Diffusion on the scaling limit of the critical percolation cluster in the diamond hierarchical lattice

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Abstract

We construct critical percolation clusters on the diamond hierarchical lattice and show that the scaling limit is a graph directed random recursive fractal. A Dirichlet form can be constructed on the limit set and we consider the properties of the associated Laplace operator and diffusion process. In particular we contrast and compare the behaviour of the high frequency asymptotics of the spectrum and the short time behaviour of the on-diagonal heat kernel for the percolation clusters and for the underlying lattice. In this setting a number of features of the lattice are inherited by the critical cluster.

1 Introduction

There has been extensive recent work on gaining a mathematical understanding of random walk on the clusters of Bernoulli bond percolation in $\mathbb{Z}^d$ for $d \geq 2$. In the percolation model each edge of $\mathbb{Z}^d$ is open independently with probability p. The system exhibits a phase transition in that at a critical probability $p_c \in (0,1)$ there exists an (unique) infinite connected component $\mathcal{C}_\infty$ of the set of open edges. In the supercritical case, where $p > p_c$, there are now annealed [16] and quenched [10, 39, 43] invariance principles, full Gaussian heat kernel bounds [4] and a local limit theorem [5] for the random walk on $\mathcal{C}_\infty$ for any $p > p_c$ in any dimension.

The transport properties of the percolation cluster 'at criticality' have been studied in the physics literature in great detail through heuristics and numerical work, [25] however they are much less well understood mathematically. Let $Y = (Y_t, t \geq 0)$ be the (continuous time) simple random walk on the critical cluster $\mathcal{C}$, and $p_t(x,y)$ be its heat kernel. Define the spectral dimension of $\mathcal{C}$ by

$$d_s(\mathcal{C}) = -2 \lim_{t\to\infty} \frac{\log p_t(x,x)}{\log t},$$

if this limit exists. Alexander and Orbach [1] conjectured that, for any $d \geq 2$, $d_s(\mathcal{C}_{\mathbb{Z}^d}) = 4/3$. While it is now thought that this is unlikely to be true for small d, it has been proved for the mean field regime [33], that is for sufficiently high dimension, or in $d > 6$ when the lattice is sufficiently spread out. The first issue is to construct a critical cluster as the probability of the existence of such an infinite cluster is 0 (this is proved for $d = 2, d \geq 19$). In the two dimensional

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case the incipient infinite cluster (IIC), the critical cluster, was first constructed in [27] and the first result on the random walk on this cluster [28], showed that it was subdiffusive. The only recent work on the random walk on the two dimensional IIC is an estimate for the resistance of the IIC, which leads to bounds on the random walk exponent[3]. Although the scaling limit will have a description in terms of SLE, there is no conjecture regarding the dynamic exponents in two dimensions. In high dimensions more detailed results such as subdiffusive heat kernel estimates are now available for the random walk on the incipient infinite cluster on d-ary trees [8], oriented and unoriented spread out percolation clusters in dimension greater than 6 [6, 33]. In all cases, it is proved that the Alexander-Orbach conjecture is true. Spectral properties [15] and heat kernel estimates [14] are also known for the continuum random tree, a set closely related to the scaling limit of the IIC on the tree.

Our aim is to investigate a simpler lattice than $\mathbb{Z}^d$ and consider the analogue of the infinite cluster from critical bond percolation on this lattice and study its transport properties. The recent progress on the high dimensional critical cluster makes use of the fact that in the mean field the percolation clusters are close to trees, in that there are very few loops, which makes resistance calculations easier. The lattice we consider here has features not seen in the mean field regime in that there are loops at all scales but, due to exact self-similarity, it is easier to handle than $\mathbb{Z}^d$ for low dimensions.

The diamond hierarchical lattice was initially investigated in the physics literature, for instance in [11], [44] and [17] and more recently for random polymers in [12, 19, 36], and random conductors [45]. It is constructed in a self-similar manner and the first few stages in the construction are shown in Figure 1. The self-similarity allows for the straightforward computation of a number of exponents for the lattice, for example the dimension is 2. Thus we may hope that the lattice has some similarities to the two-dimensional integer lattice. However detailed properties are more difficult to obtain as it is not a finitely ramified fractal lattice and can be viewed as having a multifractal structure.

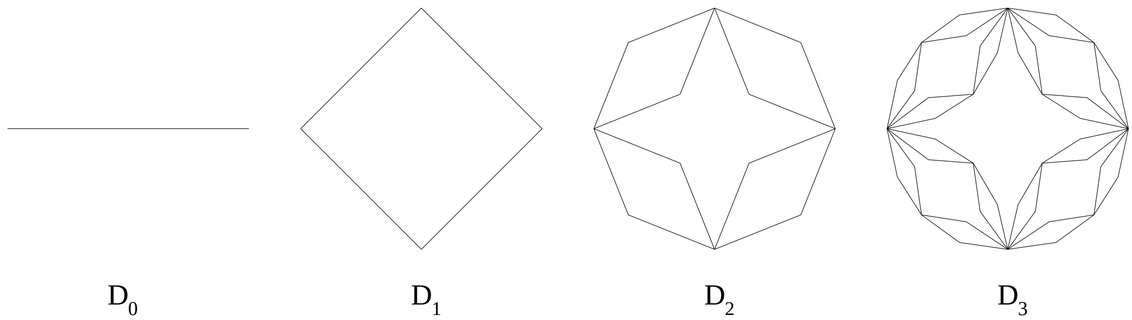

Figure 1: The first 3 stages of the construction of the diamond hierarchical lattice

At each stage of construction the lattice is a finite graph D_n and Bernoulli bond percolation can be performed. We define percolation as the existence of an open cluster in D_n joining the two vertices of D_0 in the limit as $n \to \infty$. The exact construction and definition will be given

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in Section 2. A picture of the level 3 lattice after percolation (which contains two non-trivial open clusters and 10 single point clusters) is shown in Figure 2. In this setting there is no need

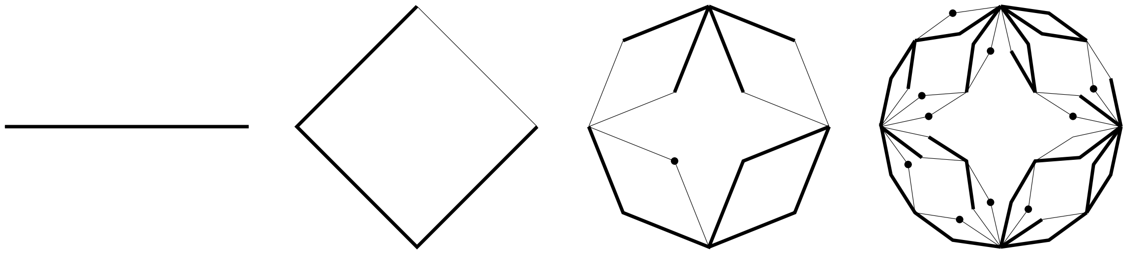

Figure 2: The first 3 stages of the diamond hierarchical lattice after percolation where a thick line indicates an open edge and a dot indicates an isolated vertex

to construct an IIC as, for our model on this lattice, an infinite cluster will exist with positive probability at the critical probability. Once we have shown that there is such a cluster we will give an alternative probabilistic description of the infinite cluster via the tree associated with a multitype branching random walk. This structure is the key to our analysis as we can apply techniques that have been developed for handling random recursive fractals. We will write $(\Omega, \mathbb{P})$ for the probability space of clusters and $\mathcal{C}(\omega)$ for the critical cluster.

We will proceed to construct a Dirichlet form on the critical percolation cluster and then to show that it can be renormalized to produce a Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathcal{C}(\omega), \mu_\omega)$ (where μ_ω is a natural measure defined later) for the scaling limit. This scaling limit is a graph directed random recursive fractal set viewed as a self-sufficient metric space. The approach is to choose weights associated with each edge in the lattice in such a way that the effective resistance across the whole lattice remains at one. This mimics the construction of Dirichlet forms on random recursive Sierpinski gaskets as in [20].

The first results we obtain are to indicate the properties of the scaling limit of the diamond hierarchical lattice (which we denote by K) itself. We focus on two aspects. Firstly the behaviour of the heat kernel, where we can obtain only weak bounds. As there is no volume doubling for the natural measure it is difficult to get sharp uniform estimates for the heat kernel and we only give an upper bound and a diagonal lower bound. The other property that we consider is the high frequency asymptotics of the spectrum of the Laplacian. As the scaling limit of the lattice is similar to a finitely ramified fractal we can show that there are strictly localized eigenfunctions and that these dominate the spectrum.

For the scaling limit of the critical percolation cluster $\mathcal{C}$, we can obtain results for the on-diagonal heat kernel and also for the spectral asymptotics.

Theorem 1.1 *Let $N_K(\lambda)$ and $N_{\mathcal{C}}(\lambda)$ be the number of eigenvalues less than λ for the Laplacian (Dirichlet or Neumann) on K and $\mathcal{C}$ respectively. Then, there exist periodic functions p and p_1, a mean one random variable $W > 0$ and a constant $\theta = 5.2654..$ such that the following hold as $\lambda \to \infty$,*

$$N_K(\lambda) = \lambda p(\log \lambda) + o(\lambda), \tag{1.1}$$

$$N_{\mathcal{C}}(\lambda) \quad = \quad W\lambda^{\theta/(\theta+1)}p_1(\log\lambda) + o(\lambda^{\theta/(\theta+1)}), \;\; \mathbb{P}-a.s.. \tag{1.2}$$

Further, p is not a constant function.

Note that θ is the dimension of the cluster with respect to the effective resistance metric.

Theorem 1.2 (i) *There exist jointly continuous heat kernels $p_t(x,y)$ for the Laplacian on K and $q_t^\omega(x,y)$ for the Laplacian on $\mathcal{C} = \mathcal{C}(\omega)$ such that the following on-diagonal estimates hold: For $\epsilon > 0$, for a.e. $x \in K$, there exists $T(x) > 0$, constants c_1, c_2 and random constants c_3, c_4 such that*

$$\begin{aligned} c_1 t^{-1}|\log t|^{-2-\epsilon} \leq & \quad p_t(x,x) \quad \leq c_2 t^{-1}, \;\; a.e.\ x \in K, \forall t < T(x), \\ c_3 t^{-\theta/(\theta+1)}|\log t|^{-2(2\theta+3)(\theta+2)-\epsilon} \leq & \quad q_t^\omega(x,x) \quad \leq c_4 t^{-\theta/(\theta+1)}|\log|\log t||^{(\theta-1)/(\theta+1)}, \\ & \mu_\omega - a.e.\ x \in \mathcal{C}(\omega),\ \mathbb{P}-a.s.,\ \forall t < 1. \end{aligned}$$

(ii) *For the vertex 0 (which is in both K and $\mathcal{C}$), there are constants $c_5, c_6 > 0$, random constants $c_7, c_8 > 0$, and $\theta' = 3.927..$ such that the following hold:*

$$\begin{aligned} c_5 t^{-1/2} \leq p_t(0,0) \leq c_6 t^{-1/2}, & \qquad \forall t < 1, \\ c_7 t^{-\theta'/(\theta'+1)} \leq q_t^\omega(0,0) \leq c_8 t^{-\theta'/(\theta'+1)}, & \qquad \mathbb{P}-a.e.\ \omega,\ \forall t < 1. \end{aligned}$$

This theorem shows that while the diamond hierarchical lattice itself behaves, at the level of exponents, like $\mathbb{Z}^2$, a version of the Alexander-Orbach conjecture does not hold for this critical cluster. We have no reason to believe that the spectral dimension of the critical cluster in the diamond lattice should be the same as that for the IIC in $\mathbb{Z}^2$. We would expect that this exponent would depend upon the local geometry which is quite different between the two lattices. The spectral exponent we have computed here is also determined by the particular Laplace operator we have chosen on our limit cluster which enables us to perform the renormalization in a straightforward fashion.

We also remark here that the diamond hierarchical lattice is just one hierarchical lattice with two boundary points which could be constructed. Our approach can be applied to other families of hierarchical substitution rules but we note that in order to apply some of the techniques used here it is important that the spectral dimension is less than 2.

The structure of the paper is as follows. In Section 2 we describe the diamond hierarchical lattice and introduce percolation on it. The percolation problem leads to an exact renormalization map and we give explicit results on the percolation probability and show that the infinite cluster at criticality will exist with positive probability and can be described by a branching process. This leads to a description of the scaling limit of both the diamond hierarchical lattice and the infinite critical percolation cluster in Section 3. In Section 4 we consider the properties of the diamond hierarchical lattice itself. Then we consider the same properties for the scaling limit of the infinite cluster in Section 5. We complete the paper by discussing some open problems in Section 6.

Note that throughout the paper we will write, c, c', C, C' for constants whose value may vary from line to line. Constants marked c_i are fixed within a given argument.

2 Percolation on the diamond hierarchical lattice and its scaling limit

The diamond hierarchical lattice is a recursively constructed graph. We begin with $D_0 = (V_0, E_0)$, where V_0 consists of two vertices and E_0 an edge between them. The graph $D_{n+1} = (V_{n+1}, E_{n+1})$ is constructed by replacing each edge in E_n of D_n by a diamond, that is two sets of edges, each set consisting of two edges in series with a vertex between them, in parallel as shown in Figure 1. We may also think of this as taking 4 copies of the graph D_n and attaching them in a diamond configuration to form D_{n+1}.

We note that E_n has 4^n edges and that the local geometry varies radically from point to point. The original two vertices, which we label 0 and 1, have 2^n edges leaving them in D_n, while each of the new vertices in $V_n \backslash V_{n-1}$ (those added at the n-th level) have only two edges leaving them.

2.1 Percolation on D_n

Now we perform percolation on the n-th graph D_n. Let $\Omega_n = \{0,1\}^{E_n}$, $p \in [0,1]$, and $\mathbb{P}_{n,p}$ be the probability measure on Ω_n which makes $\omega(e)$, for each $e \in E_n$, an i.i.d. Bernoulli r.v. with $\mathbb{P}_{n,p}(\omega(e) = 1) = p$. The edges e with $\omega(e) = 1$ are called open and the open cluster $\mathcal{C}_n(x)$ containing x is the set of $y \in D_n$ such that $x \leftrightarrow y$, that is x and y are connected by an open path in D_n. Let D_n^p be the graph whose components are the open clusters of D_n.

From D_n^p we can construct $D_{k,n}^p$ for $k = n-1, \dots, 0$ by considering each copy of D_1, a subgraph of 4 edges in D_k between a connected pair of vertices, say x, y in D_{k-1}, and setting $\omega(e) = 1$ for each edge e in $D_{k-1,n}^p$ if the edges of that subgraph of $D_{k,n}^p$ form an open path between the vertices x, y of D_{k-1}. If the subgraph does not form an open path between these vertices, then we set $\omega(e) = 0$ for that edge in $D_{k-1,n}^p$. As the edges of D_n are subject to independent Bernoulli bond percolation and the procedure of determining if the subgraphs are connected only depends on the four bonds in each subgraph, the edges of D_k will be subjected to independent Bernoulli bond percolation for each $k = n-1, \dots, 0$. However the probability of an edge being present will be modified. We can compute the effect in that at level 1 we have 7 combinations of edges which give a connection between 0 and 1 and hence the overall connection probability is $p_0 = f(p)$ where

$$\begin{aligned} f(p) &= 2p^2(1-p)^2 + 4p^3(1-p) + p^4, \\ &= 2p^2 - p^4. \end{aligned}$$

Thus we have a map on the percolation probability as the graph is decimated. We see that $\mathbb{P}_{n,p}$ induces a probability measure $\mathbb{P}_{k,f^{n-k}(p)}$ on Ω_k, where $f^m(p)$ is the m-fold composition of f with itself, which makes $\omega(e)$ independent Bernoulli random variables for each $e \in E_k$ and hence we have that $D_{k,n}^p = D_k^{f^{n-k}(p)}$ in distribution. It is easy to see that the map f has 3 fixed points in the interval $[0,1]$. Those at 0 and 1 are attracting and the one at $p_c = (\sqrt{5}-1)/2$ is repulsive. It is therefore simple to deduce the following.

Lemma 2.1 *If the graph D_n is subject to Bernoulli bond percolation with $p = p_c$, then there is percolation in the sense that the vertices 0 and 1 are connected by an open path with probability*

p_c.
If $p > p_c$, then P(0 and 1 are connected in D_n) $\to 1$ as $n \to \infty$.
If $p < p_c$, then P(0 and 1 are connected in D_n) $\to 0$ as $n \to \infty$.

2.2 A tree description of the critical percolation cluster

We can now build a branching tree model of $(D_n^{p_c})_{n=0}^{\infty}$. We first give an informal description by labelling the sequence of graphs D_n. For any graph D_n we label each edge as one of two types - a c, for connected and a d for disconnected. Now to produce the labelling on D_{n+1} we use the following reproduction rule for the two types of edges. We first observe that applying bond percolation to D_1 gives 16 possible configurations of labelled edges.

1. If we have a c, then to ensure that the graph remains connected, the replacement graph for that edge comes from one of the 7 possible connected graph structures, shown on the left of Figure 3, with the original probabilities normalized by dividing by p_c.

2. If we have a d, then the replacement graph for that non-edge is chosen from the 9 possible disconnected configurations, shown on the right of Figure 3, with the original probabilities normalized by dividing by $1 - p_c$.

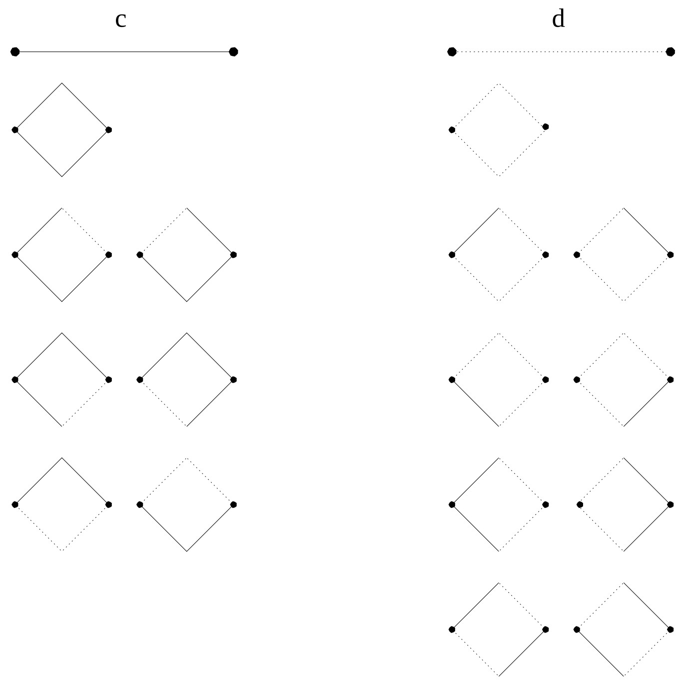

Figure 3: The 7 connected and 9 disconnected configurations

Thus we view our sequence of percolation configurations (G_n) as starting from the initial edge G_0, that is D_0 labelled with a c, and then each graph G_n is the subgraph of the labelled graph D_n where we only keep the edges with labels c.

We now set this up more formally. Let $I = \{1, 2, 3, 4\}$ and assign each number to an edge of the four which form D_1. let $I^0 = \emptyset$, and $T_n = \cup_{i=0}^{n} I^i$ denote the quaternary tree to level n and the full quaternary tree as $T = \cup_n T_n$. We write $\mathbf{i} = (i_1, \ldots, i_n)$ where $i_j \in I$ for a vertex $\mathbf{i} \in T_n$. We will write ∂T for the boundary of the tree, that is the infinite sequences of elements of I. For any $\mathbf{i} = (i_1, i_2, \ldots) \in \cup_{m \geq n} T_m \cup \partial T$ we will write $\mathbf{i}|_n$ for $(i_1, \ldots, i_n)$. Let $U = \{c, d\}$ denote the possible types for each vertex in the tree. If the tree has a c at vertex $\mathbf{i} \in T_n$ this corresponds to the edge in D_n with label $\mathbf{i}$ being present after percolation and a d at the vertex to the edge labelled $\mathbf{i}$ being absent.

We can construct a probability space $(\Omega, \mathbb{P})$ by setting $\Omega = U^T$, and taking as a probability measure $\mathbb{P}$, that for a multitype branching process where an element of U represents the type of an individual. The offspring distribution is always four and the type distribution is straightforward to compute. If we have a type c individual then we have one of the 7 possible connected configurations with probability given by the following. For each $i = 1, 2, 3, 4$ we choose independently a c with probability p_c or a d with probability $1 - p_c$ and then renormalize. Similarly for a type d individual. The distribution will be given explicitly in (2.1) for a slightly extended type space.

Lemma 2.2 *The probabilistic structure of the sequence of graphs $(G_n)_{0 \leq n \leq N}$ generated above is the same as that of the decimated sequence of Bernoulli bond percolated graphs $(D^{p_c}_{n,N})_{0 \leq n \leq N}$ derived from $D^{p_c}_N$. If the branching process starts from an individual of type c, then the vertices 0 and 1 in G_n are connected and $(G_n)_{n \geq 0}$ corresponds to a sequence of graphs $(D^{p_c}_n)_{n \geq 0}$ in which we have percolation.*

Proof: The probability measure for the percolation on D_N is $\mathbb{P}_{n,p}$, the Bernoulli product measure on the edges of D_N. The labels on level N induce a labelling on the sequence of decimated graphs $D^{p_c}_{n,N}$. At the critical probability we know that the measure induced on the tree has the property that it is invariant under decimation so $D^{p_c}_{n,N} = D^{p_c}_n$ in distribution. Indeed, $(\Omega_N, \mathbb{P}_{N,p_c})$, the probability space for D_N with critical Bernoulli product measure projects onto $(\Omega_n, \mathbb{P}_{n,p_c})$ for all $0 \leq n < N$. Thus we have the same measure on the labels as given by the multitype branching process. Thus if we start the branching process with a c corresponding to a connected edge for G_0, then this leads to each G_n having the same distribution as $D^{p_c}_n$ given that under decimation $D^{p_c}_0$ is connected. ∎

Thus, by Kolmogorov's extension theorem, there is a $D^{p_c}_\infty$ which has the property that if it is subject to Bernoulli bond percolation it produces a finite lattice $D^{p_c}_n$ with the property that the vertices 0 and 1 are connected with probability p_c. This infinite object is then described by the limiting behaviour of the multitype branching process. We will use the notation $D^{p_c}_n$ for the bond percolation graph arising at the critical probability on D_n however it is constructed.

2.3 The critical cluster

The critical cluster is now obtained by considering only the connected component of $D^{p_c}_\infty$ between 0 and 1. As the existence of the critical cluster has positive probability we can condition on its existence and thus we will work on a subset $\Omega_c \subset \Omega$ of our probability space which starts with the label c at the root of the tree corresponding to a connected structure. The critical cluster is described by a sub-branching structure contained within the full description of the diamond hierarchical lattice subject to Bernoulli bond percolation.

We now reconsider our construction of $(D_n^{p_c})_{0\leq n\leq N}$ and extend it to produce a description of the infinite cluster at criticality. Start with the sequence of graphs $(D_{n,N}^{p_c})_{0\leq n\leq N}$ which leads to a connected D_0. Now we consider $\mathcal{C}_n(0)$, which we just write as $\mathcal{C}_n$, by removing all the edges of the graph $D_N^{p_c}$ that are not connected by an open path to the vertices 0 and 1. From this form the sequence of graphs $(\mathcal{C}_n)_{0\leq n\leq N}$ in the same way as we formed $(D_{n,N}^{p_c})_{0\leq n\leq N}$. This is a sequence of graphs, each of which is the connected component of $D_n^{p_c}$ containing 0 and 1. Thus as $n\to\infty$, this leads to the infinite cluster at criticality.

This graph can also be described by a branching tree. We now choose a different labelling of the edges to ensure that the branching tree only retains and produces edges that are connected to the two outermost vertices. We label connected edges by a c as before but now split the disconnected case up into two types. Firstly $d_{(1)}$ for those disconnected edges which have one end connected to the infinite cluster and $d_{(2)}$ for those disconnected edges which have two ends connected to the infinite cluster. We now have the following replacement rules:

1. If we have a c, then the replacement graph for that edge comes from one of the 7 possible connected graph structures.

2. If we have a $d_{(1)}$, then the replacement graph for that non-edge is chosen from the 4 possible disconnected configurations which only have one vertex in the infinite cluster.

3. If we have a $d_{(2)}$, then the replacement graph for the non-edge is one of the 9 possible replacement disconnected graphs available.

4. Edges in D_N which are not connected to the infinite cluster do not reproduce.

The configurations for c and $d_{(2)}$ are the same as for the original model shown in Figure 3. The new configurations for $d_{(1)}$ are shown in Figure 4 in the case when the image of vertex 0 is part of the cluster. We also have the reflections of these configurations when the image of vertex 1 is part of the cluster.

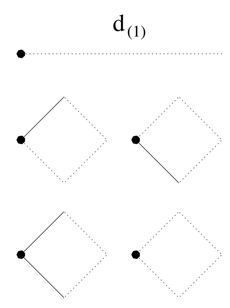

Figure 4: The 4 extra disconnected configurations

The probability distribution for the evolution of the types is given by

$$
\begin{aligned}
c \to & \begin{cases} (2c, 2d_{(1)}, 0d_{(2)}) & 2 \text{ configs} \quad p(1-p)^2 \\ (3c, 0d_{(1)}, 1d_{(2)}) & 4 \text{ configs} \quad p^2(1-p) \\ (4c, 0d_{(1)}, 0d_{(2)}) & 1 \text{ config} \quad p^3 \end{cases}, \\
d_{(1)} \to & \begin{cases} (1c, 2d_{(1)}, 0d_{(2)}) & 2 \text{ configs} \quad p(1-p) \\ (2c, 2d_{(1)}, 0d_{(2)}) & 1 \text{ config} \quad p^2(1-p) \\ (0c, 2d_{(1)}, 0d_{(2)}) & 1 \text{ config} \quad 1-p \end{cases}, \\
d_{(2)} \to & \begin{cases} (1c, 2d_{(1)}, 1d_{(2)}) & 4 \text{ configs} \quad p(1-p)^2 \\ (2c, 0d_{(1)}, 2d_{(2)}) & 4 \text{ configs} \quad p^2(1-p) \\ (0c, 4d_{(1)}, 0d_{(2)}) & 1 \text{ config} \quad (1-p)^3. \end{cases}
\end{aligned}
\tag{2.1}
$$

For example, the transition distribution for $c \to (2c, 2d_{(1)}, 0d_{(2)})$ can be computed as $p^2(1-p)^2/p = p(1-p)^2$, since the initial state is conditioned on c. Noting that $2p^2 - p^4 = p$ (because $p = p_c$), when we fix the initial state, the sum of the probabilities for each possible evolution is equal to 1.

Using these replacement rules and starting from an initial graph $\mathcal{G}_0 = D_0$ we produce a sequence of connected subgraphs $(\mathcal{G}_n)_{0 \le n \le N}$ of the diamond hierarchical lattice by retaining only those edges of D_n for $n = 0, .., N$ which are labelled c at the large scale.

Lemma 2.3 *The sequence of graphs $(\mathcal{G}_n)_{0 \le n \le N}$ has the same distribution as $(\mathcal{C}_n)_{0 \le n \le N}$, the sequence of graphs which grow to be the infinite cluster in the Bernoulli bond percolation model for the diamond hierarchical lattice conditional upon connecting the vertices 0 and 1.*

Proof: As in the case of the Bernoulli bond percolation graph constructed on the graph D_n in Lemma 2.2, this follows from the construction. ∎

From now on we will write $\mathcal{C}_n$ for the subgraph of D_n which is the level n percolation cluster connected to the origin. (Note that $G_n = \cup_x \mathcal{C}_n(x)$ is a collection of connected components, so $\mathcal{C}_n = \mathcal{C}_n(0) \subset G_n$. We will not need to use G_n any more.)

3 Scaling Limits

3.1 The scaling limit of the diamond hierarchical lattice

We begin by discussing the diamond hierarchical lattice. The sequence of graphs (D_n) can be rescaled to give each edge length 2^{-n} and the resulting limit can be regarded as a fractal in that it is a self-sufficient metric space built from 4 contraction maps. This is not a finitely ramified fractal as in the limit there will be a countable infinity of connections at any vertex in V_n for a given n. It is a simple fractal in the sense of [40] and thus there exists a diffusion on the scaling limit via the methods of [40]. We will take a different approach here.

Let (K, d) be a compact metric space containing two points labelled 0,1 and $\{\psi_i : i = 1, 2, 3, 4\}$ be a set of contractions $\psi_i : K \to K$, with contraction factor 1/2 with respect to the metric d, and the following properties: $\psi_1(0) = \psi_2(0) = 0$, $\psi_3(1) = \psi_4(1) = 1$ and $\psi_1(1) = \psi_3(0)$, $\psi_2(1) = \psi_4(0)$

and $\psi_i(\text{int}(K)) \cap \psi_j(\text{int}(K)) = \emptyset$ for all $i \neq j$, where $\text{int}K = K\backslash\{0,1\}$. This defines the scaling limit of the diamond hierarchical lattice as a self-similar set in that

$$K = \bigcup_{i=1}^{4} \psi_i(K).$$

We recall that the limit set can be regarded as the image space of the boundary of the tree ∂T via $\pi : \partial T \to K$, where $\{\pi(\mathbf{i})\} = \cap_{n\geq 0}\psi_{i_1} \circ \ldots \circ \psi_{i_n}(K)$.

We can observe that in the framework of [30] the post critical set is countably infinite (it consists of all possible addresses of the points 0 and 1) but we can still regard the fractal as having a boundary consisting of two points, as the countable collection of addresses only point to the two vertices 0 and 1. We recall that V_0 is the set of vertices of D_0 and that we can embed V_n, the vertices of D_n, in K by setting $V_n = \cup_{i=1}^{4}\psi_i(V_{n-1})$ for $n = 1, 2, \ldots$.

We note that the diamond hierarchical lattice has similar dimension properties to $\mathbb{R}^2$. If we compute the Hausdorff dimension of the set it is 2, the resistance does not scale in the sense that unit resistances on each edge lead to a unit resistance across the whole set and hence the walk dimension and the spectral dimension are also 2.

Definition 3.1 *We define the natural metric on K. For $x, y \in V_n$, let $\pi_n(x, y)$ denote the set of paths from x to y in the graph D_n. Let $d_n(x, y) = \min\{|\xi| : \xi \in \pi_n(x, y)\}$ be the number of edges in the shortest path on V_n between x and y. Then, the following limit exists;*

$$d(x, y) := \lim_{n\to\infty} 2^{-n} d_n(x_n, y_n), \qquad \forall x, y \in K$$

where $x_n, y_n \in V_n$ converge to x, y respectively as $n \to \infty$. The limit is independent of the choice of the approximating sequence. It is easy to see that d is a geodesic metric.

Definition 3.2 *Let μ be the Hausdorff measure on K. It satisfies the following for all $\mathbf{i} \in I^n$; $\mu(K_{\mathbf{i}}) = 4^{-|\mathbf{i}|}$.*

Note that μ does NOT satisfy the volume doubling property with respect to the natural distance on K. Denote the volume of a ball by $V(x, r) = \mu(B(x, r))$. Note also that the following does NOT hold; $c_1 r^2 \leq V(x, r) \leq c_2 r^2$ for all $x \in K, 0 \leq r \leq 1$, because otherwise μ would satisfy the volume doubling property.

We will discuss the properties of this set in Section 4

3.2 The scaling limit of the critical percolation cluster

The scaling limit for the critical percolation cluster itself will be a random recursive graph directed fractal. As for the diamond hierarchical lattice we define the limit as a self sufficient metric space and we take the same contraction maps as for the diamond hierarchical lattice. Now however we will only use the composition of all the maps leading to the individuals labelled c in the multitype branching process.

We recall the labelling of the infinite cluster branching process as given in Section 2.3. Each vertex $\mathbf{i} \in T$ has four edges out labelled $1, \ldots, 4$ and we associate the map ψ_i with the label i. The probability space $(\Omega, \mathbb{P})$ introduced in Section 2.2 can now be viewed as a probability space for the random recursive graph directed fractal $\mathcal{C}(\omega)$. This random recursive graph directed fractal is determined by a construction graph with three vertices, each corresponding to a type (labelled as before $c, d^{(1)}, d^{(2)}$). The edges of the construction graph determine how a given type of fractal is composed of subtypes. The random recursive set is viewed as a vector of sets, one for each type, each of which is a random set composed of copies of the random sets of the possible types, see for example [41].

We write $\omega \in \Omega$ as $\omega = \{u_{\mathbf{i}} : \mathbf{i} \in T\}$ for the tree with its labels. Then for a given $\omega \in \Omega$ the fractal $\mathcal{C}(\omega)$, which we will often denote $\mathcal{C}^{(u_\emptyset)}$ to indicate the type $u_\emptyset$ of the set, will satisfy

$$\mathcal{C}^{(u_\emptyset)} = \bigcup_{i=1}^{4} \psi_i(\mathcal{C}^{(u_i)}) = \bigcup_{i=1}^{4} \psi_i(\mathcal{C}(\sigma_i \omega)),$$

where $\mathcal{C}^{(\tau)}$ is the random recursive fractal corresponding to type τ and σ_i is the shift along the tree down the branch labelled i in that if $\omega = \{(j, \tilde{\omega}_j), j = 1, .., 4, \tilde{\omega}_j \in \Omega\}$, then $\sigma_i \omega = \tilde{\omega}_i$.

The Hausdorff measure on the limit will also satisfy a recursive formula in that for $\omega \in \Omega$,

$$\mu_\omega(.) = \mu_\omega^{u_\emptyset}(.) = \sum_{i=1}^{4} \mu_\omega^{u_i}(\psi_i(.)) = \sum_{i=1}^{4} \mu_{\sigma_i \omega}(\psi_i(.)).$$

Now μ_ω^τ is a measure on a set of type τ, the type corresponding to the root of the tree ω.

3.3 The dimension of the critical cluster

The branching structure underlying the construction means that it is possible to use branching processes to describe the volume growth of the infinite cluster. If we consider the scaling limit, in which we scale the length of each edge in $\mathcal{C}_n$, the critical cluster on D_n, by 2^{-n}, we obtain a sequence of graphs which can be embedded in a fractal. Indeed this is a random recursive graph directed simple fractal space. The computation of the Hausdorff dimension (in fact the multifractal spectrum) of such fractals is described in [41]. Here we use the connection with multitype branching processes. We note that as the length scaling is always 1/2, we just need to compute the number of edges in $\mathcal{C}_n$. These can be described by a multitype branching process with three types, corresponding to $c, d_{(1)}, d_{(2)}$. The number of edges in the graph $\mathcal{C}_n$ is the number of type c individuals in our branching process. It is straightforward to write down the mean matrix of the process and thus to compute the growth of type c individuals.

In order to compute the dimension of the set we do not need the labels for the individuals, we just record the number of each type as this is the offspring distribution for the multitype branching process which describes the growth. Let X be the random vector of the number of offspring of each type. We write $P^\tau(X_c = n_c, X_{d_{(1)}} = n_{d_{(1)}}, X_{d_{(2)}} = n_{d_{(2)}})$ for the probability that an individual of type τ has $n_c, n_{d_{(1)}}, n_{d_{(2)}}$ offspring of types $c, d_{(1)}, d_{(2)}$. From (2.1), we have the following,

$$\begin{aligned}
P^c(X_c = 2, X_{d_{(1)}} = 2, X_{d_{(2)}} = 0) &= 2p(1-p)^2 \\
P^c(X_c = 3, X_{d_{(1)}} = 0, X_{d_{(2)}} = 1) &= 4p^2(1-p) \\
P^c(X_c = 4, X_{d_{(1)}} = 0, X_{d_{(2)}} = 0) &= p^3
\end{aligned}$$

$$\begin{aligned}
P^{d_{(1)}}(X_c = 1, X_{d_{(1)}} = 2, X_{d_{(2)}} = 0) &= 2p(1-p)\\
P^{d_{(1)}}(X_c = 2, X_{d_{(1)}} = 2, X_{d_{(2)}} = 0) &= p^2(1-p)\\
P^{d_{(1)}}(X_c = 0, X_{d_{(1)}} = 2, X_{d_{(2)}} = 0) &= 1-p
\end{aligned}$$

$$\begin{aligned}
P^{d_{(2)}}(X_c = 1, X_{d_{(1)}} = 2, X_{d_{(2)}} = 1) &= 4p(1-p)^2\\
P^{d_{(2)}}(X_c = 2, X_{d_{(1)}} = 0, X_{d_{(2)}} = 2) &= 4p^2(1-p)\\
P^{d_{(2)}}(X_c = 0, X_{d_{(1)}} = 4, X_{d_{(2)}} = 0) &= (1-p)^3
\end{aligned}$$

From this we can compute the mean matrix, which simplifies by using the fact that at $p = p_c$ we have $2p - p^3 = p + p^2 = 1$, and writing $q = 1 - p = p^2$,

$$EX = \begin{bmatrix} 8q & 4pq^2 & 4q^2 \\ 2q & 2 & 0 \\ 4q & 4pq & 4q \end{bmatrix} = \begin{bmatrix} 8p^2 & 4p^5 & 4p^4 \\ 2p^2 & 2 & 0 \\ 4p^2 & 4p^3 & 4p^2 \end{bmatrix}.$$

For example, the $(1, 1)$-component of the matrix can be computed as follows,

$$2 \times 2p(1-p)^2 + 3 \times 4p^2(1-p) + 4 \times p^3 = 4pq(q + 3p + 1) = 8pq(1 + p) = 8q = 8p^2.$$

The rate of growth of the number of individuals is the maximum eigenvalue of this matrix which is the largest root of

$$x^3 + (6\sqrt{5} - 20)x^2 + (36\sqrt{5} - 68)x + 64 - 32\sqrt{5} = 0.$$

This can be computed numerically as $x_{\max} = 3.8425....$

Theorem 3.3 *The fractal which is the scaling limit of the infinite Bernoulli bond percolation cluster on the diamond hierarchical lattice has Hausdorff dimension* $d_f = \log x_{\max}/\log 2 = 1.8993....$

Remark 3.4 Thus the dimension of the critical cluster in the diamond lattice is different from that of the IIC in $\mathbb{Z}^2$ which is known to be $91/48 = 1.8959....$

The natural geometric measure on the fractal can be described by the branching process in that the limit measure will be random with the total mass given by the limit random variable in the multitype branching process. When we consider the analytic properties of the percolation cluster we will need to work in the effective resistance metric and in this setting we will use a similar construction but based on a multitype branching random walk. We discuss this further in Section 5.

4 The diamond hierarchical lattice and its properties

In this section, we will discuss the construction of the Dirichlet form on the diamond hierarchical lattice as well as the spectral asymptotics and heat kernel estimates associated with this form.

4.1 Construction of the Dirichlet form

The construction of a Dirichlet form on this limit is straightforward, even though we do not have finite ramification, as the approach of [29, 35] is still applicable.

Let

$$\mathcal{E}_0(f,g) = \frac{1}{2}(f(0)-f(1))(g(0)-g(1)). \tag{4.1}$$

We then define

$$\mathcal{E}_1(f,g) = \sum_{i=1}^{4} \mathcal{E}_0(f\circ\psi_i, g\circ\psi_i),$$

and note that $\inf\{\mathcal{E}_1(g,g) : g|_{V_0} = f\} = \mathcal{E}_0(f,f)$ for any $f : V_0 \to \mathbb{R}$. Thus we can extend this to write

$$\mathcal{E}_n(f,g) = \sum_{i=1}^{4} \mathcal{E}_{n-1}(f\circ\psi_i, g\circ\psi_i),$$

and put

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_n(f,f), \ \ \forall f \in \mathcal{F}^* := \{f : \cup_{m\geq 0} V_m \to \mathbb{R} \mid \sup_n \mathcal{E}_n(f,f) < \infty\}.$$

We denote $\mathcal{F}_D^* = \{f \in \mathcal{F} : f|_{V_0} = 0\}$.

We recall that the diamond hierarchical lattice is not a p.c.f. self-similar set in the sense of [30], and note that the harmonic structure is not regular. Nevertheless, we can construct a regular local Dirichlet form on $L^2(K,\mu)$ in the same way as the non-regular harmonic structure case (see [35] section 3 or [30] section 3.4). Below, we will state the key proposition for the construction without proof.

Proposition 4.1 (i) *For each $m \in \mathbb{N}$ and $h : V_m \to \mathbb{R}$, there exists a unique function $P_m h \in C(K)$ such that the following holds,*

$$P_m h|_{V_m} = h, \quad \text{and} \quad \mathcal{E}((P_m h)|_{\cup_{m\geq 0} V_m}, (P_m h)|_{\cup_{m\geq 0} V_m}) = \mathcal{E}_m(h,h).$$

(ii) *For any $f \in \mathcal{F}^*$, $\{P_m f\}_m$ converges in $L^2(K,\mu)$ as $m \to \infty$.*

The proof of (i) is the same as that of Corollary 3.2.15 in [30] and the proof of (ii) is the same as that of Lemma 3.4.3 in [30]. (Note that in this case $r_i = 1$ for $i = 1, \cdots, 4$ and $\mu(K_{\mathbf{i}}) = \mu_{\mathbf{i}} := 4^{-|\mathbf{i}|}$.)

For $f \in \mathcal{F}^*$, let $\iota_\mu(u)$ be the limit of $\{P_m f\}_m$ in $L^2(K,\mu)$ as $m \to \infty$.

Lemma 4.2 *$\iota_\mu : \mathcal{F}^* \to L^2(K,\mu)$ is injective and it is a compact operator. Here the norm on $\mathcal{F}^*$ is given by $\mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2}^2$.*

The proof is the same as those of Lemma 3.4.4 and Lemma 3.4.5 in [30].

Let $\mathcal{F} := \iota_\mu(\mathcal{F}^*) \subset L^2(K,\mu)$ and $\mathcal{F}_D := \iota_\mu(\mathcal{F}_D^*) \subset L^2(K,\mu)$. Then, the following can be proved in a similar way to Theorem 3.4.6 and Corollary 3.4.7 in [30].

Theorem 4.3 *The pair $(\mathcal{E},\mathcal{F})$ is a local regular Dirichlet form on $L^2(K,\mu)$ with the following self-similarity,*

$$\mathcal{E}(f,g) = \sum_{i=1}^{4} \mathcal{E}(f\circ\psi_i, g\circ\psi_i), \qquad \forall f,g \in \mathcal{F}.$$

The corresponding non-negative self-adjoint operator H_N on $L^2(K,\mu)$ has compact resolvent. Similarly $(\mathcal{E},\mathcal{F}_D)$ is a local regular Dirichlet form and the corresponding non-negative self-adjoint operator H_D on $L^2(K,\mu)$ has compact resolvent.

From the construction, it is easy to check that $\mathcal{E}(f,f)=0$ if and only if f is a constant function, in particular $1\in\mathcal{F}$ and $\mathcal{E}(1,1)=0$. So $(\mathcal{E},\mathcal{F})$ is conservative. We note that the Dirichlet form is not a resistance form.

4.2 Spectral properties

By Theorem 4.3, the self-adjoint operators H_N and H_D have compact resolvents. Therefore the Neumann eigenvalues (and also the Dirichlet eigenvalues) are non-negative, of finite multiplicity and their only accumulation point is ∞. Let $N_N(x)$ and $N_D(x)$ be the Neumann and Dirichlet eigenvalue counting functions respectively. That is, for $b=N$ and D,

$$N_b(x)=\max\{k:\lambda_k^b\le x\},$$

where $\{\lambda_i^b\}_{i\ge1}$ is the non-decreasing sequence of eigenvalues (including the multiplicity) for H_b.

Definition 4.4 *$u\in\mathcal{F}$ is called a pre-localized eigenfunction of $\mathcal{E}$ belonging to the eigenvalue λ if $u\in\mathcal{F}_D$, $u\not\equiv0$ and*

$$\mathcal{E}(u,v)=\lambda(u,v)_{L^2},\qquad\forall v\in\mathcal{F}.$$

We then have the following asymptotics for $N_b(x)$ as $x\to\infty$.

Theorem 4.5 *The following holds for $b=N$ and D,*

$$0<\liminf_{x\to\infty}\frac{N_b(x)}{x}<\limsup_{x\to\infty}\frac{N_b(x)}{x}<\infty. \tag{4.2}$$

Further, (1.1) in Theorem 1.1 holds where p in (1.1) is a non-constant periodic function.

Proof: Again we can apply the proof for p.c.f. self-similar sets in [30]. The proof of $0<\liminf_{x\to\infty}N_b(x)/x\le\limsup_{x\to\infty}N_b(x)/x<\infty$ and (1.1) without the knowledge of p being a non-constant, are the same as that of Theorem 4.1.5 (2) in [30]. To prove the strict inequality in the middle (and thus prove that p is non-constant), we use the existence of pre-localized eigenfunctions.

By Theorem 4.1.5 (2) and Theorem 4.3.4 in [30], the strict inequality in the middle of (4.2) is equivalent to the existence of a pre-localized eigenfunction. Let $h:K\to K$ be a homeomorphism such that $h(\pi(\mathbf{i}))=\pi(\bar{\mathbf{i}})$, where $\bar{\mathbf{i}}\in I^\infty$ is determined by $\mathbf{i}\in I^\infty$ by exchanging letters 1 to 2, and 3 to 4 in each element. (So, h is a "reflection" of K with respect to the "hypersurface" that contains V_0.) By Proposition 4.4.3 in [30], this h guarantees the existence of a pre-localized eigenfunction. ∎

Remark 4.6 In [30], pre-localized eigenfunctions are defined for the Laplace operators instead of the Dirichlet form. Using Definition 4.4, the above arguments still work in a similar way to those in [30].

We note that a complete description of the spectrum for the diamond lattice is given in [2].

4.3 Heat kernel estimates

In this subsection, we obtain detailed heat kernel estimates for the diffusion process $\{X_t\}$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ given in Theorem 4.3. Our main theorem is the following.

Theorem 4.7 *There exists a jointly continuous function $p_t(x,y)$, $t \in (0,1), x, y \in K$ such that*

$$P_t f(x) = \int_K p_t(x,y) f(y)\mu(dy), \qquad \forall t \in (0,1), x \in K, \text{and } f \in L^2(K,\mu). \tag{4.3}$$

Further $p_t(x,y)$ enjoys the following estimates: There are strictly positive constants c_1, c_2, c_3, c_4 such that for all $x, y \in K$, $t \in (0,1)$,

$$0 < p_t(x,y) \le \frac{c_1}{t} \exp\Big(- c_2 \frac{d(x,r)^2}{t} \Big) \tag{4.4}$$

$$p_t(x,x) \ge \frac{c_3}{V(x, c_4\sqrt{t})}. \tag{4.5}$$

In order to prove this theorem, we will discuss various properties of $\{X_t\}$.

(I) Poincaré inequality

Since the self-adjoint operator H_N has a compact resolvent (Theorem 4.3), there is a spectral gap. Thus, $0 < \lambda_{min} := \inf_{f\in\mathcal{F}\setminus\{\text{const}\}} \mathcal{E}(f,f)/\|f\|_2^2$. Since $1 \in \mathcal{F}$ and $\mathcal{E}(1,h) = 0$ for all $h \in \mathcal{F}$, we have the following.

Proposition 4.8 *There exists $c_1 > 0$ such that*

$$\int_K |f - \bar{f}|^2 d\mu \le c_1 \mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}, \tag{PI}$$

where $\bar{f} = \int_K f d\mu$.

(II) Ultracontractivity

We will use (PI) and the self-similarity of the form to establish the following ultracontractivity.

Proposition 4.9 *There exists $c_1 > 0$ such that for each $t \in (0,1)$,*

$$\|P_t\|_{1\to\infty} \le \frac{c_1}{t}.$$

Remark 4.10 Note that we *cannot* expect to obtain the following sharp upper bound:

$$p_t(x,x) \le \frac{c_1}{V(x, c_3\sqrt{t})} \qquad \forall x \in K, t \in (0,1]. \tag{4.6}$$

Indeed, Lemma 3.5.4 and Theorem C.3 in [31], (4.6), and the self-similarity of the Dirichlet form imply volume doubling, which is a contradiction.

Proof of Proposition 4.9: The following argument is a modification of the proof of Proposition 5.1 in [7]. For $\mathbf{i} \in I^m$ write $f_{\mathbf{i}} = f \circ \psi_{\mathbf{i}}$ and define

$$\bar{f}_{\mathbf{i}} = \int_K f_{\mathbf{i}}(x)\mu(dx) = \mu_{\mathbf{i}}^{-1} \int_{\psi_{\mathbf{i}}(K)} f(x)\mu(dx).$$

Note that for $v \in \mathcal{F}$ and $l \geq 0$, $\bar{v} = \int v d\mu = \sum_{\mathbf{i} \in I^l} \bar{v}_{\mathbf{i}} \mu_{\mathbf{i}}$. Let $u_0 \in \mathcal{D}(\mathcal{L})$ with $u_0 \geq 0$ and $\|u_0\|_1 = 1$. Set $u_t(x) = (P_t u_0)(x)$ and $g(t) = \|u_t\|_2^2$. We remark that g is continuous and decreasing. As the semigroup is symmetric and Markov,

$$\|u_t\|_1 = \int P_t u_0 d\mu = \int u_0 P_t 1 d\mu \leq \|u_0\|_1 = 1.$$

For each $l \geq 0$,

$$\begin{aligned}
\frac{d}{dt} g(t) &= 2(\mathcal{L}u_t, u_t) = -2\mathcal{E}(u_t, u_t) \\
&= -2 \sum_{\mathbf{i} \in I^l} \mathcal{E}(u_t \circ \psi_{\mathbf{i}}, u_t \circ \psi_{\mathbf{i}}) \\
&\leq -2c_2 \sum_{\mathbf{i}} \int (u_{t,\mathbf{i}} - \bar{u}_{t,\mathbf{i}})^2 d\mu \quad \text{(by } (PI)) \\
&= -2c_2 \sum_{\mathbf{i}} \mu_{\mathbf{i}}^{-1} \int_{\psi_{\mathbf{i}}(K)} (u_t)^2 d\mu + 2c_2 \sum_{\mathbf{i}} (\mu_{\mathbf{i}}^{-1} \int_{\psi_{\mathbf{i}}(K)} u_t d\mu)^2 \\
&= -2c_2 4^l \|u_t\|_2^2 + 2c 4^{2l} \sum_{\mathbf{i}} (\int_{\psi_{\mathbf{i}}(K)} u_t d\mu)^2 \\
&\leq -2c_2 4^l g(t) + 2c_2 4^{2l} (\sum_{\mathbf{i}} \int_{\psi_{\mathbf{i}}(K)} u_t d\mu)^2 \\
&\leq -2c_2 4^l (g(t) - 4^l).
\end{aligned}$$

Therefore

$$-\frac{d}{dt} \log(g(t) - 4^l) \geq c_3 4^l, \quad \text{if } \ g(t) > 4^l. \tag{4.7}$$

Let $s_l = \inf\{t \geq 0 : g(t) \leq 4^l\}$ for $l \in \mathbf{N}$. Thus (4.7) holds for $0 < t < s_l$. Note that $s_l \to 0$ as $l \to \infty$. Integrating (4.7) from s_{l+2} to s_{l+1} we obtain

$$\begin{aligned}
c_3 4^l (s_{l+1} - s_{l+2}) &\leq -\log(g(s_{l+1}) - 4^l) + \log(g(s_{l+2}) - 4^l) \\
&= \log(4^{l+2} - 4^l)/(4^{l+1} - 4^l) \leq c_4.
\end{aligned}$$

Thus $s_{l+1} - s_{l+2} \leq c_5 4^{-l}$, and iterating this we have

$$s_l \leq c_5 \sum_{k=l-1}^{\infty} 4^{-k} \leq c_6 4^{-l}.$$

This implies that $g(c_6 4^{-l}) \leq g(s_l) = 4^l$. Let n be such that $4^{-n} \leq t/c_6 \leq 4^{-n+1}$. Taking $l = n$, it follows that

$$g(t) \leq 4^n \leq c_7 t^{-1}.$$

Using the fact that $\|P_t\|_{1\to\infty} \leq \|P_t\|_{1\to 2}^2$, we deduce the result. ∎

(III) Exit times

For $A \subset K$, let

$$\tau_A = \tau_A(X) = \inf\{t \geq 0 : X_t \notin A\}.$$

We then have the following.

Lemma 4.11 *There exist $c_1, c_2 > 0$ such that for all $x \in K$ and $0 < r < 1$,*

$$c_1 r^2 \leq E^x \tau_{B(x,r)} \leq c_2 r^2. \qquad (E_2)$$

Proof: Let Pr be the projection from K onto $[0,1]$ defined as follows; $Pr(\pi(\mathbf{i})) = \hat{\pi}(\hat{\mathbf{i}})$, where $\hat{\mathbf{i}} \in \{1,3\}^\infty$ is determined from $\mathbf{i} \in I^\infty$ by swapping the letters 2 to 1, and 4 to 3 in each element and $\hat{\pi} : \{1,3\}^\infty \to [0,1]$ is the natural projection from the word space to $[0,1]$. It is easy to see that $Pr(X_t) =: \hat{X}_t$ is a reflected Brownian motion on $[0,1]$ and

$$Pr(B(x,r)) \subset \hat{B}(Pr(x), r),$$
$$A(x, r/4) := (\text{Connected component of } Pr^{-1}(\hat{B}(Pr(x), r/4)) \text{ containing } x) \subset B(x,r),$$

where $\hat{B}(Pr(x), r)$ is a ball in $[0,1]$ centred at $Pr(x)$ and radius r. Further, it is well known that

$$c_3 r^2 \leq E^{Pr(x)}[\tau_{\hat{B}(Pr(x),r)}(\hat{X})] \leq c_4 r^2.$$

Combining these, we have

$$\begin{aligned} E^x \tau_{B(x,r)} &\leq E^{Pr(x)}[\tau_{\hat{B}(Pr(x),r)}(\hat{X})] \leq c_4 r^2, \\ c_3 r^2/16 &\leq E^{Pr(x)}[\tau_{\hat{B}(Pr(x),r/4)}(\hat{X})] = E^x[\tau_{A(x,r/4)}(X)] \leq E^x \tau_{B(x,r)}. \end{aligned}$$

Thus we obtain (E_2). ∎

From (E_2) a standard argument gives the following. See, for example, Lemma 3.16 and (3.21) in [3].

Proposition 4.12 *There exist $c_1, c_2 > 0$ such that for all $x \in K$ and $0 < r, t < 1$,*

$$P^x(\tau_{B(x,r)} \leq t) \leq c_1 \exp(-\frac{c_2 r^2}{t}). \qquad (ELD)$$

(IV) Existence and continuity of the heat kernel

As in Proposition 4.9, the semigroup is ultracontractive. This fact together with (E_2) and the structure of K allow us to deduce that there is a jointly continuous heat kernel for $\{X_t\}$. Let $\{\lambda_k^N\}_{k\geq 1}$ be the increasing sequence of eigenvalues for H_N and $\{\varphi_k\}$ be a complete orthonormal system for $L^2(K,\mu)$ such that $H_N \varphi_k = \lambda_k^N \varphi_k$.

Proposition 4.13 *There exists $p_t(x,y)$, $t \in (0,1), x, y \in K$ that satisfies (4.3). Further $\phi_k \in C(K)$ for all $k \geq 1$ and*

$$p_t(x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k^N t} \phi_k(x)\phi_k(y) > 0, \qquad (4.8)$$

where the sum is absolutely and uniformly convergent on $[T_0, 1] \times K \times K$ for any $T_0 \in (0,1)$. In particular $p_t(x,y)$ is jointly continuous.

Proof: First, since $\mu(K) < \infty$ and $\{P_t\}_t$ is ultracontractive, by general theory we know that $\phi_k \in L^\infty$ and (4.8) holds where the sum is absolutely and uniformly convergent on $[T_0, 1] \times K \times K$ for any $T_0 \in (0,1)$ (see for example, [31, Theorem A.3]).

We next show that ϕ_k is continuous (then the joint continuity of $p_t(x,y)$ can be deduced). Note that harmonic functions are continuous in this case. (This can be proved similarly to [30, Theorem 3.2.4].) For each $\lambda > 0$, let U^λ be the λ-order Green operator, i.e. $U^\lambda f(x) = E^x[\int_0^\infty e^{-\lambda t} f(X_t)dt]$. Then, by the continuity of harmonic functions and (E_2), $U^\lambda f$ is continuous for any bounded function f. We will show this following [9, Proposition 3.3]. Fix x_0, let $r < 1/2$, and suppose $x, y \in B(x_0, r/2)$. By the strong Markov property,

$$U^\lambda f(x) = E^x[\int_0^{\tau_r} e^{-\lambda t} f(X_t)\, dt] + E^x(e^{-\lambda \tau_r} - 1)U^\lambda f(X_{\tau_r}) + E^x U^\lambda f(X_{\tau_r}) =: I_1 + I_2 + I_3,$$

where $\tau_r = \tau_{B(x,r)}$. By (E_2), we have

$$|I_1 + I_2| \le \|f\|_\infty E^x \tau_r + \lambda E^x \tau_r \|U^\lambda f\|_\infty \le c r^2 \|f\|_\infty,$$

where $\|U^\lambda f\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$ is used in the last inequality. So

$$|U^\lambda f(x) - U^\lambda f(y)| \le c r^2 \|f\|_\infty + |E^x U^\lambda f(X_{\tau_r}) - E^y U^\lambda f(X_{\tau_r})|. \tag{4.9}$$

But $z \to E^z U^\lambda f(X_{\tau_r})$ is bounded in K and harmonic in $B(x_0, r)$, so it is continuous. Set $r = d(x,y)^{1/2}$, then we see that the right hand side of (4.9) is small when $d(x,y)$ is small and the continuity of $U^\lambda f$ is deduced. Now, since $P_t \phi_k = e^{-\lambda_k^N t}\phi_k$ a.e., we have $U^\lambda \phi_k = (\lambda + \lambda_k^N)^{-1}\phi_k$ a.e., in other words $\phi_k = (\lambda + \lambda_k^N) U^\lambda \phi_k$ a.e.. Since $\phi_k \in L^\infty$, the right hand side is continuous, so we have a continuous version of ϕ_k.

Given the above results, the positivity of $p_t(x,y)$ can be deduced by a standard argument; see for example, [31, Theorem A.4]. ∎

(V) Full upper bound

By Proposition 4.9 and (ELD), a standard argument gives the full upper bound in (4.4). See, for example, the first half of the proof of Theorem 3.11 in [3].

(VI) On-diagonal lower bound

Since $\{X_t\}$ is conservative, (ELD) gives the on-diagonal lower bound of the heat kernel (4.5).

Lemma 4.14 *There exist $c_1, c_2 > 0$ such that for all $x \in K$ and $0 < r, t < 1$,*

$$p_t(x,x) \ge \frac{c_1}{V(x, c_2\sqrt{t})}. \tag{DLHK}$$

Proof: The proof is standard. Using (ELD) we have that

$$P^x(Y_t \notin B(x,r)) \le P(\tau_{B(x,r)} \le t) \le c_1 \exp(-\frac{c_2 r^2}{t}).$$

Hence by choosing r such that $c_3 r^2 < t < c_4 r^2$ for some $c_3, c_4 > 0$, we have

$$P^x(Y_t \notin B(x,r)) \le c_5 < 1.$$

Since $\{X_t\}$ is conservative, this gives $P^x(Y_t \in B(x,r)) \ge 1 - c_5 > 0$. By Cauchy-Schwarz,

$$(1-c_5)^2 \le P^x(Y_t \in B(x,r))^2 = (\int_{B(x,r)} p_t(x,z) d\mu(z))^2 \le V(x,r) p_{2t}(x,x).$$

Now, using the lower bound of our choice of t, we obtain the result. ∎

Remark 4.15 Note that the elliptic Harnack inequality (EHI for short) does not hold in this case. Recall that $(\mathcal{E}, \mathcal{F})$ satisfies EHI if there exists $c > 0$ such that for any non-negative harmonic function h on $B(x, 2r)$ and any $0 < r \le 1$,

$$\sup_{y \in B(x,r)} h(y) \le c \inf_{y \in B(x,r)} h(y). \qquad (EHI)$$

Let $x = 0$, $2r = 2^{-n}$ and let $N = 2^n$. Then $B(0, 2r)$ consists of N copies of small diamonds with length $2r$, which we label $C_0, C_1, \cdots, C_{N-1}$. Consider a harmonic function whose boundary value at each $x \in \partial B(0, 2r) \cap C_i$ is 2^i when $i \ge 1$ and the value at $\partial B(0, 2r) \cap C_0$ is 0. Then, its value at 0 is $\sum_{i=1}^{N-1} 2^i/N$ which is of order $2^N/N$. So, the value of the harmonic function at $\partial B(0, r) \cap C_0$ is of order $2^N/N$ whereas the value at $\partial B(0, r) \cap C_{N-1}$ is of order 2^N. These two values are not comparable when n (so N) varies, thus (EHI) does not hold.

(VII) Proof of Theorem 1.2 for $p_t(\cdot, \cdot)$

(i) First, a sequence $\{x_1, x_2, \cdots, x_l\} \subset V_m$ is called an m-walk if $\{x_i, x_{i+1}\} \in E_m$ for all $i = 1, 2, \cdots, l - 1$. For $x = \psi_{\mathbf{i}}(0) \in K \setminus \cup_{l \ge 0} V_l$ where $\mathbf{i} \in I^\infty$, define $\partial D_n(x) := \psi_{\mathbf{i}|n}(V_0)$. (Here $\mathbf{i}|_n = i_1 i_2 \cdots i_n$ if $\mathbf{i} = i_1 i_2 \cdots$.) Now, for $x \in K \setminus \cup_{l \ge 0} V_l$ and $n, m \ge 0$, let $n_{n,m}(x)$ be the smallest number of steps by an $(n + m)$-walk from x to $\partial D_n(x)$, where we take x_1 to be the nearest point to x in D_{n+m} (with an arbitrary choice for ties). Then, we can prove the following in the same way as Proposition 3.3 of [7]: there exists $g : K \to [0, \infty)$ such that for a.e. $x \in K$,

$$c(nm)^{-2} 2^m \le n_{n,m}(x), \qquad \forall n \ge 0, m \ge g(x). \qquad (4.10)$$

(Note that in [7] we needed to define a Λ_n-complex since the self-similar maps did not necessarily have the same contraction rates, but we do not need this notion in our setting. Further, it is easy to see that α in Proposition 3.3 of [7] is 2 in this case.) Now take $m = c' \log n$ where $c' > 0$. Then, the distance between x and V_n is no less than $2^{-n-m} \times c(nm)^{-2} 2^m = C 2^{-n} (n \log n)^{-2}$. So, taking $r = C 2^{-n} (n \log n)^{-2}$, we have

$$V(x, r) \le 4^{-n} = r^2 (n \log n)^2 / C \le C' r^2 |\log r|^2 |\log \log r|^2.$$

Using this together with (4.4) and (4.5), we obtain the desired estimate.

(ii) Since $p_t(x, y)$ is jointly continuous, we have for each $t < 1$,

$$p_t(0, 0) = \lim_{r \to 0} \frac{1}{V(0, r)} P^0(X_t \in B(0, r)).$$

It is easy to see $V(0, r) = r$. Furthermore, if we consider the projection of X_t onto $[0, 1]$ as in Lemma 4.11, then we see that

$$P^0(X_t \in B(0, r)) = P^0(\hat{X}_t \in B(0, r)).$$

So the desired estimate can be obtained from that of the heat kernel of reflected Brownian motion in $[0, 1]$. ∎

5 The critical percolation cluster

We give a multitype branching random walk description of the set and use this to construct a natural measure on the scaling limit of the critical percolation cluster. The branching random walk allows us to describe the sizes of all sets of a particular size in our cluster.

We begin with some notation. For $\mathbf{i} \in I^n$ we write $\mathcal{K}_{\mathbf{i}}$ for the set $\psi_{i_1} \circ \ldots \circ \psi_{i_n}(\mathcal{C}(\sigma_{\mathbf{i}}\omega))$ of type $u_{\mathbf{i}}$ with address $\mathbf{i}$ and call this an n-cell. We write $\mathcal{N}_n(\mathbf{i}) = \{\mathbf{j} \in I^n : \mathcal{K}_{\mathbf{j}} \cap \mathcal{K}_{\mathbf{i}} \neq \emptyset\}$ for the addresses of the n-neighbours of $\mathbf{i}$. For an n-cell we write

$$\bar{\mathcal{K}}_{\mathbf{i}} = \bigcup_{\mathbf{j} \in \mathcal{N}_n(\mathbf{i})} \mathcal{K}_{\mathbf{j}},$$

and call this the n-neighbourhood of the n-cell $\mathcal{K}_{\mathbf{i}}$.

We now set up a multitype branching random walk which describes the fractal's properties in the resistance metric (which we define later). The basic types of the individuals are given by c if the connection is present, $d_{(1)}$ where the connection is absent but a vertex is in the infinite cluster, $d_{(2)}$ where the connection is absent but both vertices are in the infinite cluster.

We need to extend the labelling from that in Section 2.2. We now split the type c individuals into types $c_{(1)}$ and $c_{(2)}$ in order to keep track of the property that the resistance of a connected edge depends on the offspring. For each edge previously labelled c we let its label be $c_{(2)}$ with probability $\tilde{p} = p_c^3$ (this corresponds to the configuration with all four edges present), otherwise it is labelled $c_{(1)}$. Note that i in $c_{(i)}$ stands for the number of connections between the two end points of the edge. Now the offspring distribution of type $c_{(2)}$ has all four offspring of type c and then the labels are determined independently according to $\tilde{p}$. For type $c_{(1)}$ the offspring are of the other 6 types of connected configuration, with all c labels determined independently according to $\tilde{p}$.

Let the type space for our branching process be $\mathcal{S} = \{c_{(1)}, c_{(2)}, d_{(1)}, d_{(2)}\}$. We consider the probability space of labelled trees $\Omega = \mathcal{S}^T$. (This is an abuse of notation since Ω was U^T, but from now on, we let $\Omega = \mathcal{S}^T$.) Thus if $\omega \in \Omega$ we have $\omega = \{u_{\mathbf{i}}\}_{\mathbf{i} \in T}$ where $u_{\mathbf{i}} \in \mathcal{S}$ for $\mathbf{i} \in T = \cup_{i=1}^{\infty} I^i \cup \{\emptyset\}$.

The distribution for the vector of types $\tilde{X} = (\tilde{X}_{c_{(1)}}, \tilde{X}_{c_{(2)}}, \tilde{X}_{d_{(1)}}, \tilde{X}_{d_{(2)}})$ can be expressed in terms of the previous distribution as $\tilde{X} = (X_c - Y, Y, X_{d_{(1)}}, X_{d_{(2)}})$ where Y is a Binomial(X_c, p^3) random variable.

In order to prove our results we let our individuals evolve as a branching random walk. Let Z_i^{kj} denote the position of the i-th individual offspring who is of type j arising from a parent of type k. We write $Z_{\mathbf{i}}^{kj}$ for the position of the individual $\mathbf{i} \in T$ of type j with initial ancestor of type k. We now define the distribution of the positions of the offspring in order that position of an individual is the logarithm of the electrical resistance of the corresponding edge.

By considering the effective resistance across the different configurations we define the resistance scale factors

$$\rho_{u_{\mathbf{i}}} = \begin{cases} 1 & u_{\mathbf{i}} = c_{(2)} \\ 2 & \text{otherwise} \end{cases} \tag{5.1}$$

Note that the resistance scale factor for $c_{(2)}$ is 1 NOT 2. Then set the position of the i-th offspring of a k-type individual, if the offspring is type j, to be $Z_i^{kj} = \log \rho_k$(which is independent of i).

Now define

$$A_{kj}(\theta) = \mathbb{E}\sum_{i=1}^{4} e^{-\theta Z_i^{kj}}, \qquad k, j \in \mathcal{S},$$

which gives the mean matrix

$$A(\theta) = \begin{bmatrix} 4pq(1+2p)2^{-\theta} & 2q^2(1+2p)2^{-\theta} & 2pq2^{-\theta} & 2q2^{-\theta} \\ 4-4p^3 & 4p^3 & 0 & 0 \\ 4p^2q2^{-\theta} & 2pq^2 2^{-\theta} & 22^{-\theta} & 0 \\ 8q^2 2^{-\theta} & 4pq^2 2^{-\theta} & 4pq2^{-\theta} & 4q2^{-\theta} \end{bmatrix}.$$

For example, the $(1,1)$-component of the matrix can be computed as follows,

$$\Big(\frac{4p^2(1-p)}{1-p^3} \times 3 + \frac{2p(1-p)^2}{1-p^3} \times 2\Big) \times (1-p^3) \times 2^{-\theta} = 2(1+2p)p(1-p^3)2^{-\theta} = 4pq(1+2p)2^{-\theta}.$$

We choose θ to be such that the maximum eigenvalue of the matrix $A(\theta) = [A_{kj}(\theta)]_{kj}$ is 1. Let φ be the corresponding right eigenvector. Now put

$$W_n^k = \sum_{j\in\mathcal{S}} \sum_{\mathrm{i}\in I^n} e^{-\theta Z_{\mathrm{i}}^{kj}} \varphi_j, \qquad \forall k \in \mathcal{S}.$$

A standard result from the theory of branching processes for the multitype branching random walk is the following.

Theorem 5.1 *For each $k \in \mathcal{S}$, the process $\{W_n^k : n = 0, 1, 2, \ldots\}$ is a positive martingale, hence has a limit such that*

$$W_n^k \to W\phi_k, \quad \text{as } n \to \infty,$$

where ϕ is the left eigenvector of A and W is a real valued random variable with mean one. The random variable W satisfies the following decomposition

$$W\phi_k = \sum_{j\in\mathcal{S}} \sum_{i=1}^{4} e^{-\theta Z_i^{kj}} W_i \phi_j, \tag{5.2}$$

where $W_i, i = 1, .., 4$ are i.i.d. copies of W.

Remark 5.2 An alternative view is that we can consider the Galton Watson branching process for the type $c_{(2)}$, which generates types $c_{(1)}$ and $c_{(2)}$. As the expected number of offspring of type $c_{(2)}$ generated by a parent of type $c_{(2)}$ is $4p_c^3 < 1$, the process is subcritical. If we consider the total progeny generated by a type $c_{(2)}$ individual, it generates a random number of type $c_{(1)}$ individuals. We then let these reproduce as usual. Now we consider the whole collection of progeny to be the offspring of the original type $c_{(2)}$ individual as they correspond to the stopping line in the branching random walk of the first hitting of the point $\log 2$ to the right from the position of the type $c_{(2)}$. In this way we have a multitype branching process where each generation counts the number of cells of size 2^{-n} in the infinite cluster. We will use this idea later.

5.1 The Dirichlet form

We give a short discussion of how to construct a Dirichlet form on the critical percolation cluster. This form is built in the same way as was done in [20]. We put resistances on each cell to ensure that the global resistance remains at 1. Thus, for $\omega = \{u_{\mathbf{i}}\}_{\mathbf{i}\in T}$, we set

$$\mathcal{E}_1^{(\omega)}(f,g) = \sum_{i:u_i\in\{c_{(1)},c_{(2)}\}} \mathcal{E}_0(f\circ\psi_i, g\circ\psi_i)\rho_{u_\emptyset},$$

where $\mathcal{E}_0(f,g)$ is the Dirichlet form corresponding to a two state Markov chain given in (4.1). The resistance scale factor was defined in (5.1) and is chosen to ensure that the resistance across level 0 corresponding to the form on the first level vertices $\mathcal{C}_1$ is 1. We now repeat this construction by setting

$$\mathcal{E}_n^{(\omega)}(f,g) = \sum_{i=1}^{4} \mathcal{E}_{n-1}^{(\sigma_i\omega)}(f\circ\psi_i, g\circ\psi_i)\rho_{u_\emptyset},$$

where $\sigma_j\omega = \{u_{j\mathbf{i}}\}_{\mathbf{i}\in T_j}$ (where T_j is the subtree of T descended from branch j). Thus we have a sequence of compatible resistance networks and we can define the limit Dirichlet form

$$\mathcal{E}^{(\omega)}(f,f) = \lim_{n\to\infty}\mathcal{E}_n^{(\omega)}(f,f),\ \ \forall f\in\mathcal{F}^{(\omega)} = \{f : \sup_n \mathcal{E}^{(\omega)}(f,f)<\infty\}.$$

We now define a measure on the scaling limit of the critical cluster. For any set measurable set $A\subset\mathcal{C}$ we define

$$\mu_\omega^k(A) = \lim_{m\to\infty}\sum_j\sum_{\mathbf{j}\in I^m} e^{-\theta Z_{\mathbf{j}}^{kj}}\varphi_j I_{\{\pi(\mathbf{j})\in A\}}. \tag{5.3}$$

First for an n-cell with address $\mathbf{i}$ the measure of type k is defined to be

$$\mu_\omega^k(\mathcal{K}_{\mathbf{i}}) = \lim_{m\to\infty}\sum_j\sum_{\mathbf{j}\in I^m} e^{-\theta Z_{\mathbf{j}}^{kj}}\varphi_j I_{\{\pi(\mathbf{j})\in \mathcal{K}_{\mathbf{i}}\}}.$$

A simple calculation and use of Theorem 5.1 gives

$$\begin{aligned}\mu_\omega^k(\mathcal{K}_{\mathbf{i}}) &= e^{-\theta Z_{\mathbf{i}}^{ku_{\mathbf{i}}}}\lim_{m\to\infty}\sum_j\sum_{\mathbf{j}\in I_m(\mathbf{i})} e^{-\theta Z_{\mathbf{j}}^{u_{\mathbf{i}}j}}\varphi_j \\ &= \rho_{-\mathbf{i}}^{\theta} W_{\mathbf{i}}\phi_k, \end{aligned} \tag{5.4}$$

where for $m\geq n$

$$I_m(\mathbf{i}) = \{\mathbf{j}\in I^m : \mathbf{j} = \mathbf{i}i_{n+1}\dots i_m\},$$

(for $m<n$ it is $\emptyset$), $\rho_{\mathbf{i}} = \rho_{u_\emptyset}\rho_{u_1}\dots\rho_{u_{\mathbf{i}}}$, and $\{W_{\mathbf{i}}\}_{\mathbf{i}\in I^n}$ are i.i.d. with the same distribution as W. By standard branching process results this is a measure with total mass $W\phi_k$. Thus for our critical cluster which is a 0-cell of type c (either $c_{(1)}$ or $c_{(2)}$) we have our random measure μ_ω^c. This has a self-similar decomposition as

$$\mu_\omega^c(.) = \sum_j\sum_{i=1}^4 e^{-\theta Z_i^{cj}}\mu_{\sigma_i\omega}^j(\psi_i^{-1}(.)).$$

If we write μ_ω without a superscript we will mean μ_ω^c. We will usually drop the ω and simply write μ for the measure we work with.

Our Dirichlet form will be defined on $L^2(\mathcal{C}(\omega),\mu_\omega)$. In order to show that we have a resistance form we need some preliminary lemmas.

Lemma 5.3 *There exist constants $c_1 = c_1(\omega), \lambda > 1$ such that $\mathbb{P}$-a.s.*

$$\inf\{\rho_{\mathbf{i}} : \mathbf{i} \in T_n\} \geq c_1 \lambda^n, \quad \forall n \geq 1.$$

Proof: We begin by estimating $\rho_{\mathbf{i}}^{-1}$ and consider, for $x > 1$, and $s > 0$,

$$\begin{aligned}
\mathbb{P}(\inf\{\rho_{\mathbf{i}} : \mathbf{i} \in T_n\} < x) &= \mathbb{P}(\inf_{\mathbf{i}\in T_n} \log \rho_{\mathbf{i}} < \log x) \\
&= \mathbb{P}(\sup_{\mathbf{i}\in T_n} \exp(-s \log \rho_{\mathbf{i}}) > x^{-s}) \\
&\leq \mathbb{P}(\sum_{\mathbf{i}\in T_n} \exp(-s \log \rho_{\mathbf{i}}) > x^{-s}) \\
&\leq x^s \mathbb{E}(\sum_{\mathbf{i}\in T_n} \exp(-s \log \rho_{\mathbf{i}})) \\
&\leq x^s \mathbb{E}(\sum_{i\in T_1} \exp(-s \log \rho_i))^n \\
&\leq x^s (4\tilde{p} + 4(1-\tilde{p})2^{-s})^n. \qquad (5.5)
\end{aligned}$$

We now observe that, for large enough s and setting $x = \lambda^n$ for λ close enough to 1, we have $\lambda^s (4\tilde{p} + 4(1-\tilde{p})2^{-s}) = c < 1$. Thus we can apply Borel-Cantelli to obtain the result. ∎

Remark 5.4 Note that we must have

$$\lambda < (4\tilde{p} + 4(1-\tilde{p})2^{-s})^{-1/s},$$

where s is large enough such that $\lambda^s(4\tilde{p} + 4(1-\tilde{p})2^{-s}) = c < 1$. Maximizing the bound on λ over s we see that $s = 8.6079...$ and hence $\lambda < 1.005718...$

We now define the effective resistance between points in the graph $\mathcal{C}_n$ as

$$R_n(x,y) = [\inf\{\mathcal{E}_n^{(\omega)}(f,f) : f(x) = 0, f(y) = 1\}]^{-1}, \quad \forall x, y \in \mathcal{C}_n.$$

Lemma 5.5 *If $x, y \in \mathcal{C}_n$ are connected by an edge and $x, y \in \mathcal{K}_{\mathbf{i}}$, then*

$$\frac{1}{2}\rho_{\mathbf{i}}^{-1} \leq R_n(x,y) \leq \rho_{\mathbf{i}}^{-1}, \quad \forall \mathbf{i} \in T_n, \quad n \geq 0.$$

Proof: By the definition of the resistance metric

$$\frac{1}{R_n(x,y)} = \inf\{\mathcal{E}_n^{(\omega)}(f,f) : f(x) = 0, f(y) = 1\}.$$

Now for the upper bound on $R_n(x,y)$ we just use the particular edge connecting x, y so that

$$\begin{aligned}
\mathcal{E}_n^{(\omega)}(f,f) &= \sum_{\mathbf{j}\in T_n : u_{\mathbf{j}} \in \{c^{(1)}, c^{(2)}\}} \mathcal{E}_0(f \circ \psi_{\mathbf{j}}, f \circ \psi_{\mathbf{j}}) \rho_{\mathbf{j}} \\
&\geq \mathcal{E}_0(f \circ \psi_{\mathbf{i}}, f \circ \psi_{\mathbf{i}}) \rho_{\mathbf{i}} \\
&\geq \rho_{\mathbf{i}}
\end{aligned}$$

for all $f : \mathcal{C}_n \to \mathbb{R}$ such that $f(x) = 0, f(y) = 1$.

For the lower bound we choose a particular function f. As one of either x or y must be a newly added vertex at level n, it will have only two neighbours. We assume without loss of generality that it is y, and that $f(y) = 1$. Then we set all the other vertices in $\mathcal{C}_n$ to have value 0. Thus for this f we have

$$\begin{aligned}\mathcal{E}_n^{(\omega)}(f,f) &= \sum_{\mathbf{j}\in T_n : u_{\mathbf{j}}\in\{c^{(1)},c^{(2)}\}} \mathcal{E}_0(f\circ\psi_{\mathbf{j}}, f\circ\psi_{\mathbf{j}})\rho_{\mathbf{j}} \\ &\leq 2\rho_{\mathbf{i}},\end{aligned}$$

which gives the lower bound. ∎

Lemma 5.6 *If $\chi_n = \sup_{x,y\in\mathcal{C}_n} R_n(x,y)$, then $\chi = \lim_{n\to\infty}\chi_n$ exists and has finite moments.*

Proof: We first observe that for any pair of points $x, y \in D_n$ there is a sequence of points $\{x = x_n, \dots, x_{k(x,y)}, y_{k(x,y)}, \dots, y_n = y\}$, where $x_i, y_i \in D_i$, the pairs x_i, x_{i+1} and y_i, y_{i+1} are either the same point or are connected by an edge in D_{i+1} and $k(x,y) := \sup\{l : (x,y) \in \mathcal{K}_{\mathbf{i}},\ \text{for some } \mathbf{i}\in I_l\}$. Thus, by the triangle inequality for the metric R, we have that

$$R_n(x,y) \leq \sum_{i=k(x,y)}^{n-1} R_n(x_i,x_{i+1}) + R(x_{k(x,y)}, y_{k(x,y)}) + \sum_{i=k(x,y)}^{n-1} R_n(y_{i+1},y_i).$$

It is straightforward to see that, by construction of the Dirichlet form and the definition of R, $R_n(x_i,x_{i+1}) = R_{i+1}(x_i,x_{i+1})$ and hence, by Lemma 5.5,

$$R_n(x_i,x_{i+1}) \leq c_2 \sup_{\mathbf{i}\in I^{i+1}} \rho_{\mathbf{i}}^{-1} \leq Cc_2\lambda^{-i},\ \ \mathbb{P}-a.s.$$

and hence for all n and $x, y \in D_n$,

$$R_n(x,y) \leq Cc_2 \sum_{i=k(x,y)}^{n} \lambda^{-i} \leq C' = \frac{Cc_2}{1-\lambda},\ \ \mathbb{P}-a.s.$$

Thus

$$\chi_n = \sup_{x,y} R_n(x,y) \leq C',$$

where C' is independent of n and hence we have the existence of χ almost surely.

Now we obtain moment bounds on χ. Firstly, an upper bound for χ_n is

$$\begin{aligned}\chi_n &\leq 2\sum_{m=0}^{n} \sup_{x,y\in D_m} R_n(x,y) \\ &\leq c_3 \sum_{m=0}^{n} \sup_{\mathbf{i}\in T_m} \rho_{\mathbf{i}}^{-1} \\ &\leq c_4 n \sup_{\mathbf{i}\in T_n} \rho_{\mathbf{i}}^{-1}. \qquad (5.6)\end{aligned}$$

From (5.5) we have that for large y

$$\mathbb{P}(n \sup_{\mathbf{i}\in T_n} \rho_{\mathbf{i}}^{-1} > y) \leq \xi(s)^n (c_4 n)^s y^{-s},$$

where $\xi(s) = 4\tilde{p} + 4(1-\tilde{p})2^{-s}$. Thus, for s sufficiently large such that $\xi(s) < 1$, we have

$$\begin{aligned}
\mathbb{P}(\chi > y) &\leq \mathbb{P}(\sup_n \chi_n > y) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}(\chi_n > y) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}(c_4 n \sup_{\mathbf{i} \in T_n} \rho_{\mathbf{i}}^{-1} > y) \\
&\leq c_5 y^{-s}
\end{aligned}$$

Thus, with this tail estimate, we see that $\mathbb{E}(\chi^\kappa) < \infty$ for all $\kappa < s$ but as s can be chosen arbitrarily large we have $\mathbb{E}(\chi^\kappa) < \infty$ for all κ. ∎

We can now show that the Dirichlet form is a resistance form as this is a question about controlling the asymptotic growth of the products of the resistance scale factors along each branch of the tree.

Theorem 5.7 (i) *There exists a Dirichlet form $(\mathcal{E}^{(\omega)}, \mathcal{F}^\omega)$ on $L^2(\mathcal{C}(\omega), \mu_\omega)$ for all $\omega \in \Omega$.*
(ii) *The form $(\mathcal{E}^{(\omega)}, \mathcal{F}^\omega)$ is a resistance form for $\mathbb{P}$-almost every $\omega \in \Omega$.*
(iii) *For each $\omega \in \Omega$ the Dirichlet form $(\mathcal{E}^{(\omega)}, \mathcal{F}^\omega)$ satisfies the self-similarity condition*

$$\mathcal{E}^{(\omega)}(f,g) = \sum_{i=1}^{4} \mathcal{E}^{(\sigma_i \omega)}(f \circ \psi_i, g \circ \psi_i)\rho_{u_\emptyset}, \quad \forall f, g \in \mathcal{F}^\omega.$$

Proof: (i) The fact that the limiting form is a Dirichlet form is a standard application of the techniques of [30].

(ii) In order to show that we have a resistance form, by [30] Section 2.3, under the conditions we have here all we need is to observe that $\mathbb{P}$-a.s.

$$\chi = \lim_{n \to \infty} \sup_{x,y \in D_n} R_n(x,y) < \infty.$$

(iii) The self-similarity is obtained by decomposing the set at the first level. ∎

5.2 The measure on the critical cluster

The Hausdorff dimension of the set in the resistance metric can be calculated by following the same procedure as for random recursive graph directed fractals. Following [41] the dimension is, $\mathbb{P}$-a.s. given by

$$d_f^r = \inf\{s : \Phi(s) = 1\},$$

where $\Phi(s)$ is the spectral radius of the matrix A_s defined to be

$$A_s = \begin{bmatrix} 4pq(1+2p)2^{-s} + 4p^3 & 4pq^2 2^{-s} & 4q^2 2^{-s} \\ 2q2^{-s} & 22^{-s} & 0 \\ 4q2^{-s} & 4pq2^{-s} & 4q2^{-s} \end{bmatrix}.$$

We note that this is effectively the same calculation required to obtain θ, the Malthusian parameter of our branching process and hence we have that $d_f^r = \theta$.

Note that the only difference between this and the previous estimate of the dimension in the natural length scale is the term in the element $A_s(1,1)$. A numerical calculation shows that in this case the dimension has changed dramatically to 5.2654...! This then gives us the walk dimension in the resistance metric as we must have $d_w = d_f + 1$, that is $d_w = 6.2654...$. The spectral dimension will therefore be $d_s = 2d_f^r/(d_f^r + 1) = 2\theta/(\theta + 1) = 1.6808....$

Let $B(x,r)$ denote a ball of radius r in the resistance metric at the point x. We now compute the volume growth of a resistance ball in our measure μ_ω, as defined in (5.3). It is easy to see that we can write

$$\mu_\omega^k(B(x,r)) = \lim_{n\to\infty} \sum_{j\in\mathcal{S}} \sum_{\mathbf{i}\in\mathcal{T}_n} e^{-\theta Z_{\mathbf{i}}^{kj}} \varphi_j I_{\{\pi(\mathbf{i})\in B(x,r)\}},$$

where

$$\mathcal{T}_n = \{\mathbf{i} \in T_m : \rho_{\mathbf{i}} = 2^n, \rho_{\mathbf{i}|(|\mathbf{i}|-1)} < 2^n\}.$$

We write $\mathcal{N}_{R,n}(\mathbf{i}) = \{\mathbf{j} \in \mathcal{T}_n : \mathcal{K}_{\mathbf{j}} \cap \mathcal{K}_{\mathbf{i}} \neq \emptyset\}$.

Lemma 5.8 *$\mathbb{P}$-a.s. there are constants $c_1, c_2 > 0$ such that for all $x \in \mathcal{C}$ for $r < 1$ with $2^{-n} \leq r < 2^{-n+1}$,*

$$c_1 r^\theta W_{\mathbf{i}} \leq \mu(B(x,r)) \leq c_2 r^\theta \sum_{\mathbf{j}\in\mathcal{N}_{R,n}(\mathbf{i})} W_{\mathbf{j}},$$

where $W_{\mathbf{i}}$ is the limit random variable associated with $\mathbf{i} \in \mathcal{T}_n$ as in (5.4), and $x \subset \mathcal{K}_{\mathbf{i}}$.

Proof: Let $\mathbf{i} \in \partial T$ be such that $x = \pi(\mathbf{i})$. We have that $\mathbf{i}|_m \in \mathcal{T}_n$ for some $m \geq n$, and by construction

$$\mathcal{K}_{\mathbf{i}|_m} \subset B(x, 2^{-n}) \subset \bar{\mathcal{K}}_{\mathbf{i}|_m},$$

hence

$$\rho_{\mathbf{i}|_m}^{-\theta} W_{\mathbf{i}|_m} \leq \mu(B(x,2^{-n})) \leq \sum_{\mathbf{j}\in\mathcal{N}_m(\mathbf{i}|_m)} \rho_{\mathbf{j}}^{-\theta} W_{\mathbf{j}}.$$

The fact that $\rho_{\mathbf{i}|_m} = 2^n$ and the comparison of r and n gives the result. ∎

We now use fluctuation results on the behaviour of the sequence $\{W_{\mathbf{i}|_n}; n \in \mathbb{N}\}$ to see the fluctuations in the volume at generic points. We begin with a preliminary lemma.

Lemma 5.9 *There exist constants c_1, c_2 such that*

$$P(W > x) \leq 1/x, \tag{5.7}$$

and

$$P(W < x) \leq c_1 \exp(-c_2 x^{-1/(\theta-1)}). \tag{5.8}$$

Proof: An application of Markov's inequality and that fact that $\mathbb{E}W = 1$ gives (5.7).

For (5.8) we regard our process slightly differently using the idea in Remark 5.2. Let the type $c_{(2)}$ individuals reproduce until they have died out (extinction is certain as $\mathbb{E}^{c_{(2)}} X_{c_{(2)}} < 1$). At each stage they produce a certain number of $c_{(1)}$ individuals and thus at the end of this process we have a random number of type $c_{(1)}$ individuals (in fact $3Y+1$, where Y is the total progeny of the $c_{(2)}$ branching process). As they are all type $c_{(1)}$ their location is $\log 2$ to the right of their parent. Let ζ_n denote the vector of the number of each type in this branching process after n generations. In terms of our previous branching random walk all the particles at generation n are at $n\log 2$ and hence we can view this as a standard multitype branching process. As the process of running the $c_{(2)}$ type to extinction is just a stopping line for the branching random walk, we know that the limit random variable W from the two processes will be the same. Let $\Phi_k(u) = \mathbb{E}(e^{-uW}|\zeta_0 = e_k)$, where e_k is the unit vector denoting a single individual of type k. The Laplace transform of W, satisfies the following identity,

$$\Phi_k(u) = f_k(\Phi_{c_{(1)}}(u2^{-\theta}), \Phi_{c_{(2)}}(u2^{-\theta}), \Phi_{d_{(1)}}(u2^{-\theta}), \Phi_{d_{(2)}}(u2^{-\theta})),$$

where f_k is the generating function for the offspring of an individual of type k.

In order to estimate the left tail we need to determine how slowly the process can grow. Using ideas in [34], [26] we can see that the minimal growth rate is determined by the first terms in the generating function f_k. A simple calculation shows that the minimal growth rate is at least 2. This can be seen from the factorization $f_{d_{(1)}}(u_{c_{(1)}}, u_{c_{(2)}}, u_{d_{(1)}}, u_{d_{(2)}}) = u_{d_{(1)}}^2 g_{d_{(1)}}(u_{c_{(1)}}, u_{c_{(2)}}, u_{d_{(1)}}, u_{d_{(2)}})$, where $g_{d_{(1)}}$ is a polynomial. This shows that, in the eigenvalue problem introduced by [34], the maximum eigenvalue is at least 2. Thus following [34] we have constants C, c'_k such that

$$\Phi_k(u) \le C\exp(-c'_k u^{1/\theta}), \quad u \ge 0.$$

Once we have this Laplace transform estimate it is straightforward to deduce

$$\begin{aligned} P(W < x|\zeta_0 = e_k) &\le e^{ux}\Phi_k(u) \\ &\le Ce^{ux}\exp(-c'_k u^{1/\theta}) \end{aligned}$$

and by optimizing u (i.e. by taking $u = c_* x^{-\theta/(\theta-1)}$ with $c_* > 0$ small), we have for each k

$$P(W < x|\zeta_0 = e_k) \le c_1\exp(-c_2 x^{-1/(\theta-1)}),$$

as required. ∎

Theorem 5.10 *$\mathbb{P}$-a.s. for $\epsilon > 0$ there are positive random constants c_1, c_2 such that, for μ-a.e. $x \in \mathcal{C}$, we have for $r < 1$,*

$$c_1 r^\theta |\log|\log r||^{1-\theta} \le \mu(B(x,r)) \le c_2 r^\theta |\log r|^{2+\epsilon}.$$

Proof: This is similar to the result in [22], where the fluctuations of the measure in such random recursive constructions are studied in detail in the single type case. We are still in the setting of a finite probability space, just that the number of types is more than one.

We begin with the lower bound. Given the tail estimate on W it is a simple application of the Borel-Cantelli Lemma to establish that there is a positive constant c such that

$$W_{\mathbf{i}|_n} \ge c(\log n)^{1-\theta}, \forall n \ge 0.$$

Combining this estimate with the estimates on the measure of $B(x,r)$ for $2^{-n} \leq r < 2^{-(n-1)}$ from Lemma 5.8, we have the lower estimate.

For the upper estimate we need to control the number of neighbouring cells in $\mathcal{N}_{R,n}(\mathbf{i})$ for μ-a.e. $\mathbf{i}$. This is done by observing that the number of neighbouring cells of a given cell depends on the lowest level vertex to which the cell is attached. In order for a cell at level n to be attached to a vertex at level $k < n$ it must be the case that the address of the cell has a string of $n-k$ symbols coming from either the pair $(1,2)$ or $(3,4)$. Let S_n be the length of the sequence of address labels coming from either $(1,2)$ or $(3,4)$. For a randomly chosen cell this is a Markov chain which evolves as $S_0 = 0, S_1 = 1$ and then

$$S_{n+1} = \begin{cases} S_n + 1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases}$$

It is easy to estimate the tail of S_n as $P(S_n = k) \leq 2^{-k}$ (it is 0 if $k > n$ and otherwise the event can only occur if we start at 1 and follow this by $k-1$ steps up which has probability 2^{-k}) and hence

$$P(S_n > m) \leq 2^{-m}.$$

Thus

$$P(S_n > (1+\epsilon)\frac{\log n}{\log 2}) \leq n^{-1-\epsilon},$$

and an application of Borel-Cantelli shows that

$$\limsup_{n\to\infty} \frac{S_n}{\log n} \leq \frac{1+\epsilon}{\log 2} \quad a.s.$$

and hence we can conclude that $\mathbb{P}$-a.s. there is an n_0 such that

$$S_n \leq \frac{1+\epsilon}{\log 2} \log n, \quad \forall n \geq n_0.$$

Now returning to the neighbours of the n-cell $\mathbf{i}$ we have $|\mathcal{N}_{R,n}(\mathbf{i})| \leq 22^{S_m}$, where m is the level for which the resistance metric is 2^{-n}. From the proof of Lemma 5.3 we see that almost surely there is a λ and an m_0, such that for all $m \geq m_0$,

$$\rho_{\mathbf{i}} \geq \lambda^m, \quad \forall \mathbf{i} \in T_m.$$

Thus $\rho_{\mathbf{i}} \geq 2^n$ a.s. for all $\mathbf{i} \in T_m$ where $m = [(n \log 2/\log\lambda] + 1$ must be larger than m_0. That is, almost surely for $m \geq m_0$, the level m at which all edges are of resistance at most 2^{-n} is $m \leq (\frac{\log 2}{\log \lambda} + 1)n$. Thus we deduce that for sufficiently large n

$$|\mathcal{N}_{R,n}(\mathbf{i})| \leq 22^{(1+\epsilon)\log m/\log 2} = 2m^{1+\epsilon} \leq 2(\frac{\log 2}{\log\lambda} + 1)^{1+\epsilon} n^{1+\epsilon} \leq C(\log r)^{1+\epsilon},$$

for $2^{-n} \leq r < 2^{-(n-1)}$.

Now we note that from our upper tail estimate for W that by Borel-Cantelli, we have almost surely for large enough n that

$$W_{\mathbf{i}|_n} \leq n^{1+\epsilon},$$

and combining these two bounds with the upper bound in Lemma 5.8 we have the required upper estimate on the measure. ∎

We note that, as in [22], there will be fluctuations as $r \to 0$.

However if we consider vertices $x \in \mathcal{C}_n(\omega)$ for some fixed n, then as there are lots of cells at such points the growth of the measure will be greater and, due to averaging effects, will not fluctuate. We begin by considering a sub-branching process of our branching process describing the cluster. Let $\mathcal{Z}$ denote the multitype branching process which considers only the edges with labels 1 and 2. The distribution for the random vector $N = (N_c, N_{d_{(1)}}, N_{d_{(2)}})$, the number of offspring with labels 1 and 2 of types $c, d_{(1)}, d_{(2)}$, descended from a parent of each type is given by the following. (Note that for $d_{(1)}$, we assume that the vertex that is contained in edges with label 1 and 2 is connected to the infinite cluster.)

$$\begin{aligned}
P^c(N_c = 2, N_{d_{(1)}} = 0, N_{d_{(2)}} = 0) &= p^2(2-p) \\
P^c(N_c = 1, N_{d_{(1)}} = 1, N_{d_{(2)}} = 0) &= 2p(1-p)^2 \\
P^c(N_c = 1, N_{d_{(1)}} = 0, N_{d_{(2)}} = 1) &= 2p^2(1-p)
\end{aligned}$$

$$\begin{aligned}
P^{d_{(1)}}(N_c = 2, N_{d_{(1)}} = 0, N_{d_{(2)}} = 0) &= p^2(1-p) \\
P^{d_{(1)}}(N_c = 1, N_{d_{(1)}} = 1, N_{d_{(2)}} = 0) &= 2p(1-p) \\
P^{d_{(1)}}(N_c = 0, N_{d_{(1)}} = 2, N_{d_{(2)}} = 0) &= 1-p
\end{aligned}$$

$$\begin{aligned}
P^{d_{(2)}}(N_c = 2, N_{d_{(1)}} = 0, N_{d_{(2)}} = 0) &= p^2(1-p) \\
P^{d_{(2)}}(N_c = 1, N_{d_{(1)}} = 1, N_{d_{(2)}} = 0) &= 2p(1-p)^2 \\
P^{d_{(2)}}(N_c = 1, N_{d_{(1)}} = 0, N_{d_{(2)}} = 1) &= 2p^2(1-p) \\
P^{d_{(2)}}(N_c = 0, N_{d_{(1)}} = 2, N_{d_{(2)}} = 0) &= (1-p)^3 \\
P^{d_{(2)}}(N_c = 0, N_{d_{(1)}} = 1, N_{d_{(2)}} = 1) &= 2p(1-p)^2 \\
P^{d_{(2)}}(N_c = 0, N_{d_{(1)}} = 0, N_{d_{(2)}} = 2) &= p^2(1-p).
\end{aligned}$$

As before we extend this to split the two types of c and write our type space now as $\{c_{(1)}, c_{(2)}, d_{(1)}, d_{(2)}\}$ and the distribution of the vector of offspring types as $\tilde{N} = (N_c - Y, Y, N_{d_{(1)}}, N_{d_{(2)}})$, where Y is a Binomial(N_c, p^3) random variable.

Again we can make a branching random walk which we also label $\mathcal{Z}$ by placing offspring of type $c_{(2)}$ at the same location as their parent, while all the other types have offspring at position $\log 2$.

Lemma 5.11 *The multitype branching random walk $\mathcal{Z}$ has a Malthusian parameter $\nu = 1.3384...$. Let $\tilde{N}_n^k = |\{\mathbf{i} : \mathbf{i} \in \mathcal{T}_n, u_\emptyset = k\}|$. Then there exists a constant vector c and a mean one random variable $\tilde{W}$ such that*

$$\lim_{n\to\infty} 2^{-n\nu} \tilde{N}_n = c\tilde{W}, \;\; \mathbb{P} - a.s.$$

Proof: In order to compute ν we find the mean matrix, $\tilde{A}(\xi) = [\tilde{A}_{kj}(\xi)]_{kj}$, where $\tilde{A}_{ij}(\xi) = \mathbb{E}(\sum_{i=1}^{2} \exp(-\xi Z_i^{kj}))$. A straightforward calculation, analogous to the discussion before Theorem 5.1, which simplifies by using the fact that at $p = p_c$ we have $2p - p^3 = 1$, and writing $q = 1 - p$, gives

$$\tilde{A}(\xi) = \begin{bmatrix} 4p - 2 + 2pq(1+2p)2^{-\xi} & 2pq^2 2^{-\xi} & 2q^2 2^{-\xi} \\ 2q2^{-\xi} & 2p2^{-\xi} & 0 \\ 2q2^{-\xi} & 2pq2^{-\xi} & 2q2^{-\xi} \end{bmatrix}$$

The Malthusian parameter of $\mathcal{Z}$ is then given by $\nu = \{\xi : \text{the maximum eigenvalue of } \tilde{A}(\xi) \text{ is } 1\}$. A calculation shows that $\nu = 1.3384...$ as claimed. We will write $\tilde{\varphi}$ and $\tilde{\phi}$ for the right and left eigenvectors of $\tilde{A}(\nu)$.

Now recall the stopping line $\mathcal{T}_n$ indexing the particles who lie strictly to the left of $n \log 2$ with offspring at $n \log 2$. The usual limit theorem for the multitype branching random walk gives that if

$$\tilde{W}_n = \sum_j \sum_{\mathbf{i} \in \mathcal{T}_n} e^{-\nu Z_i^{;j}} \tilde{\varphi}_j,$$

then $\tilde{W}_n \to \tilde{W}\tilde{\phi}$ as $n \to \infty$. We now note that on this stopping line all particles are at position $(n-1)\log 2$, giving the claimed result. ∎

We can now state a result about the volume of balls at the vertex 0.

Lemma 5.12 *$\mathbb{P}$-a.s. there are constants $c_1, c_2 > 0$ such that for $r < 1$,*

$$c_1 r^{\theta-\nu} W_0 \leq \mu(B(0,r)) \leq c_2 r^{\theta-\nu} W_0,$$

where $\nu = 1.3384...$ and W_0 is a limit random variable for the multitype branching process $\mathcal{Z}$.

Proof: Consider the point in the cluster $\mathcal{C}$ with label 0. If we look at the addresses which correspond to this point we see that they are any infinite sequence consisting entirely of 1 and 2. By considering the sub-branching process of the full multitype process describing the critical cluster we can determine the rate of growth of the number of such sequences. Let $\tilde{N}_n^j$ denote the number of such sequences in the resistance ball of radius 2^{-n} at 0, of type j. We will write here $\mathcal{N}_{R,n}(0) = \{\mathbf{i} \in \mathcal{T}_n : 0 \in \mathcal{K}_\mathbf{i}\}$. As we have for $2^{-n} \leq r < 2^{-n+1}$

$$\cup_{\mathbf{j} \in \mathcal{N}_{R,n}(0)} \mathcal{K}_\mathbf{j} \subset B(0,r) \subset \cup_{\mathbf{j} \in \mathcal{N}_{R,n-1}(0)} \mathcal{K}_\mathbf{j},$$

then

$$\sum_{\mathbf{j} \in \mathcal{N}_{R,n}(0)} \rho_\mathbf{j}^\theta W_\mathbf{j} \leq \mu(B(0,r)) \leq \sum_{\mathbf{j} \in \mathcal{N}_{R,n-1}(0)} \rho_\mathbf{j}^\theta W_\mathbf{j}.$$

Now considering the lower bound (the upper bound is exactly the same argument) we have

$$\begin{aligned} \mu(B(0,r)) &\geq \tilde{N}_n^c \frac{1}{\tilde{N}_n^c} \sum_{\mathbf{j} \in \mathcal{N}_{R,n}(0)} \rho_\mathbf{j}^\theta W_\mathbf{j} \\ &\geq \tilde{N}_n^c \frac{1}{\tilde{N}_n^c} \sum_{\mathbf{j} \in \mathcal{N}_{R,n}(0)} 2^{-n\theta} W_\mathbf{j} \\ &\geq c r^\theta \tilde{N}_n^c \frac{1}{\tilde{N}_n^c} \sum_{\mathbf{j} \in \mathcal{N}_{R,n}(0)} W_\mathbf{j}. \end{aligned}$$

The $W_{\mathbf{j}}$ are independent mean one random variables and independent of the process $\mathcal{Z}$. Thus, by Lemma 5.11, as $\tilde{N}_n^c \sim \tilde{W}\tilde{\phi}_c 2^{n\nu} \to \infty$ as $n \to \infty$ we can apply the strong law of large numbers and will have the result by letting $W_0 = \tilde{W}\tilde{\phi}_c$. ∎

5.3 Spectral properties

We begin by considering the scaling in the counting function. For this we follow the approach originally due to [32] and extended to the random case in [21, 15]. For now we fix $\omega \in \Omega$ and denote $(\mathcal{E}^{(\omega)}, \mathcal{F}^{(\omega)}, \mu_\omega)$ as $(\mathcal{E}, \mathcal{F}, \mu)$ and suppress the ω from our notation unless there is the possibility of confusion.

The Neumann eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu)$ are defined to be the numbers λ which satisfy

$$\mathcal{E}(u, v) = \lambda(u, v), \qquad \forall v \in \mathcal{F} \tag{5.9}$$

for some eigenfunction $u \in \mathcal{F}$. We write $(.,.)$ for the inner product on $L^2(\mathcal{C}, \mu)$.

The corresponding eigenvalue counting function, N, is obtained by setting

$$N(\lambda) := \#\{\text{eigenvalues of } (\mathcal{E}, \mathcal{F}, \mu) \leq \lambda\}, \tag{5.10}$$

To define the Dirichlet eigenvalues for $(\mathcal{E}, \mathcal{F}, \mu)$, we first introduce the related Dirichlet form $(\mathcal{E}, \mathcal{F}_D)$ by setting

$$\mathcal{F}_D := \{f \in \mathcal{F} : f|_{V^0} = 0\}.$$

The Dirichlet eigenvalues of the original form, $(\mathcal{E}, \mathcal{F}, \mu)$, are then defined to be the eigenvalues of $(\mathcal{E}, \mathcal{F}_D, \mu)$.

As we have a resistance form it is relatively straightforward, following the original arguments in [32, 21], to deduce the spectral asymptotics.

The first observation is that the Dirichlet and Neumann spectra of $(\mathcal{E}, \mathcal{F}, \mu)$ are discrete with the only accumulation point at ∞, and so the associated eigenvalue counting functions, $N_D(\lambda)$ and $N_N(\lambda)$, are well-defined and finite for all $\lambda \in \mathbb{R}$. We will label these functions by the type of the set with which it is associated; for example, $N_*^c(\lambda)$ with $* =$ D or N is the eigenvalue counting function for $(\mathcal{E}^{(\omega)}, \mathcal{F}^{(\omega)}, \mu_\omega)$ with $\omega = \{u_{\mathbf{i}}\}_{\mathbf{i} \in T}$, $u_\emptyset = c$.

Lemma 5.13 *The eigenvalue counting functions satisfy*

$$\sum_{i=1}^{4} N_D^{u_i}(\lambda \rho_{u_\emptyset}^{-1-\theta}) \leq N_D^{u_\emptyset}(\lambda) \leq N_N^{u_\emptyset}(\lambda) \leq \sum_{i=1}^{4} N_N^{u_i}(\lambda \rho_{u_\emptyset}^{-1-\theta}).$$

Also we have

$$N_D^{u_\emptyset}(\lambda) \leq N_N^{u_\emptyset}(\lambda) \leq N_D^{u_\emptyset}(\lambda) + 2.$$

Proof: We can prove this result using the decomposition and scaling of the form and the measure. This is a simple extension to the random recursive graph directed case of the random recursive set up as given in [21, 15]. ∎

We now let $X^{u_{\mathbf{i}}}(t) = N_D^{u_{\mathbf{i}}}(e^t)$ for $t \in \mathbb{R}$ and write

$$\eta^{u_{\mathbf{i}}}(t) = N_D^{u_{\mathbf{i}}}(e^t) - \sum_{j=1}^{4} N_D^{u_{\mathbf{i}j}}(e^t \rho_{u_{\mathbf{i}}}^{-1-\theta}).$$

Thus we have a random multitype renewal equation

$$X^{u_\emptyset}(t) = \eta^{u_\emptyset}(t) + \sum_{j=1}^{4} X^{u_j}(t - (1+\theta)\log \rho_{u_\emptyset}),$$

and, by iterating, we can write

$$X^{u_\emptyset}(t) = \sum_{\mathbf{i} \in T} \eta^{u_{\mathbf{i}}}(t - (1+\theta)\log(\rho_{u_\emptyset} \dots \rho_{u_{\mathbf{i}}})). \tag{5.11}$$

We now set $m^{u_\emptyset}(t) = e^{-\gamma t} E X^{u_\emptyset}(t)$ and $h^{u_\emptyset}(t) = e^{-\gamma t} E \eta^{u_\emptyset}(t)$. Thus

$$\begin{aligned} m^{u_\emptyset}(t) &= h^{u_\emptyset}(t) + \sum_{j=1}^{4} e^{-\gamma t} E X^{u_j}(t - (1+\theta)\log \rho_{u_\emptyset}), \\ &= h^{u_\emptyset}(t) + \sum_{j=1}^{4} E e^{-\gamma(1+\theta)\log \rho_{u_\emptyset}} e^{-\gamma(t-(1+\theta)\log \rho_{u_\emptyset})} E X^{u_j}(t - (1+\theta)\log \rho_{u_\emptyset}), \\ &= h^{u_\emptyset}(t) + \sum_{j=1}^{4} \int_0^\infty m^{u_j}(t-s) \nu_\gamma^{u_\emptyset u_j}(ds), \end{aligned}$$

where ν_γ is a matrix of measures with

$$\nu_\gamma^{u_\emptyset u_j}(ds) = E \rho_{u_\emptyset}^{-\gamma(1+\theta)} \delta_{(1+\theta)\log \rho_{u_\emptyset}}(ds).$$

We choose γ to ensure that the maximum eigenvalue of the matrix of distribution functions is 1. A simple computation shows that, as θ is the value that makes the original matrix have eigenvalue 1, we just need $\gamma(1+\theta) = \theta$ and hence $\gamma = \theta/(\theta+1)$.

Thus we have a matrix renewal equation which we can write as

$$m(t) = h(t) + m * \nu(t),$$

where we denote the operation of convolution of a function $a : \mathbb{R} \to \mathbb{R}$ with a measure b by

$$b * a(t) = a * b(t) = \sum_j \int_0^\infty a(t-s) b(ds),$$

and hence for two matrices A, B of measures we write the ij-th element of $C(t) = A * B(t)$ as $c_{ij}(t) = \sum_k a_{ik} * b_{kj}(t)$.

The next step is to apply an appropriate matrix renewal theorem, for which we need to extend slightly those due to [37] and [24]. Let $M = [m_{ij}]$ be a matrix of Radon measures on $\mathbb{R}_+$. We will write F for the matrix of distribution functions of M, that is $F_{ij}(t) = \int_0^t m_{ij}(ds)$ and we will write $F_{ij}(t, t+h] = F_{ij}(t+h) - F_{ij}(t)$. The indices of the matrix will be referred to as states and are the vertices of a graph G. The graph has a directed edge between state i and j if the measure m_{ij} is non-zero. Let $\gamma(i,j)$ denote a directed path in the graph G from vertex i to vertex j. We define the measure $m_{\gamma(i,j)}$ by taking the convolution of the measures associated with each given edge in the path. We will also write $m_{\hat{i}i}$ for the i-th column of the matrix M with the i-th element removed, similarly, $m_{i\hat{i}}$ for the i-th row of M with the i-th element removed. Finally we write the matrix of measures with both the i-th row and column of M removed as M_{ii}.

We follow [37] and define the measure

$$\nu_1 = m_{11} + m_{1\hat{1}} * \sum_{k=0}^{\infty} (M_{11})^{*k} * m_{\hat{1}1}. \tag{5.12}$$

It is not difficult to check that, if $F(\infty)$ has maximum eigenvalue 1 and is irreducible, this is a probability measure with support given by $\cup\{\text{supp}(m_\gamma) : \gamma \text{ is a simple cycle in } G\}$. If the support is contained in a discrete subgroup of $\mathbb{R}$ we will call this measure lattice. By the irreducibility we see that if ν_1 is lattice, then ν_i is lattice for all i.

We state the lattice case of the renewal theorem (Theorem 4.2 of [37]) for the case of irreducible $F(\infty)$.

Theorem 5.14 *We assume that $F(t)$ is a matrix of measures in which $F(\infty)$ is irreducible, has maximum eigenvalue 1, $F_{ij}(0-) = 0, \int_0^\infty t dF_{ij}(t) < \infty$ for all i, j and for each j there is at least one i such that $F_{ij}(0) < F_{ij}(\infty)$. Let $V(t) = \sum_{k=0}^\infty F^{*k}(t)$ denote the matrix renewal measure. If ν_1 is lattice, with period T, then*

$$\lim_{t\to\infty} [V_{ij}(t + \tau_{ij} + T) - V_{ij}(t + \tau_{ij})] = AT,$$

for any $\tau_{ij} \in supp(m_{\gamma(i,j)})$, where

$$A = \frac{\mathbf{u}^T \mathbf{v}}{\mathbf{v}\mathcal{M}\mathbf{u}},$$

and $\mathbf{u}, \mathbf{v}$ are the unique normalized right and left 1-eigenvectors of $F(\infty)$ and $\mathcal{M}$ is the matrix of first moments of F.

We also state a result concerning the asymptotic behaviour of the solution to the renewal equation on $\mathbb{R}$.

Theorem 5.15 *Let $\mathbf{z}(t)$ satisfy the estimate, that there exist positive finite constants C, σ such that*

$$|z_i(t)| \le Ce^{-\sigma|t|}, \quad \forall t \in \mathbb{R}, \forall i.$$

Let F be a matrix of measures satisfying the assumptions of Theorem 5.14, then the renewal equation

$$\mathbf{r}(t) = \mathbf{z}(t) + \mathbf{r} * F(t), \tag{5.13}$$

has a unique solution, bounded on finite intervals with the property that $\mathbf{r}(t) \to 0$ as $t \to -\infty$. If ν_1 is lattice with period T, then

$$\tilde{\mathbf{r}}(t) = \lim_{n\to\infty} [r_i(t + \tau_{1i} + nT)] = \sum_{l=-\infty}^{\infty} \mathbf{z}(t + lT)A$$

exists for every $t \in [0, T]$.

Proof: This is a simple extension of the renewal theorem of [38] to the multidimensional setting coupled with the use of Theorem 5.14 as in [37]. ∎

Recall that for each type $k \in \mathcal{S}$, $m^k(t) = e^{-\gamma t} E X^k(t)$ and $h^k(t) = e^{-\gamma t} E\eta^k(t)$.

Lemma 5.16 (i) *The functions* m^k *are bounded, measurable and satisfy* $m^k(t) \to 0$ *as* $t \to -\infty$.
(ii) *There exist constants* C, σ *such that the function* h *satisfies*

$$|h^i(t)| \leq Ce^{-\sigma|t|}, \quad \forall t \in \mathbb{R}, \forall i.$$

Proof: (i) For all $x, y \in \mathcal{C}$ we have, by the construction of the resistance

$$|f(x) - f(y)| \leq R(x,y)\mathcal{E}(f,f), \quad \forall x, y \in \mathcal{C}.$$

This yields the estimate

$$\|f\|_2^2 \leq \sup_{x,y \in \mathcal{C}} R(x,y)\mathcal{E}(f,f), \forall f \in \mathcal{F}_D,$$

and hence we have a lower bound of $1/\chi$ for the Dirichlet spectrum, where χ is defined in Lemma 5.6.

As $\eta^i(t) = 0$ for $t < -\log\chi$ we have

$$\mathbb{E}\eta^i(t) \leq 8\mathbb{P}(-\log\chi \leq t). \tag{5.14}$$

Thus we can estimate

$$\begin{aligned} m^k(t) &\leq \sum_{\mathbf{i} \in T} e^{-\gamma t} 8\mathbb{P}(-\log\chi \leq t - (1+\theta)\log\rho_{\mathbf{i}}) \\ &= 8e^{-\gamma t}\mathbb{E}(\#\{\mathbf{i} : t - (1+\theta)\log\rho_{\mathbf{i}} \geq -\log\chi\}) \end{aligned}$$

Thus we need to estimate this last expectation. By the rate of growth of the branching process we have a c such that

$$\mathbb{E}(\#\{\mathbf{i} : t \geq \log\rho_{\mathbf{i}}\}) \leq ce^{\theta t},$$

and hence

$$\mathbb{E}(\#\{\mathbf{i} : t - (1+\theta)\log\rho_{\mathbf{i}} \geq -\log\chi\}) \leq ce^{\gamma t}\mathbb{E}\chi^\gamma.$$

Putting these together we have

$$m^k(t) \leq 8e^{-\gamma t}ce^{\gamma t}\mathbb{E}\chi^\gamma,$$

which is finite by the moment estimates on χ. Thus m^k is bounded. The measurability is clear. Finally for the behaviour as $t \to -\infty$ we observe that for a $\delta > 0$, by Markov's inequality and conditional independence,

$$\begin{aligned} m^k(t) &\leq \sum_{\mathbf{i} \in T} 8e^{-\gamma t}\mathbb{P}(\rho_{\mathbf{i}}^{-(1+\theta)}\chi \geq e^{-t}) \\ &\leq 8e^{(\delta-\gamma)t}\sum_{\mathbf{i} \in T}\mathbb{E}(\rho_{\mathbf{i}}^{-\delta(1+\theta)})\mathbb{E}(\chi^\delta) \\ &\leq Ce^{(\delta-\gamma)t}\mathbb{E}(\chi^\delta)\sum_n e^{n\theta}c(\delta)^n, \end{aligned}$$

where $c(\delta) = \mathbb{E}(\rho_{u_\emptyset}^{-\delta(1+\theta)})$. The sum can be made finite by choice of $\delta > \gamma$. Observe that if $\delta > \gamma$, then $\delta(1+\theta) > \theta$. Hence $\sum_{\mathbf{i}\in T_n} \rho_{\mathbf{i}}^{-\delta(1+\theta)}$ is a supermartingale. As it decays exponentially, the sum over all n will converge. Hence we have $m^k(t) \to 0$ as $t \to -\infty$ for each type k.

(ii) By construction $h^i(t) = e^{-\gamma t}\mathbb{E}\eta^i(t)$. Thus as $\eta^i(t) \leq 8$ we see that, for $t > 0$, we have $|h^i(t)| \leq 8e^{-\gamma t}$. For $t < 0$ we use (5.14) and the tail estimate for χ to see that

$$|h^i(t)| \leq e^{-\gamma t} c e^{\kappa t}.$$

Thus as $\kappa > \gamma$ we can take $\sigma = \min(\gamma, \kappa - \gamma)$ to obtain the result. ∎

Theorem 5.17 *For each $t \in [0, (1+\theta)\log 2)$, there exists $m_\infty^{u_\emptyset}(t)$ such that*

$$\lim_{n\to\infty} m^{u_\emptyset}(t + n(1+\theta)\log 2) = m_\infty^{u_\emptyset}(t).$$

Proof: We can write our measures as

$$\nu_\gamma^{u_\emptyset u_i}(ds) = (1-\tilde{p})2^{-\gamma(1+\theta)}\delta_{(1+\theta)\log 2} + \tilde{p}\delta_0,$$

where $\tilde{p} = 0$ if $u_\emptyset \neq c$ while $\tilde{p} = p_c^3$ if $u_\emptyset = c$. If we consider the measure ν_1 as defined above it will clearly be lattice as all points are located at multiples of $(1+\theta)\log 2$. By Lemma 5.16 the conditions of the matrix renewal theorem are satisfied and the result is now a direct application of Theorem 5.15. ∎

The next step is to consider the random process itself and we prove the following almost sure limit theorem.

Theorem 5.18 *For each $\lambda \in [1, 2^{(1+\theta)})$, we have*

$$\lim_{n\to\infty} \left| \frac{N_D^{u_\emptyset}(\lambda 2^{(1+\theta)n})}{(\lambda 2^{(1+\theta)n})^{\theta/(\theta+1)}} - m_\infty^{u_\emptyset}(\log\lambda)W \right| = 0, \quad a.s.$$

Proof: This is a multidimensional version of similar results proved in [18], [21] and [15]. The study of the limiting behaviour of N_D can be viewed as the study of a multitype branching process counted with random characteristic. We note that as we are in the lattice case here we could extend the work of [18] to the multitype case but, as we can remove the need for a branching random walk by evolving the type $c_{(2)}$ individuals, we treat the problem directly.

We now fix a $t \in [0, (1+\theta)\log 2)$ and consider the lattice $t_n = t + n(1+\theta)\log 2$. We work with the multitype branching process $\{\tilde{Z}_n = (\tilde{Z}_n^{c_{(1)}}, \tilde{Z}_n^{d_{(1)}}, \tilde{Z}_n^{d_{(2)}})\}$ of three types in which the types $\bar{\mathcal{S}} = \{c_{(1)}, d_{(1)}, d_{(2)}\}$ evolve as before but if they have a type $c_{(2)}$ offspring, then this is evolved forward until it has only type $c_{(1)}$ offspring; thus the number of type $c_{(1)}$ offspring is determined by the total population size in the type $c_{(2)}$ process evolved to extinction along with those arising from the other types. We note that this is the original tree looked at with the stopping lines $\tilde{\mathcal{T}}_n$, where

$$\tilde{\mathcal{T}}_n = \{\mathbf{i} \in T_m : \rho_{\mathbf{i}} = 2^n, u_{\mathbf{i}} \neq c_{(2)}\}.$$

We write $\tilde{\mathcal{T}}_n(\mathbf{i})$ for the n-th generation in the tree descended from individual $\mathbf{i}$. We also write $\tilde{\mathcal{T}} = \cup_n \tilde{\mathcal{T}}_n$ for the whole tree and $\tilde{\mathcal{T}}(\mathbf{i})$ for the tree started from individual $\mathbf{i}$. We will be interested in the process $X_n^{u_\emptyset} = X^{u_\emptyset}(t_n)$ which, by (5.11) in this setting, we can express as

$$X_n^{u_\emptyset} = \sum_{\mathbf{i}\in\tilde{\mathcal{T}}} \bar{\eta}^{u_\emptyset}(n - |\mathbf{i}|),$$

where $\bar{\eta}^{u_\mathbf{i}}(t) = \sum_{\mathbf{j}:T_1^2} \eta^{\mathbf{j}}(t - (1+\theta)\log 2)$, (using the notation T_1^2 for the branches of T which have labels $c_{(2)}$ until the first $c_{(1)}$) is the sum of the original characteristic $\eta^{u_\emptyset}(t)$ and all the characteristics $\eta^{\mathbf{j}}(t)$ that arise from the type $c_{(2)}$ descendents which have been removed.

Firstly we truncate the functions $\bar{\eta}^{u_\mathbf{i}}$ by setting

$$\bar{\eta}^{u_\mathbf{i},n_0}(t) := \bar{\eta}^{u_\mathbf{i}}(t) I_{\{t<n_0\}},$$

where n_0 is to be chosen later. We can then construct the truncated process

$$X_n^{u_\mathbf{i},n_0} = \sum_{\mathbf{j}\in\tilde{\mathcal{T}}(\mathbf{i})} \bar{\eta}^{u_{\mathbf{ij}},n_0}(n - |\mathbf{j}|).$$

Let $m_n^{u_\mathbf{i},n_0} = e^{-\gamma n}\mathbb{E}X_n^{u_\mathbf{i},n_0}$, which converges to $m_\infty^{u_\mathbf{i},n_0}$ as $n\to\infty$ using the same arguments as in the proof of Theorem 5.17.

Using the recursive formulation and the truncation we see that

$$X_{n+n_1}^{u_\emptyset,n_0} = \sum_{\mathbf{i}\in\tilde{\mathcal{T}}_n} X_{n_1}^{u_\mathbf{i},n_0}, \tag{5.15}$$

for $n_1 > n_0$. We can take expectations through this and multiply by $e^{-\gamma(n+n_1)}$ to see $m_{n+n_1}^{u_\emptyset,n_0} = e^{-\gamma n} E\left(\sum_{\mathbf{i}\in\tilde{\mathcal{T}}_n} m_{n_1}^{u_\mathbf{i},n_0}\right)$. By letting $n_1\to\infty$ and using the convergence result Theorem 5.17 we have

$$m_\infty^{u_\emptyset,n_0} = e^{-\gamma n} E\left(\sum_{\mathbf{i}\in\tilde{\mathcal{T}}_n} m_\infty^{u_\mathbf{i},n_0}\right). \tag{5.16}$$

Using the decomposition (5.15) we have

$$|e^{-\gamma(n+n_1)} X_{n+n_1}^{u_\emptyset,n_0} - m_\infty^{u_\emptyset,n_0} W| \le S_1 + S_2, \tag{5.17}$$

where

$$S_1 = \left| e^{-\gamma n} \sum_{\mathbf{i}\in\tilde{\mathcal{T}}_n} \left(e^{-\gamma n_1} X_{n_1}^{u_\mathbf{i},n_0} - m_{n_1}^{u_\mathbf{i},n_0}\right) \right| \quad \text{and} \quad S_2 = \left| e^{-\gamma n} \sum_{\mathbf{i}\in\tilde{\mathcal{T}}_n} m_{n_1}^{u_\mathbf{i},n_0} - m_\infty^{u_\emptyset,n_0} W \right|.$$

We write $z_n = \tilde{Z}_n^{c_{(1)}} + \tilde{Z}_n^{d_{(1)}} + \tilde{Z}_n^{d_{(2)}}$ for the total population size. By standard multitype branching theory, for each $k\in\bar{\mathcal{S}}$,

$$e^{-\gamma n}\tilde{Z}_n^k \to c_k W, \quad a.s. \tag{5.18}$$

as $n \to \infty$. Observing that $A^{u_{\mathbf{i}}}_{n_1}(\mathbf{i}) = e^{-\gamma n_1} X^{u_{\mathbf{i}}, n_0}_{n_1} - m^{u_{\mathbf{i}}, n_0}_{n_1}$ are mean 0 random variables we have

$$S_1 \le |e^{-\gamma n} z_n \frac{1}{z_n} \sum_{\mathbf{i} \in \tilde{\mathcal{T}}_n} A^{u_{\mathbf{i}}}_{n_1}(\mathbf{i})|.$$

Applying the strong law of large numbers, as the numbers of each type $\tilde{Z}^k_n, k \in \bar{\mathcal{S}}$ grow exponentially,

$$\frac{1}{z_n} \sum_{\mathbf{i} \in \tilde{\mathcal{T}}_n} A_{\mathbf{i}} = \sum_{k \in \bar{\mathcal{S}}} \frac{\tilde{Z}^k_n}{z_n} \frac{1}{\tilde{Z}^k_n} \sum_{j=1}^{\tilde{Z}^k_n} A^k_{n_1}(j) \to 0$$

as $n \to \infty$. Combining this with the convergence result (5.18) we have $S_1 \to 0$ a.s. as $n \to \infty$.

For the second term we write, using (5.16),

$$\begin{aligned} S_2 &\le \left| e^{-\gamma n} \left(\sum_{\mathbf{i} \in \tilde{\mathcal{T}}_n} \left(m^{u_{\mathbf{i}}, n_0}_{n_1} - m^{u_{\mathbf{i}}, n_0}_{\infty} \right) \right) \right| + \left| e^{-\gamma n} \left(\sum_{\mathbf{i} \in \tilde{\mathcal{T}}_n} m^{u_{\mathbf{i}}, n_0}_{\infty} - E \left(\sum_{\mathbf{i} \in \tilde{\mathcal{T}}_n} m^{u_{\mathbf{i}}, n_0}_{\infty} \right) W \right) \right| \\ &\le e^{-\gamma n} \sum_{k \in \bar{\mathcal{S}}} \sum_{j=1}^{\tilde{Z}^k_n} \left| m^{k, n_0}_{n_1} - m^{k, n_0}_{\infty} \right| + e^{-\gamma n} \sum_{k \in \bar{\mathcal{S}}} \left| \left(\sum_{j=1}^{\tilde{Z}^k_n} m^{k, n_0}_{\infty} - E \sum_{j=1}^{\tilde{Z}^k_n} m^{k, n_0}_{\infty} W \right) \right| \\ &\le e^{-\gamma n} \sum_{k \in \bar{\mathcal{S}}} \tilde{Z}^k_n \left| m^{k, n_0}_{n_1} - m^{k, n_0}_{\infty} \right| + \sum_{k \in \bar{\mathcal{S}}} m^{k, n_0}_{\infty} \left| e^{-\gamma n} \tilde{Z}^k_n - e^{-\gamma n} E(\tilde{Z}^k_n) W \right|. \end{aligned}$$

Now, by (5.18), the boundedness of $m^{u_\emptyset, n_0}_{\infty}$ and the convergence as $n_1 \to \infty$ of $m^{k, n_0}_{n_1} \to m^{k, n_0}_{\infty}$ for each $k \in \bar{\mathcal{S}}$, for any $\epsilon > 0$, there is a random constant C such that $S_2 \le C\epsilon$. Thus we have

$$e^{-\gamma n} X^{u_\emptyset, n_0}_n \to m^{u_\emptyset, n_0}_{\infty} W, \quad a.s.$$

In order to remove the truncation we write

$$|e^{-\gamma n} X^{u_\emptyset}_n \to m^{u_\emptyset}_{\infty} W| \le e^{-\gamma n} |X^{u_\emptyset}_n - X^{u_\emptyset, n_0}_n| + |e^{-\gamma n} X^{u_\emptyset, n_0}_n - m^{u_\emptyset, n_0}_{\infty} W| + W |m^{u_\emptyset, n_0}_{\infty} - m^{u_\emptyset}_{\infty}|.$$

We note that $m^{u_\emptyset, n_0}_{\infty}$ is increasing in n_0 and bounded above, hence the last term converges to 0. We have established the second term above converges to 0, so all that remains is to show that

$$e^{-\gamma n} |X^{u_\emptyset}_n - X^{u_\emptyset, n_0}_n| \to 0.$$

In order to see this

$$\begin{aligned} |X^{u_\emptyset}_n - X^{u_\emptyset, n_0}_n| &\le \sum_{\mathbf{i} \in \tilde{\mathcal{T}}} |\bar{\eta}^{u_\emptyset}(n - |\mathbf{i}|)| I_{n - |\mathbf{i}| > n_0} \\ &\le \sum_{\mathbf{i} \in T} \eta^{u_\emptyset}(n - |\mathbf{i}|) I_{\log \rho_{\mathbf{i}} < (n - n_0) \log 2} \\ &\le 8 \# \{ \mathbf{i} \in T : \rho_{\mathbf{i}} < 2^{(n - n_0)} \}. \end{aligned}$$

As the branching process is supercritical with exponential growth, the total population size ever born is controlled by the number currently alive. It is straightforward to show that there is a constant C such that

$$\# \{ \mathbf{i} \in T : \rho_{\mathbf{i}} < 2^{(n - n_0)} \} \le C z_{n - n_0}.$$

Hence we have

$$e^{-\gamma n}|X_n^{u_\emptyset} - X_n^{u_\emptyset,n_0}| \leq e^{-\gamma n} 8Cz_{n-n_0} \leq C'e^{-\gamma n_0}.$$

Hence by taking n_0 large we can make this arbitrarily small completing the proof that

$$e^{-\gamma n}X_n^{u_\emptyset} \to m_\infty^{u_\emptyset} W, \ \ a.s.$$

for each $t \in [0,(1+\theta)\log 2)$ as $n \to \infty$.

Finally we have the result by rewriting in terms of λ. ∎

5.4 heat kernel estimates

As the form is a resistance form all we need to do here is to use our volume estimates in order to apply the results of Croydon [13] which give heat kernel estimates once volume estimates are known in the setting of resistance forms. We remark that the existence and joint continuity of the heat kernel are standard, see [13] Proposition 5.

We now state our estimates for the heat kernel. Let $\theta_\epsilon = 2(2\theta+3)(\theta+2)+\epsilon$

Theorem 5.19 *There is a positive constant $c_1(\omega)$ and for $\epsilon > 0$ positive constants $c_2(\omega), c_3(\omega)$ such that for $\mathbb{P}$-a.e. ω for $\mu_\omega - a.e. x \in \mathcal{C}$ and for $t < 1$,*

$$c_2|\log t|^{-\theta_\epsilon} t^{-\theta/(\theta+1)} \leq q_t^\omega(x,x)$$

and

$$0 < q_t^\omega(x,y) \leq c_1 t^{-\theta/(\theta+1)}|\log|\log t||^{(\theta-1)/(\theta+1)} \exp\left(-c_3\left(\frac{R(x,y)^{\theta+1}}{t}\right)^{1/\theta}\left|\log(\frac{t}{R(x,y)})\right|^{-\theta_\epsilon/\theta}\right).$$

Proof: We just use the volume estimate from Theorem 5.10 and [13] Theorem 1. We observe that our volume estimates are exactly of the form considered in [13] where $f_l(r)V(r) \leq V(x,r) \leq f_u(r)V(r)$. We take $V(r) = r^\theta$ and the oscillation terms are $f_l(r) = \log|\log r|^{1-\theta}$ and $f_u(r) = |\log r|^{2+\epsilon}$. The result is then that there are constants such that

$$c\left(\frac{f_l(h^{-1}(t))}{f_u(h^{-1}(t))}\right)^{c'} \frac{h^{-1}(t)}{t} \leq p_t(x,x) \leq C\frac{h_l^{-1}(t)}{t}$$

where $h_l(r) = rV(r)f_l(r)$ and the constant $c' > (2\theta+3)(\theta+2)$. Thus, as $h(r) = r^{\theta+1}$, the main term in the heat kernel is $h^{-1}(t)/t = t^{-\theta/(\theta+1)}$. The lower correction term is

$$\begin{aligned} c\left(\frac{f_l(h^{-1}(t))}{f_u(h^{-1}(t))}\right)^{c'} &\geq c\left(\frac{1}{|\log|\log t||^{\theta-1}|\log t|^{2+\epsilon}}\right)^{c'} \\ &\geq c|\log t|^{-2(2\theta+3)(\theta+2)-\epsilon}. \end{aligned}$$

For the upper correction we have

$$\frac{h_l^{-1}(t)}{t} \leq ct^{-\theta/(\theta+1)}|\log|\log t||^{(\theta-1)/(\theta+1)}$$

giving the result. ∎

We remark that the size of the exponent θ_ϵ that appears in the correction term for the lower bound and off diagonal upper bound is at least 196.534! It would be interesting to know what the size of the fluctuations actually is.

Similarly at the point 0 we can use the same approach with the volume estimate of Lemma 5.12 to obtain the following.

Theorem 5.20 *For $\mathbb{P}$-a.e. ω and for $t < 1$ there are constants $c_1(\omega), c_2(\omega)$ such that*

$$c_1 t^{-(\theta-\nu)/(\theta-\nu+1)} \leq q_t^\omega(0,0) \leq c_2 t^{-(\theta-\nu)/(\theta-\nu+1)}.$$

Note that this result will hold for any of the vertices in the approximating sequence V_n. This means that although the heat kernel fluctuates at μ-almost every point, there is a countable dense set of points in the cluster (of μ measure 0) where there are no fluctuations in the heat kernel.

The statement of (i) of Theorem 1.2 for the critical cluster is just the on-diagonal version of Theorem 5.19. While the second part (ii) of Theorem 1.2 for the critical cluster at 0, is directly Theorem 5.20.

6 Open Problems

There are a number of questions which arise naturally.

(i) In our construction we chose the weights on the edges to ensure that the total resistance across the cluster was 1. It would be natural to consider the problem where we use the same fixed resistance weight for each edge in the previous construction of a Dirichlet form for the scaling limit. We would like to think of this in the graph setting where we have a random walk moving on the graph with unit resistors for each edge. In order to understand this problem we need to consider a random hierarchical system [23]. We can view this as either for the conductance or the resistance. Using the self-similarity and independence it is clear that if R_n denotes the resistance between 0 and 1, then

$$R_n = \begin{cases} R_{n-1}(1) + R_{n-1}(2) & \text{if a single series connection,} \\ \frac{1}{\frac{1}{R_{n-1}(1)+R_{n-1}(2)} + \frac{1}{R_{n-1}(3)+R_{n-1}(4)}} & \text{if there are two series connections in parallel.} \end{cases}$$

Alternatively we can write this in terms of conductances as

$$C_n = \begin{cases} \frac{1}{\frac{1}{C_{n-1}(1)} + \frac{1}{C_{n-1}(2)}} & \text{if a single series connection,} \\ \frac{1}{\frac{1}{C_{n-1}(1)} + \frac{1}{C_{n-1}(2)}} + \frac{1}{\frac{1}{C_{n-1}(3)} + \frac{1}{C_{n-1}(4)}} & \text{if there are two series connections in parallel.} \end{cases}$$

As a first question we would like to know if there exists a λ such that $(\log R_n)/n \to \lambda$ as $n \to \infty$. If there is exponential growth, is there a limit distribution such that $R_n \lambda^n \to c$ as $n \to \infty$.

(ii) The general case of n pairs of edges in parallel for $n \geq 2$, extending our discussion from the case of $n = 2$, presents a challenge. It is clear that there will be different behaviour as the dimension of the diamond lattice is $\log(2n)/\log(2)$ and we will be in the transient case for all

$n > 2$. The percolation question can be answered by solving a suitable fixed point equation and we will see that there is a unique p_c in $(0, 1)$ at which there is percolation. When considering the infinite critical percolation cluster the techniques we applied in the case $n = 2$ revolve around the fact that we have a resistance form and can use the resistance metric. We believe that there will not be a resistance form for larger values of n and hence new techniques for establishing existence of the Dirichlet form and heat kernel estimates in the random transient case will be needed.

A simpler alternative is to consider different probability measures on the cluster generating configurations, for instance the random cluster measure, which may produce resistance forms for larger values of n. We note that if $n = 3$ has a resistance form, then a heuristic spectral analysis would suggest a smoother limit result for the high frequency eigenvalues in that the normalized limit of the eigenvalue counting function would exist.

(iii) The study of random walks on the graphs would be of interest. In particular the case where we use a random sequence of blow ups of the set in order to remove the existence of points of infinite degree in the graphs. For the spectral properties of such random blow ups in the finitely ramified fractal case see [42]. This will give nicer graphs but it is not clear what to do about the percolation problem in this setting as an incipient infinite cluster construction may be required in order to ensure that the infinite cluster contains our initial edge.

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