_s=\xi, T_{B^c}=s) \le \frac{c_0}{4V(x,n^{1/\beta})},$$

hence

$$u_n(x,y) - u_n^B(x,y) \le \frac{c_0}{2V(x,n^{1/\beta})}.$$

Together with (4.1), this yields the result. □

Proposition 4.2. *1)* $(HK(\beta)) \Rightarrow (PI(\beta))$.
2) $(HK(\beta)) \Rightarrow (SRL(\beta))$.

Proof. The proof of 1) is standard. It originates in [45]. We modify the argument given in [50] (see also Theorem 3.11 in [22]). Let $B = B(x_0,r)$. Let $\{H_n^B\}$ be the discrete time semigroup corresponding to $\mathscr{E}_B$ (the corresponding process is the random walk X_n reflected at $\{y : d(x_0,y) = r\}$). If we denote the transition density as $\tilde{h}_n^B(x,y)$, then clearly $\tilde{h}_n^B(x,y) \ge h_n^B(x,y)$. Let $B' = B(x_0, ar/2)$ where $a > 0$ is the constant in Lemma 4.1. For $y \in B'$ we have by Lemma 4.1 (with $\varepsilon = 1/2$),

$$\begin{aligned} H^B_{[(ar)^\beta]}(f - H^B_{[(ar)^\beta]}f(y))^2(y) &\ge \frac{c}{V(x,ar/2)} \sum_{z\in B'} |f(z) - H^B_{[(ar)^\beta]}f(y)|^2 \mu_z \\ &\ge \frac{c}{V(x,ar/2)} \sum_{z\in B'} |f(z) - \bar{f}_{B'}|^2 \mu_z, \end{aligned}$$

where in the last inequality, we use the fact that $\sum_{z\in B'} |f(z)-\alpha|^2 \mu_z$ attains its minimum when $\alpha = \bar{f}_{B'}$. Summing up over B', we obtain

$$\sum_{y\in B'} H^B_{[(ar)^\beta]}(f - H^B_{[(ar)^\beta]}f(y))^2(y)\mu_y \ge c \sum_{z\in B'} |f(z) - \bar{f}_{B'}|^2 \mu_z.$$

On the other hand, we have

$$\sum_{y\in B'} H^B_{[(ar)^\beta]}(f - H^B_{[(ar)^\beta]}f(y))^2(y)\mu_y \le \|f\|^2_{2,B} - \|H^B_{[(ar)^\beta]}f\|^2_{2,B} \le Cr^\beta \mathscr{E}_B(f),$$

where we write $\|f\|^2_{2,B} = \sum_{y\in B} f(y)^2 \mu_y$. Here the first inequality is a simple computation of the variance (plus the fact $\|H_n^B f^2\|_{1,B} \le \|f\|^2_{2,B}$) and the second inequality is a general estimate of Dirichlet forms (Lemma 1.3.3 (i) in [26]). We thus obtain $(PI(\beta))$.

We now prove 2). By (3.9), we have $R(x_0,B^c) = \sum_{n=0}^{\infty} h_n^B(x_0,x_0)$. Thus, by Lemma 4.1,

$$\begin{aligned} R(x_0,B^c) &\geq c \sum_{n=0}^{[(ar)^\beta]} u_n^B(x_0,x_0) \geq c' \left(\sum_{n=1}^{[(ar)^\beta]} V(x_0,n^{1/\beta})^{-1} + 1 \right) \\ &\geq \frac{(ar)^\beta}{V(x_0,[(ar)^\beta]^{1/\beta})} \geq \frac{c'' r^\beta}{V(x_0,r)}, \end{aligned}$$

where we use (VD) in the last inequality. We thus obtain $(SRL(\beta))$. □

5 Examples

In this section, we give examples of weighted graphs where $(HK(\beta))$ holds. Let (Γ,μ) be a weighted graph satisfying the (p_0) condition.

Example 1: Case $\beta = 2$

We note that $(RL(2))$ always holds. Indeed, for $x,y \in \Gamma$, define $f = f_{x,y} \in \mathbb{R}^\Gamma$ by $f(z) := d(x,z) \wedge d(x,y)$. Then clearly $\mathscr{E}(f) \leq V(x,y)$. By this and (1.9), we have

$$R(x,y) \geq \frac{|f(x)-f(y)|^2}{\mathscr{E}(f)} \geq \frac{d(x,y)^2}{V(x,y)},$$

as required. It follows that $(RL(\beta))$ can hold only if $\beta \geq 2$, and, by using Theorem 1.3, one recovers the well-known fact that $(HK(\beta))$ can hold only if $\beta \geq 2$. Also, by Theorem 1.3, we have the following equivalence under $(VG(2_-))$ and (p_0):

$$(HK(2)) \Leftrightarrow (RU(2)) \Leftrightarrow (PI(2)). \tag{5.1}$$

It is known in general that $(HK(2))$ is equivalent to $(VD)+(PI(2))$ (cf. [22]), but (5.1) gives an additional equivalence condition under $(VG(2_-))$. In the even more particular situation where $(VG(1))$ and (2.4) hold (the volume growth is then linear) $(RU(2))$ follows from Lemma 2.1. One therefore recovers the well-known fact that, on a weighted graph with $(VG(1))$, (p_0) and (2.4), $(HK(2))$ always holds (see [20]).

Example 2: Trees

A graph Γ is called a *tree* if it has no cycle, a cycle being a sequence of vertices such that $x_0 \sim \ldots \sim x_n$, with no repetition besides $x_n = x_0$, and $n \geq 2$. A connected graph is a tree if and only if any two points x,y are joined by a unique (non-oriented) self-avoiding path.

Lemma 5.1. *If* Γ *is a tree satisfying* (2.4) *and*

$$\sup_{x\sim y} \mu_{x,y} = M < \infty., \tag{5.2}$$

then

$$R(x,y) \simeq d(x,y),\ x,y \in \Gamma.$$

Proof. It follows from Lemma 2.1 that $R(x,y) \leq C\,d(x,y)$, $\forall x,y \in \Gamma$. Now take two distinct points x,y in Γ, and let $f \in \mathbb{R}^\Gamma$ be such that $f(x) = 0$, $f(y) = 1$, f is linear on the geodesic path joining x and y, and constant on any other geodesic emanating from x, y, and any intermediate point. This construction is of course possible only because Γ is a tree. Now $\mathscr{E}(f) \leq \frac{M}{d(x,y)}$, therefore

$$\inf\{\mathscr{E}(f) : f \in \mathbb{R}^\Gamma, f(x) = 1, f(y) = 0\} \leq \frac{M}{d(x,y)},$$

hence $MR(x,y) \geq d(x,y)$. □

As a consequence, if Γ is a tree satisfying (2.4) and (5.2), $(RU(\beta))$ (resp. $(RL(\beta))$) is equivalent to the volume growth condition $V(x,y) \leq C\,d(x,y)^{\beta-1}$ (resp. $V(x,y) \geq c\,d(x,y)^{\beta-1}$), i.e. to the fact that (Γ,μ) has polynomial growth of exponent $\beta-1$:

$$V(x,r) \simeq r^{\beta-1},\ x \in \Gamma,\ r \geq 0.$$

Thus, as a corollary to Theorem 1.3, we have the following.

Proposition 5.2. *Let* (Γ,μ) *be a tree such that* $0 < \inf_{x\sim y} \mu_{xy} \leq \sup_{x\sim y} \mu_{xy} < \infty$. *If* (Γ,μ) *has polynomial volume growth of exponent* $\alpha \geq 1$*, then it satisfies the following heat kernel estimates*

$$h_n(x,y) \leq \frac{C}{n^{\frac{\alpha}{\alpha+1}}} \exp\left(-\left(\frac{d(x,y)^{\alpha+1}}{Cn}\right)^{\frac{1}{\alpha}}\right),\ \textit{for all } x,y \in \Gamma,\ \in \mathbb{N}$$

and

$$h_n(x,y) + h_{n+1}(x,y) \geq \frac{c}{n^{\frac{\alpha}{\alpha+1}}} \exp\left(-\left(\frac{d(x,y)^{\alpha+1}}{cn}\right)^{\frac{1}{\alpha}}\right).$$

Conversely, if (Γ,μ) *satisfies* $(VG(\beta\_))$ *and* $(HK(\beta))$ *for some* $\beta \geq 2$*, then it must have polynomial growth of exponent* $\beta - 1$.

An interesting class of trees with polynomial growth is given by the Vicsek graphs considered for instance in [12], Section 4.

<u>Example 3: Finitely ramified fractal graphs</u>

For $\alpha > 1$ and $I = \{1,2,\cdots,N\}$, let $\{\Psi_i\}_{i\in I}$ be a family of α*-similitudes* on $\mathbb{R}^D$. An α-similitude is a map $\Psi_i\mathbf{x} = \alpha^{-1}U_i\mathbf{x} + \gamma_i$, $\mathbf{x} \in \mathbb{R}^D$ where U_i is a unitary map and $\gamma_i \in \mathbb{R}^D$. We assume the *open set condition* for $\{\Psi_i\}_{i\in I}$, that there is a non-empty, bounded open set W such that $\{\Psi_i(W)\}_{i\in I}$ are disjoint and $\cup_{i\in I}\Psi_i(W) \subset W$. Since

$\{\Psi_i\}_{i\in I}$ is a family of contraction maps, there exists a unique non-void compact set K such that $K = \cup_{i\in I}\Psi_i(K)$. We will consider the case where K is connected.

Let Fix be the set of fixed points of the Ψ_i's, $i \in I$. A point $x \in Fix$ is called an *essential fixed point* if there exist $i, j \in I$, $i \neq j$ and $y \in Fix$ such that $\Psi_i(x) = \Psi_j(y)$. We write V_0 for the set of essential fixed points, we assume $\sharp V_0 \geq 2$. Following [34], K is called a (compact) uniform finitely ramified fractal (u.f.r. fractal for short) if it satisfies the following finitely ramified property in addition to the above properties:
(FR) If $\{i_1,\ldots,i_n\},\{j_1,\ldots,j_n\}$ are distinct sequences, then

$$\Psi_{i_1,\ldots,i_n}(K)\bigcap\Psi_{j_1,\ldots,j_n}(K) = \Psi_{i_1,\ldots,i_n}(V_0)\bigcap\Psi_{j_1,\ldots,j_n}(V_0),$$

where we denote $\Psi_{i_1,\ldots,i_n} = \Psi_{i_1}\circ\cdots\circ\Psi_{i_n}$. If we further assume the following symmetry condition, then K is called a (compact) nested fractal as introduced in [47].
(SYM) If $x, y \in V_0$, then the reflection in the hyperplane $H_{xy} = \{z \in \mathbb{R}^D : |z-x| = |z-y|\}$ maps V_n to itself, where

$$V_n = \cup_{i_1,\cdots,i_n\in I}\Psi_{i_1,\ldots,i_n}(V_0). \tag{5.3}$$

Thus, u.f.r. fractals form a class of fractals which is wider than nested fractals, and is included in the class of p.c.f. self-similar sets ([40]).

We assume without loss of generality that $\Psi_1(\mathbf{x}) = \alpha_1^{-1}\mathbf{x}$ and $\mathbf{0}$ belongs to V_0. Let $\Gamma = \cup_{n=0}^{\infty}\alpha^n V_n$. We now introduce uniform finitely ramified graphs. These will be graphs with vertices Γ and a collection of edges B. In order to define the edges, we first define $B_0 := \{\{x,y\} : x \neq y \in V_0\}$. Then inside each $\alpha^n\Psi_{i_1,\ldots,i_n}(V_0)$ ($n \geq 0, i_1,\cdots,i_n \in I$), we place a copy of B_0 and we denote by B the set of all the edges determined in this way. Now, we assign $\mu_{xy} = \mu_{yx} > 0$ for each $\{x,y\} \in B$. We assume that the weights are bounded from above and below, i.e. there exist $c, C > 0$ such that

$$c \leq \mu_{xy} \leq C, \qquad \forall\{x,y\} \in B. \tag{5.4}$$

Then, (Γ,μ) is a weighted graph satisfying condition (p_0). We call the weighted graph (Γ,μ) a uniform finitely ramified (u.f.r.) graph. If we construct the graph starting from a nested fractal, then it will be called a nested fractal graph. We denote the corresponding quadratic form (1.5) as $\mathscr{E}_\mu$.

On a u.f.r. graph (Γ,μ), we can naturally define a renormalization map F as follows.

$$\mathscr{E}_{F(\mu)}(u) = \inf\{\mathscr{E}_\mu(v) : v \in \mathbb{R}^\Gamma,\ v(\alpha x) = u(x),\ x \in \Gamma\}, \qquad \forall v \in \mathbb{R}^\Gamma.$$

In [34] (together with a result in [43]), the following theorem is proved.

Theorem 5.3. *Let (Γ,μ) a u.f.r. graph and assume that there exists $\{\mu_{xy}\}$ satisfying* (5.4) *and such that*

$$F(\mu) = \rho^{-1}\mu \tag{5.5}$$

for some $\rho > 0$. Then, there exist $C, c > 0$ (which depend on μ) and $0 < \gamma_1 \le \gamma_2$ such that for each $x, y \in \Gamma$ and $n \ge d(x,y)$,

$$h_n(x,y) \le Cn^{-\frac{S}{S+1}} \exp\left(- \left(\frac{R(x,y)^{S+1}}{Cn} \right)^{\gamma_1} \right),$$

$$h_n(x,y) + h_{n+1}(x,y) \ge cn^{-\frac{S}{S+1}} \exp\left(- \left(\frac{R(x,y)^{S+1}}{cn} \right)^{\gamma_2} \right),$$

where $S = \log N / \log \rho$.

In [34], it is also shown by counter-examples that in general one cannot take $\gamma_1 = \gamma_2$, and that one cannot obtain the same type of heat kernel estimates with $d(\cdot,\cdot)$ instead of $R(\cdot,\cdot)$.

By Theorem 5.3 and Theorem 1.3, we have the following characterization of $(HK(\beta))$.

Proposition 5.4. *Let (Γ, μ) be a u.f.r. graph satisfying* (5.5). *Then, $(HK(\beta))$ holds on (Γ, μ) for some $\beta \ge 2$ if and only if the following relation between the resistance metric and the graph distance holds:*

$$R(x,y) \simeq d(x,y)^{\gamma} \qquad \forall x, y \in \Gamma, \tag{5.6}$$

for some $\gamma \ge 1$.

Proof. Suppose (5.6) holds. Note that $\mu(B_R(x,r)) \simeq r^S$ where $B_R(x,r) := \{y \in \Gamma : R(x,y) \le r\}$ (cf. Lemma 3.2 in [34]). Thus, we have $V(x,r) \simeq r^{S\gamma}$. Similarly, since $E^x[T_{B_R(x,r)}] \simeq r^{S+1}$, we have $E^x[\tau(x,r)] \simeq r^{\beta}$ where $\beta = (S+1)\gamma$. Thus $(VG(\beta\_))$, $(RU(\beta))$ and $(RL(\beta))$ hold with $\beta = (S+1)\gamma$ which implies $(HK(\beta))$ by Theorem 1.3.

Next, suppose $(HK(\beta))$ holds. Then, by comparing them with Theorem 5.3 for $x = y$, we have $n^{-S/(S+1)} \simeq V(x, n^{1/\beta})$ for all $n \in \mathbb{N}$ and all $x \in \Gamma$. Thus,

$$V(x,r) \simeq r^{S\beta/(S+1)} \qquad \forall r \in \mathbb{N},\ x \in \Gamma, \tag{5.7}$$

so that $(VG(\beta\_))$ holds. Now, by Theorem 1.3, $(RU(\beta))$ and $(RL(\beta))$ hold. So, together with (5.7), we obtain (5.6) with $\gamma = \beta/(S+1)$. □

For nested fractal graphs, it is known that (5.5) and (5.6) hold. Thus, this proposition gives another proof of the known fact that $(HK(\beta))$ holds for such graphs (cf. [34]).

Example 4: Graphical Sierpinski carpets

Let $H_0 = [0,1]^d$, and let $l \in \mathbb{N}$, $l \ge 2$ be fixed. Set $\mathcal{Q} = \{\Pi_{i=1}^d [(k_i - 1)/l, k_i/l] : 1 \le k_i \le l,\ k_i \in \mathbb{N}\ (1 \le i \le d)\}$, let $l \le N \le l^d$ and let Ψ_i, $I \in I := \{1, \cdots, N\}$ be orientation preserving affine maps of H_0 onto some element of $\mathcal{Q}$. (We assume that the sets $\Psi_i(H_0)$ are distinct.) Set $H_1 = \cup_{i \in I} \Psi_i(H_0)$. Then, there exists a unique non-void compact set $K \subset H_0$ such that $K = \cup_{i \in I} \Psi_i(K)$. K is called a (generalized)

Sierpinski carpet if the following holds (cf. [9]):
(SC1) (Symmetry) H_1 is preserved by all the isometries of the unit cube H_0.
(SC2) (Connected) H_1 is connected.
(SC3) (Non-diagonality) Let B be a cube in H_0 which is the union of 2^d distinct elements of $\mathcal{Q}$. (So B has side length $2l^{-1}$.) Then if $\mathrm{Int}(H_1 \cap B)$ is non-empty, it is connected.
(SC4) (Borders included) H_1 contains the line segment $\{x : 0 \le x_1 \le 1, x_2 = \cdots = x_d = 0\}$.

The main difference from p.c.f. self-similar sets is that Sierpinski carpets are infinitely ramified, i.e. K cannot be disconnected by removing a finite number of points.

Let V_0 be a set of vertices for H_0 and define V_n as in (5.3). Then, one can define a graphical Sierpinski carpet Γ in the same way as in Example 3 – see [10]. In [7] and [49] it is shown that there exists $\rho > 0$ such that the resistance across a cube of side l^k in Γ grows as ρ^k. In [10] it is proved that Γ satisfies $(VG(\alpha))$ for $\alpha = \log N/\log l$, and $(HK(\beta))$ with $\beta = \log(\rho N)/\log l$.

The proof in [10] relies on an elliptic Harnack inequality, which is proved in [9] by a difficult probabilistic coupling argument. In the case $\rho > 1$ it may be possible to prove $(R(\beta))$ directly using resistance bounds similar to those in [7] and [49]; this would then yield a much quicker proof of $(HK(\beta))$ for these Sierpinski carpets. We remark that such classes of infinitely ramified fractals are also studied in [46].

Example 5: A graph which is strongly recurrent but not very strongly recurrent

Choose $\alpha_i < \beta_i$, $i = 1,2$ such that $2 \le \beta_i \le \alpha_i + 1$ and if $\theta_i = \beta_i - \alpha_i$ then $\theta_1 < \theta_2$. Then, by [4] there exist (distinct) graphs Γ_i, $i = 1,2$ satisfying (p_0), $VG(\alpha_i)$ and $R(\beta_i)$, and therefore also $HK(\beta_i)$. Since each of Γ_i is very strongly recurrent, it is also strongly recurrent. Fix $x_i \in \Gamma_i$, and let Γ be the graph obtained from $\Gamma_1 \cup \Gamma_2$ by identifying the vertices x_1 and x_2. (Γ is the join of Γ_1 and Γ_2.) We can assume that the probability, starting in the common point $x = x_1 = x_2$ that X enters each of Γ_i is $1/2$. In Γ_i one has, if $1 < r < 2r \le s$, that $R(B(x,r), B(x,R)^c) \simeq s^{\theta_i}$. Using this it is not hard to check that Γ satisfies (1.11).

However, Γ cannot satisfy (1.10) since it fails to satisfy the EHI. To see this, let $B = B_1(x_1,r) \cup B_2(x_2,r)$, and let h be the harmonic function on B which is zero on $\partial B \cap B_1(x_1,r)$ and 1 on $\partial B \cap B_2(x_2,r)$. Since (in Γ_i) one has that if $y_i \sim x_i$ then $\mathbb{P}^{y_i}(T_{x_i} > \tau(x,r)) \simeq r^{-\theta_i}$, we deduce that $h(x) \simeq r^{-\theta_2+\theta_1}$. On the other hand, if z_2 is a point in Γ_2 with $d(x_2,z_2) = r/2$ then $h(z_2) \ge c_1 > 0$.

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Bibliography

[1] Auscher P.; Coulhon T. Gaussian lower bounds for random walks from elliptic regularity. *Ann. Inst. Henri Poincaré, Probab. Statist.* **35** (2001), 605–630.

[2] Barlow M. T. Diffusions and analysis on fractals. Séminaire Bourbaki, 755, 1991-1992, *Astérisque* **206** (1992), 345–368.

[3] Barlow M. T. Diffusions on fractals. *Lectures on probability theory and statistics (Saint-Flour, 1995)*, 1–121, Lecture Notes in Math., 1690, Springer, Berlin, 1998.

[4] Barlow M. T. Which values of the volume growth and escape time exponent are possible for a graph?, *Rev. Mat. Iberoamericana* **20** (2004), 1–31.

[5] Barlow M. T. Anomalous diffusion and stability of Harnack inequalities. To appear in *Surveys on differential geometry*.

[6] Barlow M. T.; Bass R. F. Construction of Brownian motion on the Sierpinski carpet. *Ann. Inst. H. Poincaré, Probab. Statist.* **25** (1989), 225–257.

[7] Barlow M. T.; Bass R. F. On the resistance of the Sierpiński carpet. *Proc. Roy. Soc. London, Ser. A* **431** (1990), 345-360.

[8] Barlow M. T.; Bass R. F. Transition densities for Brownian motion on the Sierpinski carpet. *Probab. Theory Related Fields* **91** (1992), 307-330.

[9] Barlow M. T.; Bass R. F. Brownian motion and harmonic analysis on Sierpinski carpets. *Canadian J. of Math.* **51** (1999), 673–744.

[10] Barlow M. T.; Bass R. F. Random walks on graphical Sierpinski carpets. *Random walks and discrete potential theory (Cortona, 1997)*, 26–55, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.

[11] Barlow M. T.; Bass R. F. Stability of parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1501–1533.

[12] Barlow M. T.; Coulhon T.; Grigor'yan A. Manifolds and graphs with slow heat kernel decay. *Invent. Math.* **144** (2001), 609–649.

[13] Benjamini I.; Chavel I.; Feldman E. Heat kernel lower bounds on manifolds using the old ideas of Nash. *Proc. London Math. Soc. (3)* **72** (1996), no. 1, 215–240.

[14] Coulhon T. Espaces de Lipschitz et inégalités de Poincaré. *J. Funct. Anal.* **136** (1996), 81–113.

[15] Coulhon T. Off-diagonal heat kernel lower bounds without Poincaré. *J. London Math. Soc. (2)* **68** (2003), no. 3, 795–816.

[16] Coulhon T.; Grigor'yan A. Random walks on graphs with regular volume growth. *Geom. Funct. Anal.* **8** (1998), no. 4, 656–701.

[17] Coulhon T.; Grigor'yan A.; Zucca F. The discrete integral maximum principle and its applications. To appear in *Tohoku Math. J.*.

[18] Coulhon T.; Koskela P. Geometric interpretations of L^p-Poincaré inequalities on graphs with polynomial volume growth. *Milan J. Math.* **72** (2004), 209–248.

[19] Coulhon T.; Saloff-Coste L. Puissances d'un opérateur régularisant. *Ann. Inst. Henri Poincaré, Probab. Statist.* **26** (1990), no. 3, 419–436.

[20] Coulhon T.; Saloff-Coste L. Minorations pour les chaînes de Markov unidimensionnelles. *Probab. Theory Related Fields* **97** (1993), 423–431.

[21] Coulhon T.; Saloff-Coste L. Variétés riemanniennes isométriques à l'infini. *Rev. Mat. Iberoamericana* **11** (1995), no. 3, 687–726.

[22] Delmotte, T. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[23] Delmotte, T. Graphs between the elliptic and parabolic Harnack inequalities. *Potential Anal.* **16** (2002), no. 2, 151–168.

[24] Doyle P.; Snell, J. L. *Random Walks and Electrical Networks*. Math. Assoc. America, Washington D.C. 1984.

[25] Fabes, E. B.; Stroock, D. W. A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Rational Mech. Anal.* **96** (1986), 327–338.

[26] Fukushima, M.; Oshima, Y.; Takeda, M. *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, Berlin, 1994.

[27] Grigoryan A. Heat kernel upper bounds on fractal spaces. To appear in *J. London Math. Soc.*.

[28] Grigor'yan A.; Hu J.; Lau K. S. Heat kernels on metric-measure spaces and an application to semi-linear elliptic equations. *Trans. Amer. Math. Soc.* **355** (2003), no. 5, 2065–2095.

[29] Grigor'yan A.; Telcs A. Sub-Gaussian estimates of heat kernels on infinite graphs. *Duke Math. J.* **109** (2001), 451–510.

[30] Grigor'yan A.; Telcs A. Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Ann.* **324** (2002), 521–556.

[31] Hajłasz P.; Koskela P. Sobolev meets Poincaré. *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 1211–1215.

[32] Hajłasz P.; Koskela P. Sobolev met Poincaré. *Mem. Amer. Math. Soc.* **145** (2000), no. 688.

[33] Hambly B. M.; Kumagai T. Transition density estimates for diffusion processes on post critically finite self-similar fractals. *Proc. London Math. Soc. (3)* **78** (1999), no. 2, 431–458.

[34] Hambly B. M.; Kumagai T. Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries. *Fractal geometry and applications: A Jubilee of B. Mandelbrot (San Diego, CA, 2002)*, 233–260, Proc. Sympos. Pure Math., 72, Part 2, Amer. Math. Soc., Providence, RI, 2004.

[35] Hebisch W.; Saloff-Coste L. Gaussian estimates for Markov chains and random walks on groups. *Ann. Probab.* **21** (1993), 673–709.

[36] Hebisch W.; Saloff-Coste L. On the relation between elliptic and parabolic Harnack inequalities. *Ann. Inst. Fourier (Grenoble)* **51** (2001), 1437–1481.

[37] Jones O. D. Transition probabilities for the simple random walk on the Sierpinski graph. *Stoch. Proc. Appl.* **61** (1996), 45–69.

[38] Kanai M. Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds. *J. Math. Soc. Japan* **37** (1985), 391–413.

[39] Kanai M. Analytic inequalities, and rough isometries between non-compact riemannian manifolds. *Curvature and topology of Riemannian manifolds (Katata, 1985)*, 122–137, Lecture Notes in Math., 1201, Springer, Berlin, 1986.

[40] Kigami J. *Analysis on Fractals*. Cambridge Univ. Press, Cambridge, 2001.

[41] Kigami J. Harmonic analysis for resistance forms. *J. Funct. Anal.* **204** (2003), 399–444.

[42] Kigami J. Local Nash inequality and inhomogeneity of heat kernels. *Proc. London Math. Soc. (3)* **89** (2004), no. 2, 525–544.

[43] Kumagai T. Homogenization on finitely ramified fractals. *Stochastic analysis and related topics in Kyoto*, 189–207, Adv. Stud. Pure Math., 41, Math. Soc. Japan, Tokyo, 2004.

[44] Kumagai T. Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms. *Publ. RIMS, Kyoto Univ.* **40** (2004), 793–818.

[45] Kusuoka S.; Stroock D.W. Applications of Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), 391–442.

[46] Kusuoka S.; Zhou X. Y. Dirichlet form on fractals: Poincaré constant and resistance. *Probab. Theory Related Fields* **93** (1992), 169–196.

[47] Lindstrøm T. Brownian motion on nested fractals. *Mem. Amer. Math. Soc.* **83** (1990), no. 420.

[48] Lust-Piquard F. Lower bounds on $\|K^n\|_{1\to\infty}$ for some contractions K of $L^2(\mu)$, with some applications to Markov operators. *Math. Ann.* **303** (1995), 699–712.

[49] McGillivray I. Resistance in higher-dimensional Sierpiński carpets. *Potential Anal.* **16** (2002), no. 3, 289–303.

[50] Saloff-Coste L. A note on Poincaré, Sobolev and Harnack inequality. *Internat. Math. Res. Notices* **2** (1992), 27–38.

[51] Telcs A. Random walks on graphs, electric networks and fractals. *Probab. Theory Related Fields* **82** (1989), 435-451.

[52] Telcs A. Volume and time doubling of graphs and random walks: the strongly recurrent case. *Comm. Pure Appl. Math.* **54** (2001), 975–1018.

[53] Telcs A. Local sub-Gaussian estimates on graphs: the strongly recurrent case. *Electron. J. Probab.* **6** (2001), no. 22, 33 pp.
http://www.math.washington.edu/∼ejpecp/EjpVol16/paper22.abs.html.

[54] Telcs A. A note on rough isometry invariance of resistance. *Combin. Probab. Comput.* **11** (2002), 427–432.

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