Function Spaces and Stochastic Processes on Fractals

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Abstract. We summarize recent work on function spaces and stochastic processes on fractals. We discuss relations between various non-local Dirichlet forms on fractals whose domains are Besov spaces. The corresponding stochastic processes are jump-type processes. Results on heat kernel estimates for the processes are introduced. We will also discuss how jump processes and diffusion processes are related by observing their function spaces.

1. Introduction

In this article, we summarize recent work on function spaces and stochastic processes on fractals. Mathematical study of stochastic processes and the corresponding operators on fractals has its original motivation in physics to understand the heat transfer on such disordered media ([17]). Diffusion processes on fractals typically have sub-Gaussian behaviour and detailed properties such as estimates of the heat kernels (fundamental solutions of the heat equations) are obtained for typical fractals (see, for instance, [2, 23, 24, 28, 33]). On the other hand, there has been deep study of Besov spaces (fractional extensions of Sobolev spaces) on the so called d-sets (see [22, 36, 37]). Recently, it is getting clear that the domains of Dirichlet forms (quadratic forms that determine stochastic processes and the corresponding operators) on fractals are Besov-Lipschitz spaces. This motivates further study of Besov-type function spaces using probability and potential theory. For instance, detailed estimates of heat kernels can be obtained for operators naturally defined from norms of Besov spaces ([8, 9]). The purpose of this article is to make it clearer how the recent results on this direction are connected. Because of the limitation of pages, we do not give any proof except that of Theorem 3.2. The proof and further details are given in the references cited.

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The organization of the paper is as follows. In Section 2, we first review basic facts on the Dirichlet form theory which connect function spaces and stochastic processes. We will also give a definition of d-sets, the state space we work on. In Section 3, we discuss about function spaces appear as domains of local regular Dirichlet forms on fractals, whose corresponding generators are so called Laplacians on fractals. Characterization of the domains is possible through heat kernel estimates (3.1) of the Laplacians. In Section 4, we introduce other types of function spaces appear as domains of non-local Dirichlet forms on d-sets, whose corresponding processes are stable-like jump processes. Three natural jump-type processes are introduced on d-sets and the corresponding three forms are shown to be equivalent. It turns out that one of the forms (and the corresponding operators) corresponds to the form (and the operator) studied by Triebel in [36, 37] and by Zähle in [39, 40, 16]. In Section 5, we will summarize results on heat kernel estimates for the stable-like jump processes. Finally in Section 6, we will give some sufficient conditions to construct diffusion processes on d-sets. The relation to stable-like jump processes is also discussed.

Throughout the paper, we will mainly consider compact fractals, but most of the results hold for unbounded fractals with suitable modifications of the statements.

For readers who are interested in the topics, we note that function spaces on fractals are also discussed in [14, 34]. Surveys and references on jump-type process on fractals may be found in [26, 38].

2. Dirichlet forms and d-sets

In this section, we will briefly review the definition of Dirichlet forms and the correspondence to processes following [12]. We will also introduce d-sets.

Let X be a locally compact separable metric space and ν be a positive Radon measure on X whose support is X. Let $\mathcal{E}$ be a symmetric bilinear closed form on $\mathbb{L}^2(X,\nu)$ with domain $\mathcal{F}$. $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form if it is Markovian, i.e., for each $u \in \mathcal{F}$, $v := (0 \vee u) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(v,v) \le \mathcal{E}(u,u)$. A Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular if there exists $C \subset \mathcal{F} \cap C_0(X)$ such that C is dense in $\mathcal{F}$ under $\mathcal{E}_1$-norm and C is dense in $C_0(X)$ under the uniform norm, where $C_0(X)$ is a space of continuous compact supported functions on X and $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{\mathbb{L}^2}$. $(\mathcal{E},\mathcal{F})$ is local if for each $u, v \in \mathcal{F}$ whose supports are disjoint compact sets, $\mathcal{E}(u,v) = 0$. There is a one to one correspondence between a regular Dirichlet form on $\mathbb{L}^2(X,\nu)$ and a ν-symmetric Hunt process (i.e. a strong Markov process whose paths are right continuous and quasi-left continuous w.r.t. some filtration) on X except for some exceptional set of starting points. Further, if the regular Dirichlet form is local, then the corresponding process is a diffusion process (i.e. Hunt process with continuous paths).

We next introduce our state space. Let G be a compact d-set in $\mathbb{R}^n$ ($n \ge 2, 0 < d \le n$). That is, $G \subset \mathbb{R}^n$ and there exist a positive Radon measure μ on G

and $c_{2.1}, c_{2.2} > 0$ such that

$$c_{2.1} r^d \le \mu(B(x,r)) \le c_{2.2} r^d \qquad \text{for all } \ x \in G, \ 0 \le r \le 1, \tag{2.1}$$

where $B(x,r)$ is a ball centered at x and radius r w.r.t. the Euclidean metric. Thus d is the Hausdorff dimension of G and μ the Hausdorff measure on G. We normalize the size of G so that the diameter of G is 1.

3. Lipschitz spaces and domains of Dirichlet forms

For several sub-classes of d-sets, diffusion processes, corresponding Laplace operators and Dirichlet forms have been studied extensively (see [2, 23, 24, 28, 33] etc). The most typical example is the Sierpinski gasket which we will define later. In this section, we will consider a class of d-sets which has a *fractional diffusion* in the sense of Barlow [2] and show that the domain of the Dirichlet form is a Lipschitz space. We first give a definition of the fractional diffusion.

Definition 3.1. Let (G,ρ) be a complete compact metric space. (G,ρ) is called a fractional metric space and $\{B_t^G\}_{t\ge 0}$ is called a fractional diffusion if the following holds.
1) ρ has the midpoint property; for each $x, y \in G$, there exists $z \in G$ such that $\rho(x,y) = \rho(x,z)/2 = \rho(z,y)/2$. Further, there exists a Borel measure μ which satisfies (2.1) w.r.t. ρ.
2) $\{B_t^G\}_{t\ge 0}$ is a μ-symmetric conservative Feller diffusion on G which has a symmetric jointly continuous heat kernel (fundamental solution of the heat equation) $p_t(x,y)$ $(t > 0, x, y \in G)$ satisfying the following estimate,

$$\begin{aligned} &c_{3.1} t^{-d_s/2} \exp(-c_{3.2}(\rho(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \le p_t(x,y) \\ \le\ &c_{3.3} t^{-d_s/2} \exp(-c_{3.4}(\rho(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \ \text{ for all } 0 < t < 1, \ x, y \in G, \end{aligned} \tag{3.1}$$

with some constants $d_s > 0, d_w \ge 2$.

We note that in the original definition of the fractional diffusions in [2], G is not necessarily compact. For simplicity, we further assume that $\rho(\cdot,\cdot)$ is equivalent to the Euclidean metric, i.e.,

$$c_{3.5}|x-y| \le \rho(x,y) \le c_{3.6}|x-y| \qquad \text{for all } \ x, y \in G. \tag{3.2}$$

In this case, $d_s/2 = d/d_w$ holds.
<u>Example: Sierpinski gasket</u>
Let D_n be a n-dimensional simplex whose vertices are $\{p_0, p_1, \cdots, p_n\}$. For $i = 1, 2, \cdots, n+1$, let $F_i(z) = (z - p_i)/2 + p_i$, $z \in \mathbb{R}^n$. Then, there exists a unique non-void compact set G such that $G = \cup_{i=1}^{n+1} F_i(G)$. This G is called a (n-*dimensional*) *Sierpinski gasket*. It is known that the Sierpinski gasket has a fractional diffusion with $d = \log(n+1)/\log 2$, $d_s = 2\log(n+1)/\log(n+3)$, $d_w = \log(n+3)/\log 2$. (Note that $d_s < 2$ for this example.)

In general, (affine) nested fractals, (which is a class of finitely ramified fractals including Sierpinski gaskets) and Sierpinski carpets have fractional diffusions with $c_1|x-y|^{d_c} \le \rho(x,y) \le c_2|x-y|^{d_c}$ for some $d_c \ge 1$ instead of (3.2).

We now introduce Lipschitz spaces. For $1 \le p < \infty,\ 1 \le q \le \infty,\ \beta \ge 0$ and $m \in \mathbb{N} \cup \{0\}$, set

$$a_m(\beta, f) := L^{m\beta}(L^{md} \int\int_{|x-y|<c_0L^{-m}} |f(x)-f(y)|^p d\mu(x)d\mu(y))^{1/p},\ f \in \mathbb{L}^p(G,\mu),$$

where $1 < L < \infty,\ 0 < c_0 < \infty$. Define a *Lipschitz space* $\mathrm{Lip}(\beta,p,q)(G)$ as a set of $f \in \mathbb{L}^p(G,\mu)$ such that $\bar{a}(\beta,f) := \{a_m(\beta,f)\}_{m=0}^{\infty} \in l^q$. $\mathrm{Lip}(\beta,p,q)(G)$ is a Banach space with the norm $\|f\|_{\mathrm{Lip}} := \|f\|_{\mathbb{L}^p} + \|\bar{a}(\beta,f)\|_{l^q}$. Note that the Lipschitz space is determined independently of the choice of L and c_0 as long as the former is greater than 1 and the latter is positive.

Theorem 3.2. ([20, 25, 30, 15, 14, 27]) *Let $(\mathcal{E},\mathcal{F})$ be a local regular Dirichlet form on G which corresponds to a fractional diffusion. Then, the following holds.*

$$\mathcal{F} = Lip(\frac{d_w}{2}, 2, \infty)(G).$$

Proof. We will follow the argument in [30]. We first prove $\mathcal{F} \subset \mathrm{Lip}$. For $f \in \mathbb{L}^2(G,\mu)$, let $\mathcal{E}_t(f,f) := (f - P_t f, f)_{\mathbb{L}^2}/t$, where P_t is a semigroup corresponding to $(\mathcal{E},\mathcal{F})$. Then,

$$\begin{aligned}\mathcal{E}_t(f,f) &= \frac{1}{2t}\int\int_{G\times G}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy)\\ &\ge \frac{1}{2t}\int\int_{|x-y|\le c_0 t^{1/d_w}}(f(x)-f(y))^2 p_t(x,y)\mu(dx)\mu(dy)\\ &\ge \frac{c_1}{2t}\int\int_{|x-y|\le c_0 t^{1/d_w}} t^{-d_s/2}(f(x)-f(y))^2\mu(dx)\mu(dy), \qquad (3.3)\end{aligned}$$

where we use the lower bound of (3.1) in the last inequality. Taking $t = L^{-md_w}$ and using the fact $d_s/2 = d/d_w$, we see that (3.3) is equal to $c_1 a_m(d_w/2,f)^2$. It is well known that $\mathcal{E}_t(f,f) \nearrow \mathcal{E}(f,f)$ as $t \downarrow 0$ ([12], Lemma 1.3.4). We thus obtain $\sup_m a_m(d_w/2,f) \le c_2\sqrt{\mathcal{E}(f,f)}$ and the result holds.

We next prove $\mathcal{F} \supset \mathrm{Lip}$. Set $\gamma = 1/(d_w-1)$. Since the diameter of G is 1, we have for each $g \in \mathrm{Lip}$,

$$\mathcal{E}_t(g,g) = \frac{1}{2t}\int\int_{\substack{x,y\in G\\ |x-y|\le 1}}(g(x)-g(y))^2 p_t(x,y)\mu(dx)\mu(dy)$$

$$\le \frac{1}{2t}\sum_{m=1}^{\infty} c_3 t^{-d_s/2} e^{-c_4(tL^{md_w})^{-\gamma}} \int\int_{L^{-m}<|x-y|\le L^{-m+1}}(g(x)-g(y))^2\mu(dx)\mu(dy)$$

$$\le c_3 t^{-(1+d_s/2)} \sum_{m=1}^{\infty} e^{-c_4(tL^{md_w})^{-\gamma}} L^{-m(d_w+d)} a_{m-1}(d_w/2,g)^2, \qquad (3.4)$$

where we use the upper bound of (3.1) in the first inequality. For $0 < t$ and $0 \le x$, let $\Phi_t(x) = e^{-c_4(tL^{xd_w})^{-\gamma}} L^{-x(d_w+d)}$. By elementary calculations, we see that $\Phi_t(0) > 0$, $\lim_{x\to\infty} \Phi_t(x) = 0$ and $\int_0^\infty \Phi_t(x)dx = c_5 t^{1+d_s/2}$. Further, there exists $x_t > 0$ such that $\Phi_t(x)$ is increasing for $0 \le x < x_t$, decreasing for $x_t < x < \infty$ and $\Phi_t(x_t) = c_6 t^{1+d_s/2}$. Thus, $\sum_{m=1}^\infty \Phi_t(m) \le \int_0^\infty \Phi_t(x)dx + 2\Phi_t(x_t) \le c_7 t^{1+d_s/2}$. Since (3.4) is less than or equal to $c_3 t^{-(1+d_s/2)} \|g\|^2_{\mathrm{Lip}} \sum_{m=1}^\infty \Phi_t(m)$, we conclude that $\sup_{t>0} \mathcal{E}_t(g,g) = \lim_{t\to 0} \mathcal{E}_t(g,g) \le c_8 \|g\|^2_{\mathrm{Lip}}$ and the result holds. □

Remark 3.3. 1) The argument here can be extended to a class of diffusions wider than fractional diffusions. Indeed, by checking the proof of [30, 15] carefully, we see that the same results hold for diffusions whose heat kernels satisfy estimates similar to (3.1), but with different orders $0 < \gamma_1, \gamma_2 < \infty$ on the shoulders of $\rho(x,y)^{d_w} t^{-1}$ (instead of $1/(d_w - 1)$). If we weaken the condition in this way, then we can include all diffusions on p.c.f. self-similar sets (which roughly corresponds to finitely ramified fractals) as mentioned in [27].
2) In [14], characterization of domains of Dirichlet forms is given using heat kernel estimates under a more general and abstract setting.

4. Dirichlet forms and jump type processes on d-sets

In [26], three natural non-local regular Dirichlet forms are introduced, whose corresponding processes are stable-like jump type processes on compact d-sets. Here we will survey them and give relations to other results.

4.1. Jump process as a Besov space on a d-set

We first introduce Besov spaces on G and their trace theory within the scope of our use (see [22, 36] etc. for details).

For $0 < \alpha < 1$, we introduce a Besov space $B_\alpha^{2,2}(G)$ as follows,

$$\|u|B_\alpha^{2,2}(G)\| = \|u\|_{\mathbb{L}^2(G,\mu)} + \Big(\int\int_{G\times G} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}} \mu(dx)\mu(dy)\Big)^{1/2}, \tag{4.1}$$

$$B_\alpha^{2,2}(G) = \{u : u \text{ is measurable}, \|u|B_\alpha^{2,2}(G)\| < \infty\}. \tag{4.2}$$

In [22], it is shown that for $0 < \alpha < 1$, $B_\alpha^{2,2}(G) = \mathrm{Lip}(\alpha, 2, 2)(G)$ and the two norms are equivalent (Chapter V, Proposition 3).

For each $f \in \mathbb{L}^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$Rf(x) = \lim_{r\downarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy,$$

if the limit exists, where m is the Lebesgue measure in $\mathbb{R}^n$. It is well-known that the limit exists quasi-everywhere (i.e. except a set of zero capacity) in $\mathbb{R}^n$ with respect to the Newtonian capacity if $n \ge 3$ or logarithmic capacity if $n = 2$ and coincides with $f(x)$ almost everywhere in $\mathbb{R}^n$. For each $\beta > 0$, denote by $B_\beta^{2,2}(\mathbb{R}^n)$ the classical Besov space on $\mathbb{R}^n$ (see Remark 4.2 below for its definition). The following trace theorem plays an important role in the study of Besov spaces on d-sets (see, for instance, Chapters V and VI in [22]).

Proposition 4.1. *For $0 < s < 1$, the trace operator $Tr_G : f \mapsto Rf$ is a bounded linear surjection from $B^{2,2}_{s+(n-d)/2}(\mathbb{R}^n)$ onto $B^{2,2}_s(G)$ and it has a bounded linear right inverse operator E_G (which is called the extension operator in literature) so that $Tr_G \circ E_G$ is the identity map on $B^{2,2}_s(G)$.*

Remark 4.2. Note that for $\beta > 0$ with integer $k < \beta \le k+1$, the classical Besov space $B^{2,2}_\beta(\mathbb{R}^n)$ is defined to be

$$B^{2,2}_\beta(\mathbb{R}^n) = \left\{ u \in C^k(\mathbb{R}^n) : \|u\|_{B^{2,2}_\beta} := \sum_{0\le|j|\le k} \|D^j u\|_2 + \sum_{|j|=k} \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h D^j f\|_2^2}{|h|^{n+2(\beta-k)}} dh \right)^{1/2} < \infty \right\},$$

where for $j = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n_+$, $|j| = \sum_{k=1}^n j_k$ and $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, Δ_h is a difference operator so that for $h \in \mathbb{R}^n$, $(\Delta_h f)(x) = f(x+h) - f(x)$, and $\|\cdot\|_2$ denotes the $\mathbb{L}^2$-norm in $\mathbb{L}^2(\mathbb{R}^n, m)$ (see, for instance, section I.1.5 in [22]). It is known (cf. Section V.1.1 in [22]) that when $0 < \beta < 1$, the norm $\|u\|_{B^{2,2}_\beta}$ is equivalent to $\left\| u \mid B^{2,2}_\beta(\mathbb{R}^n) \right\|$ defined by (4.2) with $G = \mathbb{R}^n$, and therefore $B^{2,2}_\beta(\mathbb{R}^n)$ is the same as the space defined by (4.1) with $G = \mathbb{R}^n$. Furthermore, the space $B^{2,2}_\beta(\mathbb{R}^n)$ coincides with the classical Bessel potential space on $\mathbb{R}^n$ (also called the fractional Sobolev space or the Liouville space); see, for instance, p. 8 in Section I.1.5 of [22].

Now, for $0 < \alpha < 1$ and $u, v \in B^{2,2}_\alpha(G)$, define

$$\mathcal{E}_{Y^{(\alpha)}}(u,v) = \int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\alpha}} \mu(dx)\mu(dy).$$

By standard properties of Besov spaces, it is easy to check that $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_\alpha(G))$ is a regular Dirichlet space on $\mathbb{L}^2(G,\mu)$ (a detailed proof is given, for instance, in Theorem 3 of [32]). We denote $\{Y_t^{(\alpha)}\}_{t\ge0}$ the corresponding Hunt process on G. We note that when $G = \mathbb{R}^n$, this is a (2α)-stable process on $\mathbb{R}^n$. When G is an open n-set, it is the so called reflected (2α)-stable process on G proposed in [5].

4.2. Jump process as a subordination of a diffusion

In this subsection, we assume that there exists a fractional diffusion $\{B_t^G\}_{t\ge0}$ on G.

For $0 < \alpha < 1$, let $\{\xi_t\}_{t>0}$ be the strictly α-stable subordinator, i.e., it is a one dimensional non-negative Lévy process independent of $\{B_t^G\}_{t\ge0}$ with the generating function $E[\exp(-u\xi_t)] = \exp(-tu^\alpha)$. Let $\{\eta_t(u) : t > 0, u \ge 0\}$ be the distribution density of $\{\xi_t\}_{t>0}$. Using $p_t(x,y)$ in (3.1), we define

$$q_t(x,y) = \int_0^\infty p_u(x,y)\eta_t(u)du \qquad \text{for all } t > 0,\ x,y \in G.$$

Then, by a general theory, $q_t(x,y)$ is a heat kernel of some Markov process which we denote by $\{X_t^{(\alpha)}\}_{t\ge 0}$, called the subordinate process (see [4, 31]). In our case, $\{X_t^{(\alpha)}\}_{t\ge 0}$ is a μ-symmetric Hunt process and we denote the corresponding Dirichlet form on $\mathbb{L}^2(G,\mu)$ as $(\mathcal{E}_{X^{(\alpha)}},\mathcal{F}_{X^{(\alpha)}})$.

4.3. Jump process as a time change of a stable process on $\mathbb{R}^n$

We first briefly give a result by Triebel. In [36], Triebel defines a Besov space on d-set G as follows. (In order to distinguish it from the previous one, we put $\hat{\cdot}$.)

$$\|u|\hat{B}^{2,2}_\alpha(G)\| = \inf_{Tr_G g=u}\|g|B^{2,2}_{\alpha+(n-d)/2}(\mathbb{R}^n)\|, \tag{4.3}$$

$$\hat{B}^{2,2}_\alpha(G) = Tr_G\ B^{2,2}_{\alpha+(n-d)/2}(\mathbb{R}^n), \tag{4.4}$$

where $\alpha>0$. Here Tr_G is the trace operator given in Proposition 4.1 and the norm of $B^{2,2}_{\alpha+(n-d)/2}(\mathbb{R}^n)$ is the one given in Remark 4.2. (To be precise, the definition of the trace operator in [36] is different from the one given here, but it turns out to be the same; see p 261 in [37].) In general, this Besov space is different from the one defined by Jonsson-Wallin ([22]), but it is known that for $0<\alpha<1$, the two spaces coincide and the two norms are equivalent. Let H_α be the corresponding self-adjoint operator on G so that

$$(H_\alpha^{1/2}u,H_\alpha^{1/2}u)_{\mathbb{L}^2(G,\mu)} = \|u|\hat{B}^{2,2}_\alpha(G)\|^2,\ \text{Dom}\ (H_\alpha^{1/2}) = \hat{B}^{2,2}_\alpha(G). \tag{4.5}$$

Theorem 4.3. ([36]; Theorem 25.2) *H_α is a positive definite self-adjoint operator on $\mathbb{L}^2(G,\mu)$ with pure point spectrum. Let μ_k be its k-th eigenvalue (including multiplicities). Then there exist $c_{4.1},c_{4.2}>0$ such that the following holds.*

$$c_{4.1}k^{2\alpha/d}\le \mu_k\le c_{4.2}k^{2\alpha/d}\ \textit{for all}\ \ k\in\mathbb{N}.$$

We now give the third jump process on d-sets ([26]). It will be shown that the corresponding operator of the process is the one given by Triebel when $0<\alpha<1$.

We construct a jump process though a time change of (2α)-stable process on $\mathbb{R}^n$. Let G be a d-set on $\mathbb{R}^n$ with a Radon measure μ which satisfies (2.1). Also, let $\{B_t^{(\alpha)}\}_{t\ge 0}$ $(0<\alpha\le 1)$ be a rotation invariant (2α)-stable process on $\mathbb{R}^n$ (when $\alpha=1$ it is a Brownian motion) and let $(\mathcal{E}^{(\alpha)},H^\alpha(\mathbb{R}^n))$ be the corresponding Dirichlet form. (Note that $H^\alpha(\mathbb{R}^n)=B^{2,2}_\alpha(\mathbb{R}^n)$ when $0<\alpha<1$.) Then, it is proved in [26] that when $2\alpha>n-d$, then μ is a smooth measure w.r.t. $\{B_t^{(\alpha)}\}_t$, i.e. μ charges no set of zero capacity w.r.t. the form corresponding to $\{B_t^{(\alpha)}\}_t$. Thus, by a general theory (see [12]), there exists a unique positive continuous additive functional $\{A_t^{(\alpha)}\}_{t\ge 0}$ which is in Revuz correspondence with μ (thus, $A_t^{(\alpha)}$ increases only when $B_t^{(\alpha)}\in G$). Set $\tau_t=\inf\{s>0: A_s^{(\alpha)}>t\}$ and define $Z_t^{(\alpha)}=B_{\tau_t}^{(\alpha)}$. Then, again by a general theory, $\{Z_t^{(\alpha)}\}_{t\ge 0}$ is a μ-symmetric jump process and the corresponding regular Dirichlet form can be expressed as

$$\begin{aligned}\mathcal{F}_Z &= \{\phi\in\mathbb{L}^2(G,\mu):\phi=u\ \mu-\text{a.e. for some}\ u\in H_e^\alpha(\mathbb{R}^n)\},\\ \mathcal{E}_Z(\phi,\phi) &= \mathcal{E}^{(\alpha)}(H_{\mathcal{S}}u,H_{\mathcal{S}}u),\ \phi\in\mathcal{F}_Z,\phi=u\ \mu-\text{a.e.},u\in H_e^\alpha(\mathbb{R}^n).\end{aligned}$$

Here $\mathcal{S}$ is a quasi-support of μ (see p168 of [12]), $H_{\mathcal{S}}u(x) = E^x[u(B^{(\alpha)}_{\sigma_{\mathcal{S}}})]$ where $\sigma_{\mathcal{S}} = \inf\{t \geq 0 : B^{(\alpha)}_t \in \mathcal{S}\}$ and $H^{\alpha}_e(\mathbb{R}^n)$ is the extended Dirichlet space. See Theorem 6.2.1 in [12] for details.

In subsection 2.3 of [26], it is proved that for $2\alpha > n-d$,

$$P^x(\sigma_{\mathcal{S}} = 0) = 1 \qquad \text{for all} \ \ x \in G,$$

so $H_{\mathcal{S}}u(x) = u(x)$ for μ-a.e. $x \in G$. As μ is the smooth measure, we can prove $Tr_G H_{\mathcal{S}}u = u$, μ-a.e.. Since $H_{\mathcal{S}}u$ is harmonic outside G, we obtain

$$\mathcal{E}^{(\alpha)}(H_{\mathcal{S}}u, H_{\mathcal{S}}u) = \inf_{Tr_G g = u} \|g|B^{2,2}_{\alpha}(\mathbb{R}^n)\|.$$

Therefore, $\mathcal{E}_Z$, the trace of the Dirichlet form of the (2α)-stable process on $\mathbb{R}^n$, is equal to $\|\cdot|\hat{B}^{2,2}_{\alpha-(n-d)/2}(G)\|$ in (4.3) so that the corresponding operator is $H_{\alpha-(n-d)/2}$ given above.

Remark 4.4. When $A \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂A, a similar time changed form (with exponential killing) is studied in [18], where the original form lies on A and the time changed form lies on ∂A. In [11], a similar problem is treated when A is a bounded (ϵ, δ)-domain with boundary ∂A which is a d-set with $n-1 \leq d < n$.

There is an alternative analytic approach by Zähle using Riesz potentials ([39, 40, 16]). For $0 < 2\alpha < d$, define a Riesz potential of order 2α as follows

$$I^{2\alpha}_{\mu} f(x) := c_{n,2\alpha+n-d} \int \frac{f(y)d\mu(y)}{|x-y|^{d-2\alpha}}, \qquad \forall f \in \mathbb{L}^2(G,\mu),$$

where $c_{n,s} := \Gamma((n-s)/2)/(2^s \pi^{n/2} \Gamma(s/2))$. In [39], it is proved that $I^{2\alpha}_{\mu}$ is invertible. Define $D^{2\alpha}_{\mu} := (I^{2\alpha}_{\mu})^{-1}$; the domain of this operator is $\hat{B}^{2,2}_{2\alpha}(G)$ ([40]). Let

$$\mathcal{E}^{2\alpha}_{\mu}(f,g) = \int_G \sqrt{D^{2\alpha}_{\mu}} f(x) \sqrt{D^{2\alpha}_{\mu}} g(x) d\mu(x), \qquad \forall f, g \in \hat{B}^{2,2}_{\alpha}(G).$$

It is proved in [40] that $(\mathcal{E}^{2\alpha}_{\mu}, \hat{B}^{2,2}_{\alpha}(G))$ is a regular Dirichlet form on $\mathbb{L}^2(G,\mu)$ for $0 < \alpha + (n-d)/2 \leq 1$. In [16], it is shown that $(\mathcal{E}^{2\alpha}_{\mu}, \hat{B}^{2,2}_{\alpha}(G))$ coincides with $(\mathcal{E}_{Z^{(s)}}, \mathcal{F}_{Z^{(s)}})$ where $s = \alpha + (n-d)/2$. Thus, $\sqrt{D^{2\alpha}_{\mu}} = H^{1/2}_{\alpha}$ given in (4.5).

4.4. Comparison of the forms and heat kernel bounds

Define $\bar{\alpha} = \alpha d_w/2$ and $\hat{\alpha} = \alpha - (n-d)/2$. We then have the following.

Proposition 4.5. ([26, 32])
Let G be a d-set. For $(n-d)/2 < \alpha < 1$ or $\alpha = 1, n-2 < d < n$,

$$c_{4.3}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f) \leq \mathcal{E}_{Z^{(\alpha)}}(f,f) \leq c_{4.4}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f) \ \text{for all} \ f \in \mathbb{L}^2(G,\mu). \tag{4.6}$$

Assume further that there exists a fractional diffusion on G. For $0 < \alpha < 1$,

$$c_{4.5}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f) \leq \mathcal{E}_{X^{(\alpha)}}(f,f) \leq c_{4.6}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f) \ \text{for all} \ f \in \mathbb{L}^2(G,\mu). \tag{4.7}$$

In particular, under the conditions,

$$\mathcal{F}_{Z^{(\alpha)}} = B^{2,2}_{\hat{\alpha}}(G),\ \mathcal{F}_{X^{(\alpha)}} = B^{2,2}_{\tilde{\alpha}}(G).$$

Remark 4.6. 1) In [26], (4.7) is stated for $0 < \alpha < 2/d_w$. But the proof there shows that it actually holds for $0 < \alpha < 1$. From this fact, we see that when G is a d-set on which there exists a fractional diffusion, $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_{\alpha}(G))$ is a regular Dirichlet form for $0 < \alpha < d_w/2$.
2) Note that in general the three-type Dirichlet forms introduced are different and the corresponding processes cannot be obtained by time changes of others by positive continuous additive functionals (see [26]).

5. Heat kernel estimates for stable-like processes on d-sets

In [8], detailed estimates of heat kernels for $\{Y^{(\alpha)}\}$ are obtained. There exists a non-negative bounded heat kernel $p_t(x,y)$ on $(t,x,y) \in (0,\infty) \times G \times G$ with

$$P^{Y^{(\alpha)}}_t f(x) = \int_G p_t(x,y) f(y) \mu(dy) \qquad \text{for all} \quad x \in G,\ f \in \mathbb{L}^2(G,\mu),$$

(where $P^{Y^{(\alpha)}}_t$ is the heat semigroup w.r.t. $\mathcal{E}_{Y^{(\alpha)}}$), satisfying the following.

Theorem 5.1. ([8]) *For $0 < \alpha < 1$, the following holds.*
1) There are constants $c_{5.1}, c_{5.2} > 0$ such that for all $x, y \in G,\ 0 < t < 1$,

$$c_{5.1}\Big(t^{-\frac{d}{2\alpha}} \wedge \frac{t}{|x-y|^{d+2\alpha}}\Big) \le p_t(x,y) \le c_{5.2}\Big(t^{-\frac{d}{2\alpha}} \wedge \frac{t}{|x-y|^{d+2\alpha}}\Big).$$

2) There are constants $c_{5.3} > 0$ and $\beta > 0$ such that for any $0 < t, s < 1$ and $(x_i, y_i) \in G \times G$ with $i = 1, 2$,

$$|p_s(x_1,y_1) - p_t(x_2,y_2)| \le c_{5.3}\,(t \wedge s)^{-\frac{d+\beta}{2\alpha}} \left(|t-s|^{\frac{1}{2\alpha}} + |x_1 - x_2| + |y_1 - y_2|\right)^{\beta}.$$

Theorem 5.2. ([8]) *For every $x \in F$, $\mathbb{P}^x$-a.s., the Hausdorff dimension of $Y[0,1] := \{Y_t : 0 \le t \le 1\}$ is $d \wedge (2\alpha)$.*

Note that above theorems hold under a wider framework, i.e. when the form is expressed as

$$\mathcal{E}(f,f) = \int\int_{G\times G} (f(x) - f(y))^2 n(x,y) d\mu(x) d\mu(y),$$

where $n(x,y)$, $x, y \in G$ is a jointly measurable function such that $n(x,y) = n(y,x)$ for all $x, y \in G$ and satisfies

$$\frac{c_{5.4}}{|x-y|^{d+2\alpha}} \le n(x,y) \le \frac{c_{5.5}}{|x-y|^{d+2\alpha}}.$$

In [8], it is also proved that all the parabolic functions (two variable functions which satisfy the heat equation) satisfy the parabolic Harnack inequality.

Remark 5.3. 1) When G is a global d-set (i.e. an unbounded closed d-set satisfying $\mu(B(x,r)) \le c_1 r^d$ for all $x \in G$ and $r > 0$), Theorem 5.1 holds for all $0 < t, s < \infty$ ([9]). Thus $Y^{(\alpha)}$ is transient if and only if $d > 2\alpha$, it is point recurrent if and only if $d < 2\alpha$.
2) In [7, 6], similar heat kernel estimates for the subordinate process $\{X^{(\alpha)}\}$ ($0 < \alpha < 1$) of the fractional diffusion (see subsection 4.2) are obtained. There the original diffusion processes *is assumed* to satisfy the properties in Definition 3.1.
3) In relation to Theorem 5.2, results on ranges of stochastic processes are surveyed in [38]. In [3], among others, Hausdorff dimensions of boundary traces of reflecting Brownian motions and stable-like processes are computed on various Euclidean domains including the von Koch snowflake domain.

6. From jump-type processes to diffusion processes

6.1. Construction of diffusion processes on fractals

In [27], some sufficient conditions for constructing diffusion processes on metric measure spaces are given. In this subsection, we will briefly explain some of the results in the paper. (The results in the paper are more general, but we simplify to avoid complicated settings.)

Let

$$\mathcal{E}^{\beta,r}(u,u) = \frac{1}{r^{2\beta}} \int_G \fint_{B(x,r)} [u(x) - u(y)]^2 \, \mu(dy) \, \mu(dx) \tag{6.1}$$

for all $u \in \mathbb{L}^2(G,\mu)$. Here $\fint_A \dots \mu(dy) := \mu(A)^{-1} \int_A \dots \mu(dy)$ denotes the normalized integral. We fix a sequence $(r_n)_n$ of positive numbers decreasing to 0 and put

$$\begin{aligned} \mathcal{E}^{\beta,*}(u,u) &:= \limsup_{n\to\infty} \mathcal{E}^{\beta,r_n}(u,u), \\ \mathcal{C}^{\beta,*} &:= \left\{ u \in \mathcal{C}_0(G) : \ \mathcal{E}^{\beta,*}(u,u) < \infty \right\}. \end{aligned}$$

Let

$$\bar\beta := \sup\{\beta : \{u \in \mathbb{L}^2(G,\mu) : \mathcal{E}^{\beta,*}(u,u) < \infty\} \text{ is dense in } \mathbb{L}^2(G,\mu).\}.$$

In the following, we assume

- $\{u \in \mathbb{L}^2(G,\mu) : \mathcal{E}^{\bar\beta,*}(u,u) = 0\}$ is not dense in $\mathbb{L}^2(G,\mu)$,
- $\{u \in \mathbb{L}^2(G,\mu) : \mathcal{E}^{\bar\beta,*}(u,u) < \infty\}$ is dense in $\mathbb{L}^2(G,\mu)$,

and denote $d_w = 2\bar\beta$.

Theorem 6.1. ([27]) *(i) Assume that $\mathcal{C}^{d_w/2,*}$ is dense in $\mathcal{C}_0(G)$. Then for a suitable subsequence (r'_n) of (r_n) the Γ-limit*

$$\mathcal{E}^0 := \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}^{d_w/2, r'_n}$$

exists, is dominated by $\mathcal{E}^*$ *on* $\mathbb{L}^2(G,\mu)$ *and is the extension of a regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $\mathbb{L}^2(G,\mu)$ *with core* $\mathcal{C}^{d_w/2,*}$.
Hence, there exists a μ*-reversible strong Markov process associated with* $(\mathcal{E},\mathcal{F})$.
(ii) Assume in addition the following;
There exists $\delta > 0$ *such that*

$$\liminf_{n\to\infty} \mathcal{E}^{d_w/2,r_n}(u_n,u_n) \geq \delta \cdot \mathcal{E}^{d_w/2,*}(u,u) \tag{6.2}$$

for all $u \in \mathbb{L}^2(G,\mu)$ *and all* $(u_n)_n \subset \mathbb{L}^2(G,\mu)$ *with* $u_n \to u$ *in* $\mathbb{L}^2(G,\mu)$.
Let $\mathcal{E}^{(c)}$ *denote the diffusion part of* $\mathcal{E}$. *Then* $\mathcal{C}^{d_w/2,*} = \mathcal{F} \cap \mathcal{C}_0(G)$,

$$\mathcal{E}^{d_w/2,*} \geq \mathcal{E} \geq \mathcal{E}^{(c)} \geq \delta \cdot \mathcal{E}^{d_w/2,*}$$

on $\mathcal{F}$ *and* $(\mathcal{E}^{(c)},\mathcal{F})$ *is a strongly local regular Dirichlet form.*
Hence, there exists a diffusion process associated with $(\mathcal{E}^{(c)},\mathcal{F})$. *It is* μ*-reversible and strongly Markovian.*

Let

$$\mathrm{Lip}(\frac{d_w}{2},2,\infty)(G) = \left\{ u \in \mathbb{L}^2(G,\mu) : \sup_n \mathcal{E}^{d_w/2,r_n}(u,u) < \infty \right\}.$$

If we assume the following instead of (6.2),

$$\liminf_{n\to\infty} \mathcal{E}^{d_w/2,r_n}(u_n,u_n) \geq \delta \cdot \sup_n \mathcal{E}^{d_w/2,r_n}(u,u), \tag{6.3}$$

then $\mathcal{C}^{d_w/2,*} = \mathrm{Lip}(d_w/2,2,\infty)(G) \cap \mathcal{C}_0(G)$ holds. Note that when $r_n = L^{-n}$, then $\mathrm{Lip}(d_w/2,2,\infty)(G)$ coincides with the Lipschitz space discussed in Section 3. When there is a fractional diffusions on G, then the assumptions given in this section always hold.

6.2. Jump-type processes for $1 \leq \alpha < d_w/2$

It is natural to ask whether the results in Section 5 can be extended to $\alpha \geq 1$. The problem is that in this case Davies methods cannot be applied, likewise the case of diffusions on fractals. Let

$$\bar{d}_w := \sup\{2\alpha : (\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_\alpha(G)) \text{ is regular in } \mathbb{L}^2(G,\mu)\}.$$

It is easy to show $\bar{d}_w \leq d_w$. We do not know whether the equality holds in general or not (clearly the equality holds when there is a fractional diffusion on G).

We believe we can prove the theorems in Section 5 for all $\alpha < \bar{d}_w/2$ when $\bar{d}_w > d$, i.e. when the process is point recurrent ([9]). It is still currently under investigation whether or not these results hold in general for all $\alpha < \bar{d}_w/2$ or not.

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