HEAT KERNEL ESTIMATES FOR TIME FRACTIONAL EQUATIONS

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Abstract. In this paper, we establish existence and uniqueness of weak solutions to general time fractional equations and give their probabilistic representations. We then derive sharp two-sided estimates for fundamental solutions of general time fractional equations in metric measure spaces.

Keywords: Dirichlet form; subordinator; Caputo derivative; heat kernel estimates; time fractional equation

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1. Introduction

1.1. **Motivation.** Let (M, d) be a locally compact separable metric space, and μ be a Radon measure on M with full support. Suppose that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M; \mu)$ and $X = \{X_t, t \geqslant 0; \mathbb{P}^x, x \in M \setminus \mathcal{N}\}$ is its associated Hunt process. Here $\mathcal{N}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$ in the sense that $\mu(\mathcal{N}) = 0$ and

$$\mathbb{P}^x(X_t \in M \setminus \mathcal{N} \text{ and } X_{t-} \in M \setminus \mathcal{N} \text{ for every } t > 0) = 1 \quad \text{for all } x \in M \setminus \mathcal{N}.$$

Throughout this paper, we assume that $\mathcal{N} = \emptyset$; otherwise, we can use $M \setminus \mathcal{N}$ in place of M. Denote by $\{T_t : t \geqslant 0\}$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the strongly continuous contraction semigroup and the infinitesimal generator associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M; \mu)$, respectively. Let $S = \{S_t : t \geqslant 0\}$ be a subordinator (that is, a non-decreasing real valued Lévy process with $S_0 = 0$) without drift and having the Laplace exponent ϕ:

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)} \quad \text{for all } t, \lambda > 0.$$

It is well known (see, e.g., [20]) that there exists a unique Borel measure ν on $(0, \infty)$ with $\int_0^\infty (1 \wedge s)\, \nu(ds) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s})\, \nu(ds). \tag{1.1}$$

The measure ν is called the Lévy measure of the subordinator S. Define for $t > 0$, $E_t = \inf\{s > 0 : S_s > t\}$, the inverse subordinator. We assume that S is independent of X and that $\nu(0, \infty) = \infty$, excluding compound Poisson processes. Thus, almost surely, the function $t \mapsto S_t$ is strictly increasing, and hence $t \mapsto E_t$ is continuous.

Recently, it is established in [5, Theorem 2.1] that for any $f \in \mathcal{D}(\mathcal{L}) \subset L^2(M; \mu)$,

$$u(t, x) := \mathbb{E}\left[T_{E_t} f(x)\right] = \mathbb{E}^x\left[f(X_{E_t})\right]$$

is the unique strong solution (in some suitable sense) to the equation

$$\partial_t^w u(t, x) = \mathcal{L}u(t, x) \quad \text{with } u(0, x) = f(x), \tag{1.2}$$

where $w(s) = \nu(s,\infty)$ for $s > 0$, and ∂_t^w is the fractional derivative defined as follows: for a function $\psi : [0,\infty) \to \mathbb{R}$,

$$\partial_t^w \psi(t) := \frac{d}{dt}\int_0^t w(t-s)(\psi(s)-\psi(0))\,ds. \tag{1.3}$$

See Theorem 2.1 in Section 2 for a precise statement. If the semigroup $\{T_t : t \geqslant 0\}$ (or equivalently, the Hunt process X) has a heat kernel $q(t,x,y)$ with respect to the measure μ, then by Fubini's theorem, for any bounded function $f \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned} u(t,x) = \mathbb{E}\left[T_{E_t} f(x)\right] &= \int_0^\infty T_r f(x)\, d_r \mathbb{P}(E_t \leqslant r) = \int_0^\infty T_r f(x)\, d_r \mathbb{P}(S_r \geqslant t) \qquad (1.4)\\ &= \int_0^\infty \int_M f(y) q(r,x,y)\,\mu(dy)\, d_r \mathbb{P}(S_r \geqslant t)\\ &= \int_M f(y)\left(\int_0^\infty q(r,x,y)\, d_r\mathbb{P}(S_r \geqslant t)\right)\mu(dy). \end{aligned}$$

Here and in what follows, d_r denotes the (generalized) derivative with respect to r. This says that

$$p(t,x,y) := \int_0^\infty q(r,x,y)\, d_r \mathbb{P}(S_r \geqslant t) \tag{1.5}$$

is the "fundamental solution" to the time fractional equation (1.2). Note that in the PDE literatures, the most standard approach to analyze $p(t,x,y)$ is to use the Mittag-Leffler function, and then take the inverse Fourier transform (see for instance [8], about detailed estimates of $p(t,x,y)$ when $\{S_t\}$ is a β-stable subordinator). We emphasize that the expression (1.5) is more intuitive, simple, and general (in the sense that we do not rely on the Fourier transform).

When $S = \{S_t : t \geqslant 0\}$ is a β-stable subordinator with the Laplace exponent $\phi(\lambda) = \lambda^\beta$ for some $0 < \beta < 1$, S has no drift and its Lévy measure is given by $\nu(ds) = \frac{\beta}{\Gamma(1-\beta)} s^{-(1+\beta)}\,ds$. In this case

$$w(s) = \nu(s,\infty) = \int_s^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)}\,dy = \frac{s^{-\beta}}{\Gamma(1-\beta)},$$

and so the time fractional derivative $\partial_t^w f$ defined by (1.3) is just the Caputo derivative of order β in literature.

The time fractional diffusion equation (1.2) with $\mathcal{L} = \Delta$ has been widely used to model anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see e.g. [15, 21]). It can be used to model "ultraslow diffusion" where a plume spreads at a logarithmic rate, for example when S is a subordinator of mixed stable subordinators; see [14] for details. The time fractional diffusion equation also appears as a scaling limit of random walk on $\mathbb{Z}^d$ with heavy-tailed random conductance: Let $\{C_{xy} : x,y \in \mathbb{Z}^d, |x-y| = 1\}$ be positive i.i.d. random variables such that $C_{xy} = C_{yx}$, $\mathbb{P}(C_{xy} \geqslant 1) = 1$ and

$$\mathbb{P}(C_{xy} \geqslant u) = c_1 u^{-\alpha}(1+o(1)) \qquad \text{as } u \to \infty$$

for some constants $c_1 > 0$ and $\alpha \in (0,1)$. Let $\{Y_t\}_{t\geqslant 0}$ be the Markov chain whose transition probability from x to y is equal to $C_{xy}/\sum_{z\in\mathbb{Z}^d} C_{xz}$. Then, for $d \geqslant 3$, $\{\varepsilon Y_{t\varepsilon^{-2/\alpha}}\}_{t\geqslant 0}$ converges to a multiple of the Caputo time fractional diffusion process on the path space equipped with the Skorokhod J_1-topology $\mathbb{P}$-almost surely as $\varepsilon \to 0$; see [2]. For $d = 2$, the same result holds by changing the scaling as $\{\varepsilon Y_{t(\log(1/\varepsilon))^{1-1/\alpha}\varepsilon^{-2/\alpha}}\}_{t\geqslant 0}$; see [4].

Time fractional diffusion equations have possible applications to anomalous diffusions in soil; see for instance [18]. The ultimate goal in the application is to determine the microstructure of soil through the averaged spatial data analysis, and to predict the progress of soil contamination. For such analysis, there is no reason that the operator in the master equation (1.2) is the classical Laplace operator in Euclidean space, and it would be useful to consider more general operators in metric measure spaces ([17]). In fact there are literatures that discuss the time fractional equation (1.2) in which $\mathcal{L}$ is a fractional Laplacian; see [3, 19, 23]. In [3] the authors discuss applications to laws of human travels, and in [19, 23] applications to chaotic Hamiltonian dynamics are discussed in typical low dimensional systems. Therefore, it is interesting and desirable to obtain explicit two-sided estimates of $p(t, x, y)$ for more general operators in non-Euclidean spaces.

The goal of this paper is to accomplish this, assuming general apriori estimates (see (1.13) and (1.15) below) for the fundamental solution of the heat equation of the infinitesimal spatial generator $\mathcal{L}$, and some weak scaling property on the subordinator S (see (1.10)). Moreover, we will show that for every $f \in L^2(M;\mu)$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ is the unique weak solution to (1.2); see Theorem 2.4 for details.

In what follows, we write $h(s) \simeq f(s)$ if there exist constants $c_1, c_2 > 0$ such that $c_1 f(s) \leqslant h(s) \leqslant c_2 f(s)$, for the specified range of the argument s. Similarly, we write $h(s) \asymp f(s)g(s)$ if there exist constants $C_1, c_1, C_2, c_2 > 0$ such that $f(C_1 s)g(c_1 s) \leqslant h(s) \leqslant f(C_2 s)g(c_2 s)$ for the specified range of s. c (without subscripts) denotes a strictly positive constant whose value is unimportant and which may change from line to line. Constants $c_0, c_1, c_2, \ldots$ with subscripts denote strictly positive constants and the labeling of the constants $c_0, c_1, c_2, \ldots$ starts anew in the statement of each result and the each step of its proof. We will use ":=" to denote a definition, which is read as "is defined to be". For any $a, b \in \mathbb{R}$, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Sometimes we use the notation $\frac{\partial v(t,r)}{\partial t} = \partial_t v(t, r)$.

1.2. **Special case: d-set setting.** Before giving our main results in full generality, we first give a version of them which can be described in a tidy way.

Throughout the paper, let (M, d) be a locally compact separable metric space and μ be a Radon measure on (M, d) that has full support. We say that the metric space (M, d) satisfies the *chain condition* if there exists a constant $C > 0$ such that, for any $x, y \in M$ and for any $n \in \mathbb{N}$, there exists a sequence $\{x_i\}_{i=0}^n \subset M$ such that $x_0 = x$, $x_n = y$, and

$$d(x_i, x_{i+1}) \leqslant C\frac{d(x, y)}{n} \qquad \text{for all } i = 0, 1, \cdots, n-1.$$

We assume the Hunt process X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ has a transition density function $q(t, x, y)$ with respect to the measure μ. We call $q(t, x, y)$ the heat kernel of the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Suppose that the heat kernel enjoys the following estimates

$$q(t, x, y) \asymp \frac{1}{t^{d/\alpha}} F\left(\frac{d(x, y)}{t^{1/\alpha}}\right), \quad t > 0, x, y \in M, \tag{1.6}$$

where $d, \alpha > 0$ and $F : [0, +\infty) \to [0, +\infty)$ is a non-increasing function such that $F(s_0) > 0$ for some $s_0 > 0$. In [9, Theorem 4.1], it is proved that if (M, d) satisfies the chain condition and $(\mathcal{E}, \mathcal{F})$ is conservative, then there are only two possible shapes of F.

Theorem 1.1. ([9, Theorem 4.1]) *Assume that the metric space (M, d) satisfies the chain condition and all balls are relatively compact. Assume further that $(\mathcal{E}, \mathcal{F})$ is regular, conservative and (1.6) holds with some $d, \alpha > 0$ and non-increasing function F. Then $\alpha \leqslant d + 1$, $\mu(B(x, r)) \simeq r^d$ for all $x \in M$ and $r > 0$, and the following dichotomy holds: either the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local, $\alpha \geqslant 2$, M is connected, and*

$$F(s) \asymp \exp\left(-s^{\alpha/(\alpha-1)}\right),$$

or the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of pure jump type and

$$F(s) \simeq (1 + s)^{-(d+\alpha)}.$$

In other words, Theorem 1.1 assets that under assumptions in the theorem, (M, d, μ) is an Alfhors d-regular set and the heat kernel $q(t, x, y)$ has the following estimates:

$$q(t, x, y) \asymp t^{-d/\alpha} \exp\left(-\left(\frac{d(x, y)^\alpha}{t}\right)^{1/(\alpha-1)}\right), \quad t > 0, x, y \in M \tag{1.7}$$

for some $\alpha \geqslant 2$ when $(\mathcal{E}, \mathcal{F})$ is local and M is connected, or

$$q(t, x, y) \simeq t^{-d/\alpha}\left(1 + \frac{d(x, y)}{t^{1/\alpha}}\right)^{-(d+\alpha)} \simeq t^{-d/\alpha} \wedge \frac{t}{d(x, y)^{d+\alpha}}, \quad t > 0, x, y \in M \tag{1.8}$$

for some $\alpha > 0$ when $(\mathcal{E}, \mathcal{F})$ is of pure jump type. Property (1.7) is called the sub-Gaussian heat kernel estimates, and (1.8) is called the α-stable-like heat kernel estimates.

Definition 1.2. Suppose that $0 < \alpha_1 \leqslant \alpha_2 < \infty$. We say that a non-decreasing function $\Psi : (0, \infty) \to (0, \infty)$ satisfies the *weak scaling property with* (α_1, α_2) if there exist constants $c_1, c_2 > 0$ such that

$$c_1(R/r)^{\alpha_1} \leqslant \Psi(R)/\Psi(r) \leqslant c_2(R/r)^{\alpha_2} \quad \text{for all } 0 < r \leqslant R < \infty. \tag{1.9}$$

We say that a family of non-decreasing functions $\{\Psi_x\}_{x\in\Lambda}$ satisfies the *weak scaling property uniformly with* (α_1, α_2) if each Ψ_x satisfies the weak scaling property with constants $c_1, c_2 > 0$ and $0 < \alpha_1 \leqslant \alpha_2 < \infty$ independent of the choice of $x \in \Lambda$.

Throughout the paper, we assume that the Laplace exponent ϕ of the driftless subordinator $S = \{S_t : t \geqslant 0\}$ satisfies the weak scaling property with (β_1, β_2) such that $0 < \beta_1 \leqslant \beta_2 < 1$; namely, for any $\lambda > 0$ and $\kappa \geqslant 1$,

$$c_1\kappa^{\beta_1} \leqslant \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \leqslant c_2\kappa^{\beta_2}. \tag{1.10}$$

Note that under (1.10), the Lévy measure ν of S is infinite as $\nu(0, \infty) = \lim_{\lambda\to\infty} \phi(\lambda) = \infty$, excluding compound Poisson processes.

The following is the main result in this subsection on the two-sided sharp estimates for the fundamental solution $p(t, x, y)$ of the time fractional equation (1.2).

Theorem 1.3. *Assume conditions in Theorem* 1.1 *and* (1.10) *hold. Let* $p(t,x,y)$ *be given by* (1.5)*. Then, we have*

(i) *If* $d(x,y)\phi(t^{-1})^{1/\alpha}\leqslant 1$*, then*

$$p(t,x,y)\simeq\begin{cases}\phi(t^{-1})^{d/\alpha} & \text{if } d<\alpha,\\ \phi(t^{-1})\log\left(\dfrac{2}{d(x,y)\phi(t^{-1})^{1/\alpha}}\right) & \text{if } d=\alpha,\\ \phi(t^{-1})^{d/\alpha}\left(d(x,y)\phi(t^{-1})^{1/\alpha}\right)^{-d+\alpha}=\phi(t^{-1})/d(x,y)^{d-\alpha} & \text{if } d>\alpha.\end{cases}$$

(ii) *Suppose* $d(x,y)\phi(t^{-1})^{1/\alpha}\geqslant 1$*. When the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *is local,*

$$p(t,x,y)\asymp\phi(t^{-1})^{d/\alpha}\exp\Big(-t\bar{\phi}_\alpha^{-1}((d(x,y)/t)^\alpha)\Big),\tag{1.11}$$

where $\bar{\phi}_\alpha(\lambda)=\lambda^\alpha/\phi(\lambda)$*, and* $\bar{\phi}_\alpha^{-1}(\lambda)$ *is the inverse function of* $\bar{\phi}_\alpha(\lambda)$*, i.e.,* $\bar{\phi}_\alpha^{-1}(\lambda):=\inf\{s>0:\bar{\phi}_\alpha(s)\geqslant\lambda\}$ *for all* $\lambda\geqslant 0$*; when* $(\mathcal{E},\mathcal{F})$ *is of pure jump type,*

$$p(t,x,y)\simeq\phi(t^{-1})^{d/\alpha}(d(x,y)\phi(t^{-1})^{1/\alpha})^{-d-\alpha}=\frac{1}{\phi(t^{-1})d(x,y)^{d+\alpha}}.$$

Remark 1.4. At first glance, the estimate (1.11) may look odd since the term $d(x,y)/t$ appears instead of the scaling term $d(x,y)\phi(t^{-1})^{1/\alpha}$ which appears in the rest of the estimates in Theorem 1.3. However, since

$$t\bar{\phi}_\alpha^{-1}((d(x,y)/t)^\alpha)=\frac{\bar{\phi}_\alpha^{-1}(d(x,y)^\alpha/t^\alpha)}{\bar{\phi}_\alpha^{-1}(\bar{\phi}_\alpha(t^{-1}))}=\frac{\bar{\phi}_\alpha^{-1}(d(x,y)^\alpha/t^\alpha)}{\bar{\phi}_\alpha^{-1}(1/(\phi(t^{-1})t^\alpha))},$$

they are consistent.

Let us consider a special case of Theorem 1.3 where $\{S_t:t\geqslant 0\}$ is a β-stable subordinator for some $\beta\in(0,1)$. In this case, $\phi(s)=s^\beta$. Define

$$H_{\leqslant 1}(t,d(x,y))=\begin{cases}t^{-\beta d/\alpha}, & d<\alpha,\\ t^{-\beta}\log\left(\dfrac{2}{d(x,y)t^{-\beta/\alpha}}\right), & d=\alpha,\\ t^{-\beta d/\alpha}\left(d(x,y)t^{-\beta/\alpha}\right)^{-d+\alpha}=t^{-\beta}/d(x,y)^{d-\alpha}, & d>\alpha,\end{cases}$$

$$H^{(c)}_{\geqslant 1}(t,d(x,y))=t^{-\beta d/\alpha}\exp\Big((d(x,y)t^{-\beta/\alpha})^{\alpha/(\alpha-\beta)}\Big),$$

$$H^{(j)}_{\geqslant 1}(t,d(x,y))=t^{-\beta d/\alpha}(d(x,y)t^{-\beta/\alpha})^{-(d+\alpha)}=t^\beta/d(x,y)^{d+\alpha}.$$

Corollary 1.5. *Assume that conditions in Theorem* 1.1 *hold and* $\phi(s)=s^\beta$ *for* $0<\beta<1$*. Let* $p(t,x,y)$ *be given by* (1.5)*.*

(i) *Suppose* $F(s)=\exp(-s^{\alpha/(\alpha-1)})$ *with* $\alpha\geqslant 2$*. Then*

$$p(t,x,y)\simeq H_{\leqslant 1}(t,d(x,y))\quad \text{if } d(x,y)t^{-\beta/\alpha}\leqslant 1,$$
$$p(t,x,y)\asymp H^{(c)}_{\geqslant 1}(t,d(x,y))\quad \text{if } d(x,y)t^{-\beta/\alpha}\geqslant 1.$$

(ii) *Suppose* $F(s)=(1+s)^{-d-\alpha}$*. Then,*

$$p(t,x,y)\simeq H_{\leqslant 1}(t,d(x,y))\quad \text{if } d(x,y)t^{-\beta/\alpha}\leqslant 1,$$
$$p(t,x,y)\simeq H^{(j)}_{\geqslant 1}(t,d(x,y))\quad \text{if } d(x,y)t^{-\beta/\alpha}\geqslant 1.$$

1.3. **General case.** In this subsection, we give a general version of the heat kernel estimates for the time fractional equation (1.2).

Recall that (M, d, μ) is a locally compact separable metric measure space such that μ is a Radon measure on (M, d) that has full support. Throughout this paper we assume that $(\mathcal{E}, \mathcal{F})$ is a regular, conservative Dirichlet form on $L^2(M; \mu)$. For $x \in M$ and $r \geqslant 0$, define

$$V(x, r) = \mu(B(x, r)).$$

We further assume that for each $x \in M$, $V(x, \cdot)$ satisfies the weak scaling property uniformly with (d_1, d_2) for some $d_2 \geqslant d_1 > 0$; that is, for any $0 < r \leqslant R$ and $x \in M$,

$$c_1 \left(\frac{R}{r}\right)^{d_1} \leqslant \frac{V(x, R)}{V(x, r)} \leqslant c_2 \left(\frac{R}{r}\right)^{d_2}. \tag{1.12}$$

Note that (1.12) is equivalent to the so-called volume doubling and reverse volume doubling conditions. As in the previous section, we also assume that the Laplace exponent ϕ of the driftless subordinator $S = \{S_t : t \geqslant 0\}$ satisfies (1.10).

1.3.1. **Pure jump case**. We first consider the case that the process associated with $(\mathcal{E}, \mathcal{F})$ is a pure jump process. In this case, we assume that the heat kernel of the associated process enjoys the following two-sided estimates:

$$q(t, x, y) \simeq \frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{t}{V(x, d(x, y))\Phi(d(x, y))}, \quad t > 0, x, y \in M, \tag{1.13}$$

where $\Phi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function with $\Phi(0) = 0$ that satisfies the weak scaling property with (α_1, α_2), i.e., (1.9) is satisfied.

Examples of such Dirichlet forms can be found in [6, 7]. Note that when $V(x, r) \simeq r^d$ and $\Phi(s) = s^\alpha$ for $r, s > 0$ and $x \in M$, then for any $x, y \in M$ and $t > 0$, $V(x, \Phi^{-1}(t)) \simeq t^{d/\alpha}$ and $V(x, d(x, y))\Phi(d(x, y)) \simeq d(x, y)^{d+\alpha}$, so (1.13) boils down to (1.8).

Here is the heat kernel estimates for the time fractional equation (1.2).

Theorem 1.6. *Suppose that the heat kernel of the non-local Dirichlet form has estimates* (1.13). *Let $p(t, x, y)$ be given by* (1.5). *Then we have the following two statements:*

(i) *If $\Phi(d(x, y))\phi(t^{-1}) \leqslant 1$, then*

$$\begin{aligned} p(t, x, y) &\simeq \phi(t^{-1}) \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{1}{V(x, \Phi^{-1}(r))}\, dr \\ &= \int_{\Phi(d(x,y))\phi(t^{-1})}^{2} \frac{1}{V(x, \Phi^{-1}(r/\phi(t^{-1})))}\, dr. \end{aligned} \tag{1.14}$$

(ii) *If $\Phi(d(x, y))\phi(t^{-1}) \geqslant 1$, then*

$$p(t, x, y) \simeq \frac{1}{\phi(t^{-1})V(x, d(x, y))\,\Phi(d(x, y))}.$$

Remark 1.7. (1) Note that, by some elementary calculations (see (4.10) and (4.11) below), we have

$$\left(\frac{1}{V(x, \Phi^{-1}(1/\phi(t^{-1})))} \vee \frac{\Phi(d(x, y))\phi(t^{-1})}{V(x, d(x, y))}\right) \leqslant c\phi(t^{-1}) \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{1}{V(x, \Phi^{-1}(r))}\, dr.$$

Roughly speaking, when $s \mapsto s^{-1}V(x,\Phi^{-1}(s))$ is strictly increasing,

$$p(t,x,y) \simeq \frac{\Phi(d(x,y))\phi(t^{-1})}{V(x,d(x,y))} \quad \text{if } \Phi(d(x,y))\phi(t^{-1}) \leqslant 1.$$

When $s \mapsto s^{-1}V(x,\Phi^{-1}(s))$ is strictly deceasing,

$$p(t,x,y) \simeq \frac{1}{V(x,\Phi^{-1}(1/\phi(t^{-1})))} \quad \text{if } \Phi(d(x,y))\phi(t^{-1}) \leqslant 1.$$

For the critical case, the logarithmic factor will appear, see Theorem 1.3 (i) and Corollary 1.5, or Corollary 5.2 for the explicit statements.

(2) As the proof shows, if we assume the upper (resp. lower) bound in (1.13), then the upper (resp. lower) bounds of $p(t,x,y)$ hold in the statement of Theorem 1.6.

1.3.2. **Diffusion case**. We next consider the case that the process associated with $(\mathcal{E},\mathcal{F})$ is a diffusion. In this case, we assume further that the metric space (M,d) is connected and satisfies the chain condition. Moreover, the heat kernel of the diffusion enjoys the following two-sided estimates

$$q(t,x,y) \asymp \frac{1}{V(x,\Phi^{-1}(t))} \exp\left(-m(t,d(x,y))\right), \quad t>0, x,y \in M. \tag{1.15}$$

Here, $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\Phi(0)=0$, and satisfies the weak scaling property with (α_1,α_2) such that the constants $\alpha_2 \geqslant \alpha_1 > 1$ in (1.9); the function $m(t,r)$ is strictly positive for all $t,r>0$, non-increasing on $(0,\infty)$ for fixed $r>0$, and determined by

$$\frac{t}{m(t,r)} \simeq \Phi\left(\frac{r}{m(t,r)}\right), \quad t,r>0. \tag{1.16}$$

In particular, by (1.9) with $\alpha_1>1$ and (1.16), there are constants $c_1,c_2>0$ such that for all $r>0$,

$$c_1\left(\frac{T}{t}\right)^{-1/(\alpha_1-1)} \leqslant \frac{m(T,r)}{m(t,r)} \leqslant c_2\left(\frac{T}{t}\right)^{-1/(\alpha_2-1)}, \quad 0<t\leqslant T. \tag{1.17}$$

On the other hand, by (1.16) we have

$$m(\Phi(r),r) \simeq 1, \quad r>0. \tag{1.18}$$

Using this and the fact that $m(\cdot,r)$ is non-increasing, we have

$$q(t,x,y) \simeq \frac{1}{V(x,\Phi^{-1}(t))} \quad \text{when } \Phi(d(x,y)) \leqslant c_3 t. \tag{1.19}$$

Note that when $V(x,r)\simeq r^d$ and $\Phi(s)=s^\alpha$ for $r,s>0$ and $x\in M$, then for every $x,y\in M$ and $t>0$, $V(x,\Phi^{-1}(t))\simeq t^{d/\alpha}$ and $m(t,d(x,y))\simeq (d(x,y)^\alpha/t)^{1/(\alpha-1)}$, and so (1.15) is reduced to (1.7). Examples of such Dirichlet forms include diffusions on fractals such as Sierpinski gaskets and Sierpinski carpets. For example, Brownian motion on the 2-dimensional Sierpinski gasket enjoys (1.7) (hence (1.15)) with $d=\log 3/\log 2$ and $\alpha=\log 5/\log 2>2$. See [10] and [22, Section 13] for more examples. Intuitively, $m(t,d(x,y))$ in (1.15) is an optimal number of steps for diffusions to reach from x to y at time t. As one sees in (1.16), the time and the distance are divided by $m(t,d(x,y))$ so that the relation between them is given by Φ. Then one decomposes the path from x to y into $m(t,d(x,y))$-th 'most probable' paths on which the near-diagonal heat kernel estimates hold, and uses the chain argument. This is how the off-diagonal estimates (exponential part of (1.15)) can be deduced on various concrete examples such as diffusions on fractals.

Here is the heat kernel estimates for the time fractional equation (1.2).

Theorem 1.8. *Suppose that the heat kernel of the local Dirichlet form enjoys estimates* (1.15)*. Let $p(t,x,y)$ be given by* (1.5)*. Then we have the following two statements:*

(i) *If $\Phi(d(x,y))\phi(t^{-1}) \leqslant 1$, then*

$$p(t,x,y) \simeq \int_{\Phi(d(x,y))\phi(t^{-1})}^{2} \frac{1}{V(x,\Phi^{-1}(r/\phi(t^{-1})))}\,dr.$$

(ii) *If $\Phi(d(x,y))\phi(t^{-1}) \geqslant 1$, then there exist constants $c_i > 0$ $(i = 1,\dots,4)$ such that*

$$\begin{aligned}\frac{c_1}{V(x,\Phi^{-1}(1/\phi(t^{-1})))} & \exp(-c_2 n(t,d(x,y)))\\ &\leqslant p(t,x,y)\\ &\leqslant \frac{c_3}{V(x,\Phi^{-1}(1/\phi(t^{-1})))}\exp(-c_4 n(t,d(x,y))),\end{aligned}$$

where $n(\cdot,r)$ is a non-increasing function on $(0,\infty)$ determined by

$$\frac{1}{\phi(n(t,r)/t)} \simeq \Phi\left(\frac{r}{n(t,r)}\right), \quad t,r>0. \tag{1.20}$$

Remark 1.9. (1) As mentioned above, $p(t,x,y)$ given by (1.5) is the "fundamental solution" to the time fractional equation (1.2), and so $p(t,x,y)$ closely relates to the process $X_E := \{X_{E_t} : t \geqslant 0\}$, where $\{E_t : t \geqslant 0\}$ is the inverse subordinator with respect to S. Estimates for the distribution of subordinator S collected in Proposition 3.3 (i) below show that, from the process X to the time-change process X_E, the time scale will be changed from t to $1/\phi(t^{-1})$. By this observation, we can partly give the intuitive explanation of the shape of the heat kernel estimates in Theorems 1.6 and 1.8. In particular, the case that $\Phi(d(x,y))\phi(t^{-1}) \leqslant 1$ corresponds to "near-diagonal" estimates of $p(t,x,y)$, while the case that $\Phi(d(x,y))\phi(t^{-1}) \geqslant 1$ can be regarded as "off-diagonal" estimates.

(2) When $\Phi(d(x,y))\phi(t^{-1}) \leqslant 1$, two-sided estimates of $p(t,x,y)$ for time fractional diffusion processes enjoy the same form as these for time fractional jump processes, see Theorems 1.6(i) and 1.8(i). Similar to Theorem 1.6, as the proof shows, if we assume the upper (resp. lower) bound in (1.15), then the upper (resp. lower) bounds of $p(t,x,y)$ hold in the statement of Theorem 1.8.

The rest of the paper is organized as follows. In Section 2, we show that for every $f \in L^2(M;\mu)$, the general time fractional equation (1.2) has a unique weak solution $u(t,x)$ in $L^2(M;\mu)$ with initial value f, and the solution has a representation $u(t,x) = \mathbb{E}\left[f(X_{E_t})\right]$. This result relaxes the condition that $f \in \mathcal{D}(\mathcal{L})$ imposed in [5, Theorem 2.3] at the expense of formulating the solution to (1.2) in the weak sense rather than in the strong sense. In Section 3, we present some preliminary estimates about Bernstein functions and subordinators. In particular, we establish the relation between the weak scaling property and Bernstein functions, which is interesting of its own. Section 4 and Section 5 are devoted to proofs of the main results of this paper, Theorems 1.6 and 1.8, respectively. Theorem 1.3 is then obtained as a corollary of Theorems 1.6 and 1.8.

2. Time fractional equations

Recall that $S = \{S_t : t \geqslant 0\}$ is a subordinator with the Laplace exponent ϕ given by (1.1) with the infinite Lévy measure ν. Define $w(x) = \nu(x, \infty)$ for $x > 0$. Since $\nu(0, \infty) = \infty$, almost surely, $t \mapsto S_t$ is strictly increasing. The following is a particular case of a recent result established in [5, Theorem 2.1].

Theorem 2.1. ([5, Theorem 2.1]) *For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ is a solution in $L^2(M;\mu)$ to the time fractional equation* (1.2) *in the following sense:*

(i) *$x \mapsto u(t,x)$ is in $\mathcal{D}(\mathcal{L})$ for each $t \geqslant 0$, and both $t \mapsto u(t,\cdot)$ and $t \mapsto \mathcal{L}u(t,\cdot)$ are continuous in $L^2(M;\mu)$. Consequently,*

$$I_t^w(u(\cdot,x)) := \int_0^t w(t-s)(u(s,x) - f(x))\, ds$$

is absolutely convergent in $L^2(M;\mu)$ for every $t > 0$.

(ii) *For every $t > 0$,*

$$\lim_{\delta \to 0} \frac{1}{\delta}\left(I_{t+\delta}^w(u(\cdot,x)) - I_t^w(u(\cdot,x))\right) = \mathcal{L}u(t,x) \quad \textit{in } L^2(M;\mu).$$

Conversely, if $u(t,x)$ is a solution to (1.2) *in the sense of* (i) *and* (ii) *above with $f \in \mathcal{D}(\mathcal{L})$, then $u(t,x) = \mathbb{E}\left[T_{E_t} f(x)\right]$ in $L^2(M;\mu)$ for every $t \geqslant 0$.*

When $S = \{S_t : t \geqslant 0\}$ is a β-stable subordinator with the Laplace exponent $\phi(\lambda) = \lambda^\beta$ for $0 < \beta < 1$, its Lévy measure $\nu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)}\, dx$ and so $w(x) = \mu(x,\infty) = \frac{x^{-\beta}}{\Gamma(1-\beta)}$. Hence Theorem 2.1 recovers the main result of [1] and [13, Theorem 5.1] for parabolic equations with Caputo time derivative of order β. For other related results, see [5, Remark 2.1].

In this section, we show that the initial condition $f \in \mathcal{D}(\mathcal{L})$ can be weakened to $f \in L^2(M;\mu)$ if we formulate the solution to the time fractional equation (1.2) in weak sense. First we recall the following result from [5, Lemma 2.1 and Corollary 2.2 (i)].

Lemma 2.2. *There is a Borel set $\mathcal{N} \subset (0,\infty)$ having zero Lebesgue measure so that*

$$\mathbb{P}(S_s \geqslant t) = \int_0^s \mathbb{E}\left[w(t-S_r)\mathbf{1}_{\{t \geqslant S_r\}}\right] dr \quad \textit{for every } s > 0 \textit{ and } t \in (0,\infty) \setminus \mathcal{N}$$

and

$$\int_0^\infty \mathbb{E}\left[w(t-S_r)\mathbf{1}_{\{t \geqslant S_r\}}\right] dr = 1 \quad \textit{for every } t \in (0,\infty) \setminus \mathcal{N}.$$

Define $G(0) = 0$ and $G(x) = \int_0^x w(t)\, dt$ for all $x > 0$. We also need the following lemma, which is [5, (2.5) and Corollary 2.1 (ii)].

Lemma 2.3. *For every $t, s > 0$,*

$$\int_0^t w(t-r)\mathbb{P}(S_s > r)\, dr = G(t) - \mathbb{E}(G(t-S_s)\mathbf{1}_{\{t \geqslant S_s\}})$$

and

$$\int_0^\infty \mathbb{E}(G(t-S_r)\mathbf{1}_{\{S_r\leqslant t\}})\,dr = t.$$

Now we can present the main result of this section on the existence and the uniqueness of weak solutions to equation (1.2).

Theorem 2.4. *For any $f \in L^2(M;\mu)$, $u(t,x) := \mathbb{E}\,[T_{E_t}f(x)]$ is a weak solution to*

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x) \tag{2.1}$$

in the following sense:

(i) $t \mapsto u(t,x)$ *is continuous in $L^2(M;\mu)$. Consequently, for every $t>0$,*

$$I_t^w(u(\cdot,x)) := \int_0^t w(t-s)(u(s,x)-f(x))\,ds$$

is absolutely convergent in $L^2(M;\mu)$.

(ii) *For every $g \in D(\mathcal{L})$ and $t>0$,*

$$\frac{d}{dt}\int_M g(x) I_t^w(u(\cdot,x))\,\mu(dx) = \int_M u(t,x)\mathcal{L}g(x)\,\mu(dx). \tag{2.2}$$

Conversely, if $u(t,x)$ is a weak solution to (2.1) *in the sense of* (i) *and* (ii) *above with $f \in L^2(M;\mu)$, then $u(t,x) = \mathbb{E}\,[T_{E_t}f(x)]$ μ-a.e. on M for every $t \geqslant 0$.*

Proof. The proof is motivated by that of [5, Theorem 2.1].

(1) **(Existence)** Since $\{T_t : t\geqslant 0\}$ is a strongly continuous contraction semigroup in $L^2(M;\mu)$ and $t\mapsto E_t$ is continuous a.s., we have by the bounded convergence theorem that $t\mapsto u(t,x) = \mathbb{E}\,[T_{E_t}f(x)]$ is continuous in $L^2(M;\mu)$ and $\|u(t,\cdot)\|_2 \leqslant \|f\|_2$. Since

$$\int_0^t w(s)\,ds = \int_0^\infty (z\wedge t)\,\nu(dz) < \infty \quad \text{for every } t>0 \tag{2.3}$$

by [5, (2.2)], $I_t^w(u(\cdot,x))$ is absolutely convergent in $L^2(M;\mu)$ for every $t>0$ with $\|I_t^w(u(\cdot,x))\|_2 \leqslant 2\|f\|_2\int_0^t w(s)\,ds < \infty$.

In the following, denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(M;\mu)$. By (1.4), the integration by parts formula, Lemma 2.3 and the self-adjointness of $\mathcal{L}$ in $L^2(M;\mu)$, we have for every $t>0$,

$$\begin{aligned}
&\int_M g(x) I_t^w(u(\cdot,x))\,\mu(dx)\\
&= \int_M g(x)\int_0^t w(t-r)(u(r,x)-u(0,x))\,dr\,\mu(dx)\\
&= \int_0^t w(t-r)\int_0^\infty (\langle T_s f,g\rangle - \langle f,g\rangle)\,d_s\mathbb{P}(S_s\geqslant r)\,dr\\
&= \int_0^\infty (\langle T_s f,g\rangle - \langle f,g\rangle)\,d_s\left(\int_0^t w(t-r)\mathbb{P}(S_s\geqslant r)\,dr\right)\\
&= -\int_0^\infty (\langle T_s f,g\rangle - \langle f,g\rangle)\,d_s\mathbb{E}(G(t-S_s)\mathbf{1}_{\{S_s\leqslant t\}})\\
&= \int_0^\infty \mathbb{E}(G(t-S_s)\mathbf{1}_{\{S_s\leqslant t\}})\langle \mathcal{L}T_s f,g\rangle\,ds
\end{aligned}$$

$$= \int_0^\infty \mathbb{E}(G(t-S_s)\mathbf{1}_{\{S_s\leqslant t\}})\langle T_s f, \mathcal{L}g\rangle\, ds.$$

On the other hand, according to (1.4), the integration by parts formula and Lemma 2.2, we find that for almost all $t>0$,

$$\begin{aligned}\int_0^t\int_M u(s,x)\mathcal{L}g(x)\,\mu(dx)\,ds &= \int_0^t \Big\langle \mathcal{L}g, \int_0^\infty T_u f\, d_u\mathbb{P}(S_u\geqslant s)\Big\rangle\, ds\\ &= \int_0^\infty \langle \mathcal{L}g, T_u f\rangle\, d_u\left(\int_0^t \mathbb{P}(S_u\geqslant s)\,ds\right)\\ &= \int_0^\infty \langle \mathcal{L}g, T_u f\rangle \int_0^t \mathbb{E}(w(s-S_u)\mathbf{1}_{\{S_u\leqslant s\}})\,ds\,du\\ &= \int_0^\infty \langle \mathcal{L}g, T_u f\rangle \mathbb{E}(G(t-S_u)\mathbf{1}_{\{S_u\leqslant t\}})\,du.\end{aligned}$$

Thus we conclude that for every $t\geqslant 0$,

$$\int_M g(x) I_t^w(u(\cdot,x))\,\mu(dx) = \int_0^t\int_M u(s,x)\mathcal{L}g(x)\,\mu(dx)\,ds.$$

This establishes (2.2) as $s\mapsto u(s,x)$ is continuous in $L^2(M;\mu)$.

(2) **(Uniqueness)** Suppose that $u(t,x)$ is a weak solution to (2.1) in the sense of (i) and (ii) with $f\in L^2(M;\mu)$. Then $v(t,x) := u(t,x)-\mathbb{E}\left[T_{E_t}f(x)\right]$ is a weak solution to (2.1) with $v(0,x)=0$. Note that by (2.3),

$$\lim_{t\to 0}\|I_t^w(v(\cdot,x))\|_2 \leqslant 2\max_{s\in[0,1]}\|v(s,\cdot)\|_2\cdot \lim_{t\to 0}\int_0^t w(s)\,ds = 0.$$

Hence we have for every $t>0$ and $g\in\mathcal{D}(\mathcal{L})$,

$$\int_M g(x)\left(\int_0^t w(t-r)v(r,x)\,dr\right)\mu(dx) = \int_M\left(\int_0^t v(s,x)\,ds\right)\mathcal{L}g(x)\,\mu(dx). \tag{2.4}$$

Let $V(\lambda,x) := \int_0^\infty e^{-\lambda t}v(t,x)\,dt$, $\lambda>0$, be the Laplace transform of $t\mapsto v(t,x)$. Taking the Laplace transform in t on both sides of (2.4) yields that for every $\lambda>0$,

$$\int_M g(x)V(\lambda,x)\left(\int_0^\infty e^{-\lambda s}w(s)\,ds\right)\mu(dx) = \frac{1}{\lambda}\int_M V(\lambda,x)\mathcal{L}g(x)\,\mu(dx).$$

Note that the Laplace transform of $w(t)$ is $\phi(\lambda)/\lambda$; see [5, (2.3)]. Hence we have from the above display that for every $\lambda>0$,

$$\int_M V(\lambda,x)\left(\phi(\lambda)-\mathcal{L}\right)g(x)\,\mu(dx)=0.$$

Denote by $\{G_\alpha : \alpha>0\}$ be the resolvent of the regular Dirichlet for $(\mathcal{E},\mathcal{F})$. For each fixed $\lambda>0$ and $h\in L^2(M;\mu)$, take $g := G_{\phi(\lambda)}h$, which is in $\mathcal{D}(\mathcal{L})$. Since $(\phi(\lambda)-\mathcal{L})g=h$, we deduce that $\int_M V(\lambda,x)h(x)\,\mu(dx)=0$ for every $h\in L^2(M;\mu)$. Therefore $V(\lambda,x)=0$ μ-a.e. for every $\lambda>0$. By the uniqueness of the Laplace transform and the fact that $t\mapsto v(t,x)$ is continuous in $L^2(M;\mu)$, it follows that $v(t,x)=0$ a.e. for every $t>0$. In other words, $u(t,x)=\mathbb{E}\left[T_{E_t}f(x)\right]$ μ-a.e. on M for every $t>0$. $\square$

3. Preliminary estimates

In this section, we give some preliminary estimates needed for the proofs of Theorems 1.6 and 1.8.

3.1. **Bernstein functions and the weak scaling property.** A non-negative C^∞ function ϕ on $(0,\infty)$ is called a Bernstein function if $(-1)^n\phi^{(n)}(\lambda)\leqslant 0$ for every $n\in\mathbb{N}$ and $\lambda>0$. According to [16, (2.3)] and [12, Lemma 1.3], the following properties hold for the Bernstein function ϕ satisfying condition (1.10).

Lemma 3.1. *Let ϕ be a Bernstein function such that* (1.10) *is satisfied, i.e., there are constants $0<\beta_1\leqslant\beta_2<1$ such that for any $\lambda>0$ and $\kappa\geqslant 1$,*

$$c_1\kappa^{\beta_1}\leqslant\frac{\phi(\kappa\lambda)}{\phi(\lambda)}\leqslant c_2\kappa^{\beta_2}.$$

Then there exists a constant $C_*\geqslant 1$ such that the following holds

$$\lambda\,\phi'(\lambda)\leqslant\phi(\lambda)\leqslant C_*\lambda\,\phi'(\lambda),\quad \lambda>0. \tag{3.1}$$

In particular, there exist constants $c_i>0$ $(i=3,4,5,6)$ such that

$$c_3\kappa^{1-\beta_2}\leqslant\frac{\phi'(\lambda)}{\phi'(\kappa\lambda)}\leqslant c_4\kappa^{1-\beta_1},\quad \lambda>0,\kappa\geqslant 1, \tag{3.2}$$

and

$$c_5\kappa^{1/(1-\beta_1)}\leqslant\frac{(\phi')^{-1}(\lambda)}{(\phi')^{-1}(\kappa\lambda)}\leqslant c_6\kappa^{1/(1-\beta_2)},\quad \lambda>0,\kappa\geqslant 1, \tag{3.3}$$

where $(\phi')^{-1}(\lambda):=\inf\{s>0:\phi'(s)\leqslant\lambda\}$ for all $\lambda\geqslant 0$.

A function $f:(0,\infty)\to\mathbb{R}$ is said to be a completely monotone function if f is smooth and $(-1)^nf^{(n)}(\lambda)\geqslant 0$ for all $n\in\mathbb{N}$ and $\lambda>0$. A Bernstein function is said to be a complete Bernstein function if its Lévy measure has a completely monotone density with respect to Lebesgue measure. The next lemma is concerned with the weak scaling property, which is interesting of its own.

Lemma 3.2. *Suppose that $0<\alpha_1\leqslant\alpha_2<\infty$ and that a family of non-negative functions $\{\Phi(x,\cdot)\}_{x\in M}$ satisfies the weak scaling property uniformly with (α_1,α_2), i.e., there exist constants $c_1,c_2>0$ such that for any $x\in M$,*

$$c_1(R/r)^{\alpha_1}\leqslant\Phi(x,R)/\Phi(x,r)\leqslant c_2(R/r)^{\alpha_2},\quad 0<r\leqslant R<\infty. \tag{3.4}$$

Then for any $\alpha_3>\alpha_2$, there is a family of complete Bernstein functions $\{\varphi(x,\cdot)\}_{x\in M}$ such that

$$\Phi(x,r)\simeq\frac{1}{\varphi(x,r^{-\alpha_3})},\quad r>0,x\in M.$$

Consequently, $\{\varphi(x,\cdot)\}_{x\in M}$ enjoys the weak scaling property uniformly with $(\alpha_1/\alpha_3,\alpha_2/\alpha_3)$, i.e., there are constants $c_3,c_4>0$ such that for all $x\in M$,

$$c_3(R/r)^{\alpha_1/\alpha_3}\leqslant\varphi(x,R)/\varphi(x,r)\leqslant c_4(R/r)^{\alpha_2/\alpha_3},\quad 0<r\leqslant R<\infty \tag{3.5}$$

Proof. For any fixed $\alpha_3>\alpha_2$ and $x\in M$, define

$$\varphi(x,\lambda)=\int_0^\infty\frac{\lambda}{\lambda+s}\,\frac{1}{s\Phi(x,s^{-1/\alpha_3})}\,ds,\quad \lambda\geqslant 0,$$

and

$$\hat\Phi(x,u)=\int_0^\infty\frac{e^{-us}}{\Phi(x,s^{-1/\alpha_3})}\,ds,\quad u\geqslant 0.$$

Since

$$\int_0^\infty (1-e^{-\lambda u})e^{-us}\,du = \int_0^\infty e^{-su}\,du - \int_0^\infty e^{-(\lambda+s)u}\,du = \frac{\lambda}{s(\lambda+s)},$$

we have

$$\varphi(x,\lambda) = \int_0^\infty \int_0^\infty (1-e^{-\lambda u})e^{-us}\,du \frac{1}{\Phi(x,s^{-1/\alpha_3})}\,ds = \int_0^\infty (1-e^{-\lambda u})\hat{\Phi}(x,u)\,du.$$

In particular, $\hat{\Phi}(x,\cdot)$ is a completely monotone function, and so $\varphi(x,\cdot)$ is a complete Bernstein function.

By the change of variable $u = s^{-1/\alpha_3}$, we have that for any $x \in M$ and $\lambda > 0$,

$$\varphi(x,\lambda) = \alpha_3 \int_0^\infty \frac{\lambda u^{\alpha_3}}{\lambda u^{\alpha_3}+1} \frac{1}{u\Phi(x,u)}\,du \simeq \psi(x,\lambda),$$

where

$$\psi(x,\lambda) := \int_0^\infty (1 \wedge (\lambda u^{\alpha_3}))\frac{1}{u\Phi(x,u)}\,du.$$

Note that for all $x \in M$ and $\lambda > 0$,

$$\begin{aligned}\psi(x,\lambda)\Phi(x,\lambda^{-1/\alpha_3}) &= \int_0^\infty (1 \wedge (\lambda u^{\alpha_3}))\frac{\Phi(x,\lambda^{-1/\alpha_3})}{u\Phi(x,u)}\,du \\ &= \lambda \int_0^{\lambda^{-1/\alpha_3}} u^{\alpha_3-1}\frac{\Phi(x,\lambda^{-1/\alpha_3})}{\Phi(x,u)}\,du + \int_{\lambda^{-1/\alpha_3}}^\infty \frac{\Phi(x,\lambda^{-1/\alpha_3})}{u\Phi(x,u)}\,du.\end{aligned}$$

Using (3.4), we can find that for all $x \in M$ and $\lambda > 0$,

$$\begin{aligned}\frac{c_1}{\alpha_3-\alpha_1}\lambda^{-1} &= c_1\lambda^{-\alpha_1/\alpha_3}\int_0^{\lambda^{-1/\alpha_3}} u^{\alpha_3-1-\alpha_1}\,du \\ &\leqslant \int_0^{\lambda^{-1/\alpha_3}} u^{\alpha_3-1}\frac{\Phi(x,\lambda^{-1/\alpha_3})}{\Phi(x,u)}\,du \\ &\leqslant c_2\lambda^{-\alpha_2/\alpha_3}\int_0^{\lambda^{-1/\alpha_3}} u^{\alpha_3-1-\alpha_2}\,du = \frac{c_2}{\alpha_3-\alpha_2}\lambda^{-1}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{c_2\alpha_2} &= c_2^{-1}\lambda^{-\alpha_2/\alpha_3}\int_{\lambda^{-1/\alpha_3}}^\infty u^{-1-\alpha_2}\,du \\ &\leqslant \int_{\lambda^{-1/\alpha_3}}^\infty \frac{\Phi(x,\lambda^{-1/\alpha_3})}{u\Phi(x,u)}\,du \leqslant c_1^{-1}\lambda^{-\alpha_1/\alpha_3}\int_{\lambda^{-1/\alpha_3}}^\infty u^{-1-\alpha_1}\,du = \frac{1}{c_1\alpha_1}.\end{aligned}$$

Therefore, for all $x \in M$ and $\lambda > 0$,

$$\varphi(x,\lambda) \simeq \frac{1}{\Phi(x,\lambda^{-1/\alpha_3})},$$

which along with (3.4) yields (3.5). The proof is complete. □

By Lemma 3.2 above, for any function $\Phi(x,r)$ satisfying (3.4), we have

$$\Phi(x,r) \simeq \tilde{\Phi}(x,r) := 1/\varphi(x,r^{-\alpha_3})$$

for some complete Bernstein function $\varphi(x,\cdot)$ and $\alpha_3 > \alpha_2$. According to (3.1), for all $x \in M$ and $r > 0$,

$$r\partial_r \tilde{\Phi}(x,r) = \alpha_3 \frac{\partial_r \varphi(x, r^{-\alpha_3})}{\varphi(x, r^{-\alpha_3})^2} r^{-\alpha_3} \simeq \frac{1}{\varphi(x, r^{-\alpha_3})} = \tilde{\Phi}(x,r)$$

and so, by the inverse function theorem with $t = \tilde{\Phi}(x,s)$, for all $x \in M$ and $t > 0$,

$$\frac{(\partial_t \tilde{\Phi}^{-1}(x,\cdot))(t)}{\tilde{\Phi}^{-1}(x,\cdot)(t)} = \frac{(\partial_t \tilde{\Phi}^{-1}(x,\cdot))(\tilde{\Phi}(x,s))}{s} = \frac{1}{s\partial_s \tilde{\Phi}(x,s)} \simeq \frac{1}{\tilde{\Phi}(x,s)} = \frac{1}{t}. \tag{3.6}$$

3.2. **Estimates for subordinator.**

Proposition 3.3. *Let $\{S_t : t \geqslant 0\}$ be a subordinator whose Laplace exponent ϕ satisfies assumption* (1.10)*.*

(i) *There are constants $c_1, c_2 > 0$ such that for all $r, t \geqslant 0$,*

$$\mathbb{P}\left(S_r \geqslant t(1 + er\phi(t^{-1}))\right) \leqslant c_1 r\phi(t^{-1}) \tag{3.7}$$

and

$$\mathbb{P}(S_r \geqslant t) \geqslant 1 - e^{-c_2 r\phi(t^{-1})}. \tag{3.8}$$

In particular, for each $L > 0$, there exist constants $c_{1,L}, c_{2,L} > 0$ such that for all $r\phi(t^{-1}) \leqslant L$,

$$c_{1,L} r\phi(t^{-1}) \leqslant \mathbb{P}(S_r \geqslant t) \leqslant c_{2,L} r\phi(t^{-1}).$$

(ii) *There is a constant $c_1 > 0$ such that for all $r, t > 0$,*

$$\mathbb{P}(S_r \leqslant t) \leqslant \exp(-c_1 r\phi \circ [(\phi')^{-1}](t/r)) \leqslant \exp(-c_1 t(\phi')^{-1}(t/r)).$$

Moreover, there is a constant $c_0 > 0$ such that for each $L > 0$, there exists a constant $c_{c_0,L} > 0$ so that for $r\phi(t^{-1}) > L$

$$\mathbb{P}(S_r \leqslant t) \geqslant c_{c_0,L} \exp\left(-c_0 r\phi \circ [(\phi')^{-1}](t/r)\right) \geqslant c_{c_0,L} \exp(-c_0 C_* t(\phi')^{-1}(t/r)),$$

where $C_* > 0$ is the constant in (3.1)*.*

Proof. (i) (3.7) and (3.8) follow from [16, Propositions 2.3 and 2.9] and [16, Proposition 2.5, Lemma 2.6 and Proposition 2.9], respectively. The last assertion is a direct consequence of (1.10), (3.7) and (3.8).

(ii) According to (3.2), we have $\phi'(0) = \infty$ and so

$$\int_0^\infty s\,\nu(ds) = \phi'(0) = \infty.$$

Since

$$r \cdot \phi' \circ [(\phi')^{-1}](t/r) \cdot (\phi')^{-1}(t/r) = t \cdot (\phi')^{-1}(t/r),$$

by (3.1)

$$t \cdot (\phi')^{-1}(t/r) \leqslant r\phi \circ [(\phi')^{-1}](t/r) \leqslant C_* t \cdot (\phi')^{-1}(t/r). \tag{3.9}$$

Now, the first assertion follows from (3.9), [11, Lemma 5.2] and [16, Proposition 2.9].

On the other hand, by [11, Lemma 5.2] and [16, Proposition 2.9] again, there exist constants $c_0, c_1, c_2 > 0$ (c_0 is independent of c_1 and c_2) such that for $r\phi \circ [(\phi')^{-1}](t/r) \geqslant c_1$,

$$\mathbb{P}(S_r \leqslant t) \geqslant c_2 \exp\Big(- c_0 r\phi \circ [(\phi')^{-1}](t/r)\Big).$$

Thus, according to (3.3), (3.9) and (3.1), we see that there exists a constant $c_3 > 0$ such that for $r\phi(t^{-1}) \geqslant c_3$ (so that $\phi'(t^{-1}) \geqslant (c_3/C_*)t/r$)

$$\mathbb{P}(S_r \leqslant t) \geqslant c_2 \exp\Big(- c_0 r\phi \circ [(\phi')^{-1}](t/r)\Big) \geqslant c_2 \exp\Big(- c_0 C_* t(\phi')^{-1}(t/r)\Big). \quad (3.10)$$

We observe that, if $L < r\phi(t^{-1}) \leqslant c_3$ for a constant $L > 0$, then by (1.10) and (3.1)

$$r\phi \circ [(\phi')^{-1}](t/r) \leqslant \frac{c_3\phi \circ [(\phi')^{-1}](t\phi(t^{-1})/c_3)}{\phi(t^{-1})} = \frac{c_3\phi \circ [(\phi')^{-1}]\left(\frac{\phi(t^{-1})}{c_3 t^{-1}}\right)}{\phi \circ [(\phi')^{-1}](\phi'(t^{-1}))} \simeq 1,$$

and

$$r\phi \circ [(\phi')^{-1}](t/r) \geqslant \frac{L\phi \circ [(\phi')^{-1}](t\phi(t^{-1})/L)}{\phi(t^{-1})} = \frac{L\phi \circ [(\phi')^{-1}]\left(\frac{\phi(t^{-1})}{L t^{-1}}\right)}{\phi \circ [(\phi')^{-1}](\phi'(t^{-1}))} \simeq 1,$$

Thus, using (3.10) for $r\phi(t^{-1}) = c_3$, we have that for any $L > 0$ such that $L < r\phi(t^{-1}) \leqslant c_3$,

$$\mathbb{P}(S_r \leqslant t) \geqslant \mathbb{P}(S_{c_3/\phi(t^{-1})} \leqslant t) \geqslant c_2 e^{-c_4} \geqslant c_2 \exp\big(- c_5 r\phi \circ [(\phi')^{-1}](t/r)\big).$$

This completes the proof. □

3.3. **Preliminary lower bound estimates for** $p(t, x, y)$**.**

The next statement is a key lemma used in the proof of the lower bound for $p(t, x, y)$, which is defined in (1.5).

Lemma 3.4. *Suppose that for each $T > 0$ there exists a constant $c_0 = c_0(T) > 0$ such that*

$$q(t, x, y) \geqslant \frac{c_0}{V(x, \Phi^{-1}(t))} \quad \textit{for all } x, y \in M \textit{ and } t \in (0, T\Phi(d(x, y))], \quad (3.11)$$

where $\Phi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function with $\Phi(0) = 0$ and satisfies the weak scaling property with (α_1, α_2) for some constants $0 < \alpha_1 \leqslant \alpha_2 < \infty$. Then for every $L > 0$, there is a constant $c_1 := c_1(L) > 0$ such that for all $x, y \in M$ and $t > 0$ with $\Phi(d(x, y))\phi(t^{-1}) \leqslant L$,

$$p(t, x, y) \geqslant c_1 \left(\frac{1}{V(x, \Phi^{-1}(1/\phi(t^{-1})))} \vee \frac{\Phi(d(x, y))\phi(t^{-1})}{V(x, d(x, y))} \right).$$

Proof. By (3.7) and (3.8) in Proposition 3.3 (i), we can choose constants $\kappa_1 > \kappa_2 > 0$ such that for all $t > 0$,

$$\mathbb{P}(S_{\kappa_1/\phi(t^{-1})} \geqslant t) - \mathbb{P}(S_{\kappa_2/\phi(t^{-1})} \geqslant t) \geqslant c_0. \quad (3.12)$$

Inequality (3.12) along with (1.5) yields that for every $L > 0$, $x, y \in M$ and $t > 0$ with $\Phi(d(x, y))\phi(t^{-1}) \leqslant L$

$$\begin{aligned} p(t, x, y) &\geqslant c \int_{\kappa_2/\phi(t^{-1})}^{\kappa_1/\phi(t^{-1})} q(r, x, y)\, d_r\mathbb{P}(S_r \geqslant t) \\ &\geqslant c \left(\min_{\kappa_2/\phi(t^{-1}) \leqslant r \leqslant \kappa_1/\phi(t^{-1})} q(r, x, y) \right) \big(\mathbb{P}(S_{\kappa_1/\phi(t^{-1})} \geqslant t) - \mathbb{P}(S_{\kappa_2/\phi(t^{-1})} \geqslant t)\big) \\ &\geqslant \frac{c}{V(x, \Phi^{-1}(1/\phi(t^{-1})))}, \end{aligned} \quad (3.13)$$

where in the last inequality we have used (3.11) and the fact that $\Phi(d(x,y))\phi(t^{-1}) \leqslant L$. Similarly, according to (3.7) and (3.8) in Proposition 3.3 (i), one can choose constants $\kappa_3 > \kappa_4 > 0$ such that for all $t > 0$ and $z \geqslant 0$,

$$\mathbb{P}(S_{\kappa_3\Phi(z)} \geqslant t) - \mathbb{P}(S_{\kappa_4\Phi(z)} \geqslant t) \geqslant c_1\Phi(z)\phi(t^{-1}). \tag{3.14}$$

Using (3.14) and the argument of (3.13), we find that for every $L > 0$, $x, y \in M$ and $t > 0$ with $\Phi(d(x,y))\phi(t^{-1}) \leqslant L$

$$\begin{aligned} p(t,x,y) &\geqslant c\int_{\kappa_4\Phi(d(x,y))}^{\kappa_3\Phi(d(x,y))} q(r,x,y)\,d_r\mathbb{P}(S_r \geqslant t) \\ &\geqslant c\left(\min_{\kappa_4\Phi(d(x,y))\leqslant r\leqslant\kappa_3\Phi(d(x,y))} q(r,x,y)\right)\left(\mathbb{P}(S_{\kappa_3\Phi(d(x,y))} \geqslant t) - \mathbb{P}(S_{\kappa_4\Phi(d(x,y))} \geqslant t)\right) \\ &\geqslant \frac{c\Phi(d(x,y))\phi(t^{-1})}{V(x,d(x,y))}. \end{aligned}$$

The proof is complete. □

4. Non-local spatial motions

4.1. **Time derivative of heat kernel estimates for jump process.** In this section, we consider the pure jump case where $p(t,x,y)$ satisfies (1.13). First, note that since Φ is strictly increasing and satisfies the weak scaling property with (α_1,α_2), there are constants $c_1, c_2 > 0$ such that for all $\kappa \geqslant 1$ and $\lambda > 0$,

$$c_1\kappa^{1/\alpha_2} \leqslant \frac{\Phi^{-1}(\kappa\lambda)}{\Phi^{-1}(\lambda)} \leqslant c_2\kappa^{1/\alpha_1}. \tag{4.1}$$

Set

$$\bar q(t,x,r) := \frac{t}{tV(x,\Phi^{-1}(t)) + \Phi(r)\,V(x,r)}, \quad t, r > 0 \text{ and } x \in M. \tag{4.2}$$

Note that by (1.13) and the fact $1 \wedge (1/r) \simeq 1/(1+r)$ for $r > 0$, we have

$$q(t,x,y) \simeq \bar q(t,x,d(x,y)) \quad \text{for every } t > 0 \text{ and } x, y \in M. \tag{4.3}$$

According to Lemma 3.2 and the remark at the end of Subsection 3.1, we may and do assume that both $V(x,\cdot)$ and $\Phi(\cdot)$ are differentiable and satisfy the property like (3.6).

We next give a lemma concerning the time derivative of $\bar q(t,x,r)$.

Lemma 4.1. *Under assumptions above, there is a constant $c_1 > 0$ such that for all $t, r > 0$ and $x \in M$,*

$$\left|\frac{\partial \bar q(t,x,r)}{\partial t}\right| \leqslant c_1\frac{\bar q(t,x,r)}{t}, \tag{4.4}$$

and that there exist constants $c_2, c_3 > 0$, $c_* \in (0,1)$ and $c^* \in (1,\infty)$ such that for all $x \in M$,

$$\frac{\partial \bar q(t,x,r)}{\partial t} \leqslant -c_2\frac{\bar q(t,x,r)}{t} \quad \text{if } \Phi(r) \leqslant c_* t, \tag{4.5}$$

and

$$\frac{\partial \bar q(t,x,r)}{\partial t} \geqslant c_3\frac{\bar q(t,x,r)}{t} \quad \text{if } \Phi(r) \geqslant c^* t. \tag{4.6}$$

Proof. By elementary calculations, we have

$$\begin{aligned}
&\frac{\partial \bar q(t,x,r)}{\partial t}\\
&=\frac{(tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r))-t[V(x,\Phi^{-1}(t))+t\partial_r V(x,\Phi^{-1}(t))(\Phi^{-1}(t))']}{(tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r))^2}\\
&=\frac{\Phi(r)\,V(x,r)-t^2\partial_r V(x,\Phi^{-1}(t))(\Phi^{-1}(t))'}{(tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r))^2}\\
&=\frac{\bar q(t,x,r)}{t}\left(\frac{\Phi(r)\,V(x,r)-t^2\partial_r V(x,\Phi^{-1}(t))(\Phi^{-1}(t))'}{tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r)}\right).
\end{aligned}$$

Since $t^2\partial_r V(x,\Phi^{-1}(t))(\Phi^{-1}(t))'\simeq tV(x,\Phi^{-1}(t))$ by (3.6), we have

$$\begin{aligned}
&\frac{\bar q(t,x,r)}{t}\left(\frac{\Phi(r)\,V(x,r)-c_1tV(x,\Phi^{-1}(t))}{tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r)}\right)\\
&\qquad\leqslant\frac{\partial \bar q(t,x,r)}{\partial t}\leqslant\frac{\bar q(t,x,r)}{t}\left(\frac{\Phi(r)\,V(x,r)-c_2tV(x,\Phi^{-1}(t))}{tV(x,\Phi^{-1}(t))+\Phi(r)\,V(x,r)}\right).
\end{aligned}$$

Thus, the desired assertion follows from the estimate above. □

4.2. **Two-sided estimates for** $p(t,x,y)$**.** Recall that for $t>0$ and $x,y\in M$,

$$p(t,x,y)=\int_0^\infty q(r,x,y)\,d_r\mathbb{P}(E_t\leqslant r)=\int_0^\infty q(r,x,y)\,d_r\mathbb{P}(S_r\geqslant t).$$

Proof of Theorem 1.6. Throughout the proof, we fix $x,y\in M$. By (4.3),

$$p(t,x,y)=\int_0^\infty q(r,x,y)\,d_r\mathbb{P}(S_r\geqslant t)\simeq\int_0^\infty \bar q(r,x,d(x,y))\,d_r\mathbb{P}(S_r\geqslant t).$$

Then, for $t>0$ and $x,y\in M$,

$$\begin{aligned}
p(t,x,y)\simeq&\int_0^{2/\phi(t^{-1})}\bar q(r,x,d(x,y))\,d_r\mathbb{P}(S_r\geqslant t)\\
&-\int_{2/\phi(t^{-1})}^\infty \bar q(r,x,d(x,y))\,d_r\mathbb{P}(S_r\leqslant t)\\
=:&\,I_1+I_2.
\end{aligned}\tag{4.7}$$

For simplicity, in the following we fix $x\in M$ and let $z=d(x,y)$. Then by definition, $\bar q(t,x,d(x,y))=\bar q(t,x,z)$. We also write $\bar q(t,x,z)$ and $V(x,r)$ as $\bar q(t,z)$ and $V(r)$, respectively. The proof is divided into two parts.

Proof of the upper bound of $p(t,x,y)$**.** For I_1, since $\mathbb{P}(S_0\geqslant t)=0$ for $t>0$ and $\bar q(0,\cdot)=\delta_{\{0\}}$ (this is understood in the usual way and $\delta_{\{0\}}$ is the Dirac measure at the point 0), we have by Proposition 3.3 (i) and (4.4)

$$\begin{aligned}
I_1=&\bar q(r,z)\mathbb{P}(S_r\geqslant t)\Big|_0^{2/\phi(t^{-1})}-\int_0^{2/\phi(t^{-1})}\mathbb{P}(S_r\geqslant t)\,d_r\bar q(r,z)\\
\leqslant& c\bar q(2/\phi(t^{-1}),z)-\int_0^{2/\phi(t^{-1})}\mathbb{P}(S_r\geqslant t)\,d_r\bar q(r,z)\\
\leqslant& c\bar q(2/\phi(t^{-1}),z)+c\int_0^{2/\phi(t^{-1})}r\phi(t^{-1})\cdot\frac{1}{r}\cdot\bar q(r,z)\,dr\\
=:&\,c\bar q(2/\phi(t^{-1}),z)+cI_{1,1}.
\end{aligned}\tag{4.8}$$

For I_2, since $\bar q(\infty,z)=0$,

$$\begin{aligned}I_2=&-\int_{2/\phi(t^{-1})}^\infty \bar q(r,z)\,d_r\mathbb{P}(S_r\leqslant t)\\ =&-\bar q(r,z)\mathbb{P}(S_r\leqslant t)|_{2/\phi(t^{-1})}^\infty+\int_{2/\phi(t^{-1})}^\infty \mathbb{P}(S_r\leqslant t)\,d_r\bar q(r,z)\\ \leqslant& c\bar q(2/\phi(t^{-1}),z)+c\int_{2/\phi(t^{-1})}^\infty \exp(-c_1t(\phi')^{-1}(t/r))\cdot\frac{1}{r}\cdot \bar q(r,z)\,dr\\ =:& \ c\bar q(2/\phi(t^{-1}),z)+cI_{2,1},\end{aligned}\tag{4.9}$$

where in the inequality above we used Proposition 3.3 (ii) and (4.4). Therefore, in order to get upper bound of $p(t,x,y)$, we need to derive upper bound for $I_{1,1}$ and $I_{2,1}$.

(1-a) Suppose that $\Phi(z)\phi(t^{-1})\leqslant 1$. Then by (4.2)

$$\begin{aligned}I_{1,1}&\leqslant c\phi(t^{-1})\left(\frac{1}{V(z)\Phi(z)}\int_0^{\Phi(z)} r\,dr+\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr\right)\\ &=c\phi(t^{-1})\left(\frac{\Phi(z)}{V(z)}+\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr\right)\leqslant c\phi(t^{-1})\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr,\end{aligned}$$

where in the last inequality we used the fact that

$$\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr\geqslant \int_{\Phi(z)}^{2\Phi(z)}\frac{1}{V(\Phi^{-1}(r))}\,dr\geqslant c\,\frac{\Phi(z)}{V(z)}.\tag{4.10}$$

By changing the variable $s=r\phi(t^{-1})$ and using (1.12) and (4.1), we find that

$$\begin{aligned}&\phi(t^{-1})\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr=\int_{\Phi(z)\phi(t^{-1})}^2\frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds\\ &=\frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))}\int_{\Phi(z)\phi(t^{-1})}^2\frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds\\ &\geqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}\int_1^2 s^{-d_2/\alpha_1}\,ds\geqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}\geqslant c\bar q(2/\phi(t^{-1}),z).\end{aligned}\tag{4.11}$$

Hence

$$I_1\leqslant c\phi(t^{-1})\int_{\Phi(z)}^{2/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr.$$

(1-b) If $\Phi(z)\phi(t^{-1})\geqslant 1$, then by (4.2)

$$I_{1,1}\leqslant c\phi(t^{-1})\int_0^{2/\phi(t^{-1})}\bar q(r,z)\,dr\leqslant \frac{c\phi(t^{-1})}{V(z)\Phi(z)}\int_0^{2/\phi(t^{-1})} r\,dr\leqslant \frac{c}{\phi(t^{-1})V(z)\Phi(z)}.$$

Since by (4.2) again

$$\bar q(2/\phi(t^{-1}),z)\leqslant \frac{c}{V(z)\Phi(z)\phi(t^{-1})},$$

we obtain

$$I_1\leqslant \frac{c}{\phi(t^{-1})V(z)\Phi(z)}.$$

(2-a) If $\Phi(z)\phi(t^{-1}) \leqslant 1$, then by changing variable $s = r\phi(t^{-1})$, and using (4.2), (1.12), (4.1), (3.1) and (3.3),

$$\begin{aligned}
I_{2,1} &\leqslant c\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \bar{q}(r,z)\,dr \\
&\leqslant c\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \frac{1}{V(\Phi^{-1}(r))}\,dr \\
&= c\int_{2}^{\infty} \exp(-c_1 t(\phi')^{-1}(t\phi(t^{-1})/s)) \cdot s^{-1} \cdot \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds \\
&\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \\
&\quad\times \int_{2}^{\infty} \exp(-c_1 t(\phi')^{-1}(\phi'(t^{-1})/s)) \cdot \frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(s/\phi(t^{-1})))} \cdot s^{-1}\,ds \\
&\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{2}^{\infty} \exp(-c_1 t(\phi')^{-1}(\phi'(t^{-1})/s))\, s^{-((d_1/\alpha_2)+1)}\,ds \qquad (4.12)\\
&= \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \\
&\quad\times \sum_{n=1}^{\infty} \int_{2^n}^{2^{n+1}} \exp(-c_1 t(\phi')^{-1}(\phi'(t^{-1})/s))\, s^{-((d_1/\alpha_2)+1)}\,ds \\
&\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \sum_{n=1}^{\infty} \exp(-c_1 t(\phi')^{-1}(\phi'(t^{-1})/2^n))\, 2^{-n((d_1/\alpha_2)+1)} \\
&\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \sum_{n=1}^{\infty} \exp(-c_2 2^{n(1-\beta_2)})\, 2^{-n((d_1/\alpha_2)+1)} \\
&\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}.
\end{aligned}$$

Thus

$$I_2 \leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}.$$

(2-b) Next, we suppose that $\Phi(z)\phi(t^{-1}) \geqslant 1$. Following the same argument as (4.12), we find that

$$\begin{aligned}
I_{2,1} &\leqslant c\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \bar{q}(r,z)\,dr \\
&\leqslant \frac{c}{\Phi(z)V(z)} \int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r))\,dr \\
&\leqslant \frac{c}{\phi(t^{-1})\Phi(z)V(z)} \int_{2}^{\infty} \exp(-c_1 t(\phi')^{-1}(\phi'(t^{-1})/s))\,ds \leqslant \frac{c}{\phi(t^{-1})\Phi(z)V(z)}.
\end{aligned}$$

Thus,

$$I_2 \leqslant \frac{c}{\phi(t^{-1})V(z)\Phi(z)}.$$

Combining all the estimates above, we have proved the desired upper bounded estimates for $p(t,x,y)$.

Proof of the lower bound of $p(t,x,y)$. (1) Assume that $\Phi(z)\phi(t^{-1})\leqslant 1$. By (4.4) and (4.5), we can find a constant $c_1>1$ such that when $c_1\Phi(z)\leqslant r$,

$$\frac{\partial \bar q(r,z)}{\partial r}\leqslant -\frac{c_2}{r}\bar q(r,z)\leqslant -\frac{c_3}{rV(\Phi^{-1}(r))};$$

and when $0<r\leqslant c_1\Phi(z)$,

$$\frac{\partial \bar q(r,z)}{\partial r}\leqslant \frac{c_4}{r}\bar q(r,z).$$

Then since

$$\int_{r_0}^\infty \bar q(r,z)\,d_r\mathbb{P}(S_r\leqslant t)\leqslant 0,\quad r_0>0,$$

according to the arguments of (4.7) and (4.8), we have

$$\begin{aligned}p(t,x,y)\geqslant& c_5\int_0^{2c_1/\phi(t^{-1})}\bar q(r,z)\,d_r\mathbb{P}(S_r\geqslant t)\\ \geqslant&-c_5\int_{c_1\Phi(z)}^{2c_1/\phi(t^{-1})}\mathbb{P}(S_r\geqslant t)\,d_r\bar q(r,z)-c_5\int_0^{c_1\Phi(z)}\mathbb{P}(S_r\geqslant t)\,d_r\bar q(r,z)\\ \geqslant& c_6\phi(t^{-1})\int_{c_1\Phi(z)}^{2c_1/\phi(t^{-1})}\frac{1}{V(\Phi^{-1}(r))}\,dr-\frac{c_7\phi(t^{-1})}{V(z)\Phi(z)}\int_0^{c_1\Phi(z)}r\,dr\\ =&:I_{1,1}-I_{1,2}.\end{aligned}$$

Noting that

$$I_{1,2}\leqslant \frac{c_7c_1^2\phi(t^{-1})\Phi(z)}{2V(z)},$$

and changing the variable $s=r\phi(t^{-1})$, we have

$$p(t,x,y)\geqslant c_6\int_{c_1\Phi(z)\phi(t^{-1})}^{2c_1}\frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds-\frac{c_7c_1^2\phi(t^{-1})\Phi(z)}{2V(z)}.$$

Combining this with Lemma 3.4 yields

$$\begin{aligned}(c_*+1)p(t,x,y)\geqslant& c_*c_6\int_{c_1\Phi(z)\phi(t^{-1})}^{2c_1}\frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds\\ &-\frac{c_*c_7c_1^2\phi(t^{-1})\Phi(z)}{2V(z)}+\frac{c_8\phi(t^{-1})\Phi(z)}{V(z)}\\ \geqslant& c_*c_6\int_{c_1\Phi(z)\phi(t^{-1})}^{2c_1}\frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds\\ \geqslant& c_9\int_{\Phi(z)\phi(t^{-1})}^{2}\frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds,\end{aligned}$$

where $c_*>0$ is chosen small enough so that $c_*c_7c_1^2<2c_8$.
(2) Next we assume that $\Phi(z)\phi(t^{-1})\geqslant 1$. Due to (4.2), (4.4) and (4.6) we can find a constant $0<c_1<1$ such that when $r\leqslant c_1\Phi(z)$,

$$\frac{\partial \bar q(r,z)}{\partial r}\geqslant \frac{c_2}{r}\bar q(r,z);$$

when $r \geqslant c_1\Phi(z)$,

$$\frac{\partial \bar q(r,z)}{\partial r} \geqslant -\frac{c_3}{rV(\Phi^{-1}(r))}.$$

Then according to (3.1) and the arguments of (4.7) and (4.9), we know that

$$\begin{aligned}
p(t,x,y) \geqslant & -c_4\int_{c_1/(2\phi(t^{-1}))}^{\infty} \bar q(r,z)\, d_r\mathbb{P}(S_r \leqslant t)\\
\geqslant & c_4\int_{c_1/(2\phi(t^{-1}))}^{c_1\Phi(z)} \mathbb{P}(S_r \leqslant t)\, d_r\bar q(r,z) + c_4\int_{c_1\Phi(z)}^{\infty} \mathbb{P}(S_r \leqslant t)\, d_r\bar q(r,z)\\
\geqslant & \frac{c_5}{V(z)\Phi(z)}\int_{c_1/(2\phi(t^{-1}))}^{c_1\Phi(z)} \exp(-c_6 t(\phi')^{-1}(t/r))\, dr\\
& - c_7\int_{c_1\Phi(z)}^{\infty}\frac{1}{rV(\Phi^{-1}(r))}\exp(-c_8 t(\phi')^{-1}(t/r))\, dr\\
=: & I_{2,1} - I_{2,2}.
\end{aligned}$$

Changing the variable $s = r\phi(t^{-1})$ and using (3.1) and (3.3), we have

$$\begin{aligned}
I_{2,1} \geqslant & \frac{c_9}{\phi(t^{-1})V(z)\Phi(z)}\int_{c_1/2}^{c_1\Phi(z)\phi(t^{-1})} \exp(-c_6 t(\phi')^{-1}(t\phi(t^{-1})/s))\, ds\\
\geqslant & \frac{c_9}{\phi(t^{-1})V(z)\Phi(z)}\int_{c_1/2}^{c_1} \exp(-c_6 t(\phi')^{-1}(t\phi(t^{-1})/s))\, ds\\
\geqslant & \frac{c_9}{\phi(t^{-1})V(z)\Phi(z)}\exp(-c_6 t(\phi')^{-1}(t\phi(t^{-1})/c_1)) \geqslant \frac{c_{10}}{\phi(t^{-1})V(z)\Phi(z)},
\end{aligned} \tag{4.13}$$

where we used (3.1) and (3.3) in the last inequality above. Here, we observe that by (1.12) and (4.1)

$$\begin{aligned}
\frac{1}{\phi(t^{-1})V(z)\Phi(z)} = & \frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))}\frac{1}{\Phi(z)\phi(t^{-1})}\frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(\Phi(z)))}\\
\geqslant & \frac{c_{11}}{V(\Phi^{-1}(1/\phi(t^{-1})))}\frac{1}{(\Phi(z)\phi(t^{-1}))^{1+d_2/\alpha_1}}.
\end{aligned} \tag{4.14}$$

On the other hand, changing the variable $s = r\phi(t^{-1})$ we have

$$\begin{aligned}
I_{2,2} = & \frac{c_7}{V(\Phi^{-1}(1/\phi(t^{-1})))}\\
& \times\int_{c_1\Phi(z)\phi(t^{-1})}^{\infty}\frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, s^{-1}\, \exp(-c_8 t(\phi')^{-1}(t\phi(t^{-1})/s))\, ds\\
\leqslant & \frac{c_{12}}{V(\Phi^{-1}(1/\phi(t^{-1})))}\\
& \times\int_{c_1\Phi(z)\phi(t^{-1})}^{\infty} s^{-1-d_1/\alpha_2}\exp(-c_8 t(\phi')^{-1}(t\phi(t^{-1})/s))\, ds\\
\leqslant & \frac{c_{13}}{V(\Phi^{-1}(1/\phi(t^{-1})))}\frac{1}{(\Phi(z)\phi(t^{-1}))^{d_1/\alpha_2}}\\
& \times\exp(-c_8 t(\phi')^{-1}(t\phi(t^{-1})/(\Phi(z)\phi(t^{-1}))))\\
\leqslant & \frac{c_{14}}{V(\Phi^{-1}(1/\phi(t^{-1})))}\exp\Big(-c_{15}(\Phi(z)\phi(t^{-1}))^{1/(1-\beta_2)}\Big),
\end{aligned} \tag{4.15}$$

where we used (1.12) and (4.1) again in the first inequality, and used (3.1) and (3.3) in the last inequality. From (4.13)–(4.15) we can choose a constant $c^* > 0$ large enough such that for all $t > 0$ and $z \geqslant 0$ with $\Phi(z)\phi(t^{-1}) > c^*$, it holds that

$$I_{2,2} \leqslant 2^{-1} I_{2,1}.$$

Thus,

$$p(t, x, y) \geqslant \frac{2^{-1} c_{10}}{\phi(t^{-1}) V(z) \Phi(z)}.$$

Moreover, if $c^* \geqslant \Phi(z)\phi(t^{-1}) \geqslant 1$, we see from Lemma 3.4 that

$$p(t, x, y) \geqslant \frac{c_{16}}{\phi(t^{-1}) V(z) \Phi(z)}.$$

Therefore, when $\Phi(z)\phi(t^{-1}) \geqslant 1$,

$$p(t, x, y) \geqslant \frac{c_{17}}{\phi(t^{-1}) V(z)\, \Phi(z)}.$$

This completes the proof. □

5. Local spatial motions

5.1. **Time derivative of heat kernel estimates for diffusion processes.** In this section, we consider the diffusion case where the associated heat kernel $q(t, x, y)$ satisfies (1.15). In the following, set

$$\bar{q}(t, x, r) := \frac{1}{V(x, \Phi^{-1}(t))} \exp\left(-m(t, r)\right), \quad t, r > 0 \text{ and } x \in M.$$

Applying Lemma 3.2 to $V(x, \cdot)$, $\Phi(\cdot)$ and $1/m(\cdot, r)$, we may and do assume that all $V(x, \cdot)$, $\Phi(\cdot)$ and $m(\cdot, r)$ are differentiable, that $V(x, \cdot)$ and $\Phi(\cdot)$ satisfy the property like (3.6), and that $m(\cdot, r)$ satisfies

$$m(t, r) \simeq -t \frac{\partial m(t, r)}{\partial t} \quad \text{for all } t, r > 0. \tag{5.1}$$

Then similar to those in Lemma 4.1, we have the following time derivative estimates for $\bar{q}(t, x, r)$ defined above.

Lemma 5.1. *Under all assumptions above, there exist constants $c_0, c_0^* > 0$ such that for all $t, r > 0$ and $x \in M$,*

$$\left| \frac{\partial \bar{q}(t, x, r)}{\partial t} \right| \leqslant \frac{c_0}{t V(x, \Phi^{-1}(t))} \exp\left(-c_0^* m(t, r)\right) =: \frac{c_0}{t} \bar{q}^*(t, x, r), \tag{5.2}$$

and that there exist constants $c_1, c_2 > 0$, $c_* \in (0, 1)$ and $c^* \in (1, \infty)$ such that for all $x \in M$,

$$\frac{\partial \bar{q}(t, x, r)}{\partial t} \leqslant -c_1 \frac{\bar{q}(t, x, r)}{t} \quad \text{if } \Phi(r) \leqslant c_* t, \tag{5.3}$$

and

$$\frac{\partial \bar{q}(t, x, r)}{\partial t} \geqslant c_2 \frac{\bar{q}(t, x, r)}{t} \quad \text{if } \Phi(r) \geqslant c^* t. \tag{5.4}$$

Proof. Since

$$\frac{\partial \bar{q}(t,x,r)}{\partial t} = \bar{q}(t,x,r)\left(-\frac{\partial_r V(x,\Phi^{-1}(t))(\Phi^{-1}(t))'}{V(x,\Phi^{-1}(t))} - \partial_t m(t,r)\right),$$

we have by (3.6) and (5.1)

$$\frac{\bar{q}(t,x,r)}{t}\left(-c_1 + c_2 m(t,r)\right) \leqslant \frac{\partial \bar{q}(t,x,r)}{\partial t} \leqslant \frac{\bar{q}(t,x,r)}{t}\left(-c_1^{-1} + c_2^{-1} m(t,r)\right). \tag{5.5}$$

This along with the fact that $re^{-r} \leqslant 2e^{-r/2}$ for all $r > 0$ immediately yields (5.2).

Note that, by (1.18), $m(\Phi(r), r) \simeq 1$. If $\Phi(r) \leqslant c_* t$, then by (1.17),

$$\frac{1}{m(t,r)} \simeq \frac{m(\Phi(r),r)}{m(t,r)} \geqslant c_3 \left(\frac{t}{\Phi(r)}\right)^{1/(\alpha_2-1)} \geqslant c_3 \left(\frac{1}{c_*}\right)^{1/(\alpha_2-1)}$$

and so

$$m(t,r) \leqslant c_4 {c_*}^{1/(\alpha_2-1)}.$$

By this and (5.5), we can take $c_* > 0$ small enough such that (5.3) is satisfied.

Similarly, if $\Phi(r) \geqslant c^* t$ for $c^* > 1$ large enough, then $m(t,r) \geqslant 2c_1/c_2$ and so

$$-c_1 + c_2 m(t,r) \geqslant c_1,$$

which combined with (5.5) in turn gives us (5.4). The proof is complete. □

5.2. **Two-sided estimates for $p(t,x,y)$.**

Proof of Theorem 1.8. We will closely follow the approach of Theorem 1.6 but need to carry out some non-trivial modifications. We fix $x \in M$ and, for simplicity, we again denote $z = d(x,y)$, and write $\bar{q}(t,x,z)$, $\bar{q}^*(t,x,z)$ and $V(x,r)$ as $\bar{q}(t,z)$, $\bar{q}^*(t,z)$ and $V(r)$, respectively. The proof is divided into two parts again.

Proof of the upper bound of $p(t,x,y)$. By (1.5) and (1.15) we have

$$p(t,x,y) \asymp \int_0^\infty \bar{q}(r,z)\, d_r \mathbb{P}(S_r \geqslant t) = I_1 + I_2,$$

where

$$I_1 := \int_0^{2/\phi(t^{-1})} \bar{q}(r,z)\, d_r \mathbb{P}(S_r \geqslant t) \quad \text{and} \quad I_2 := -\int_{2/\phi(t^{-1})}^\infty \bar{q}(r,z)\, d_r \mathbb{P}(S_r \leqslant t).$$

Following the same arguments as (4.8) and (4.9), and using Proposition 3.3 and (5.2), we have

$$\begin{aligned} I_1 \leqslant & c\bar{q}(2/\phi(t^{-1}),z) - \int_0^{2/\phi(t^{-1})} \mathbb{P}(S_r \geqslant t)\, d_r \bar{q}(r,z) \\ \leqslant & c\bar{q}(2/\phi(t^{-1}),z) + c\int_0^{2/\phi(t^{-1})} r\phi(t^{-1}) \cdot \frac{1}{r} \cdot \bar{q}^*(r,z)\, dr \\ =: & \ c\bar{q}(2/\phi(t^{-1}),z) + cI_{1,1} \end{aligned}$$

and

$$\begin{aligned} I_2 = & -\bar{q}(r,z)\mathbb{P}(S_r \leqslant t)\big|_{2/\phi(t^{-1})}^\infty + \int_{2/\phi(t^{-1})}^\infty \mathbb{P}(S_r \leqslant t)\, d_r \bar{q}(r,z) \\ \leqslant & c\bar{q}(2/\phi(t^{-1}),z) + c\int_{2/\phi(t^{-1})}^\infty \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot \frac{1}{r} \cdot \bar{q}^*(r,z)\, dr \end{aligned}$$

$$=: c\bar{q}(2/\phi(t^{-1}), z) + cI_{2,1}.$$

(1-a) Suppose that $\Phi(z)\phi(t^{-1}) \leqslant 1$. Then

$$\begin{aligned} I_{1,1} = \phi(t^{-1}) &\int_0^{\Phi(z)} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_0^* m(r,z))\, dr \\ &+ \phi(t^{-1}) \int_{\Phi(z)}^{2/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))}\, dr \\ &=: I_{1,1,1} + I_{1,1,2}. \end{aligned}$$

According to (1.12), (4.1) and (1.17), we have

$$\begin{aligned} I_{1,1,1} =& \phi(t^{-1}) \int_0^{\Phi(z)} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_0^* m(r,z))\, dr \\ =& \phi(t^{-1}) \sum_{n=0}^{\infty} \int_{\Phi(z)/2^{n+1}}^{\Phi(z)/2^n} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_0^* m(r,z))\, dr \\ \leqslant& \phi(t^{-1}) \sum_{n=0}^{\infty} \frac{\Phi(z)/2^{n+1}}{V(\Phi^{-1}(\Phi(z)/2^{n+1}))} \exp(-c_0^* m(\Phi(z)/2^n, z)) \\ \leqslant& \frac{c\phi(t^{-1})\Phi(z)}{V(z)} \sum_{n=0}^{\infty} 2^{n((d_2/\alpha_1)-1)} \exp(-c_1 m(\Phi(z),z) 2^{n/(\alpha_2-1)}) \\ \leqslant& \frac{c\phi(t^{-1})\Phi(z)}{V(z)}, \end{aligned} \tag{5.6}$$

where in the last inequality we used the fact that $m(\Phi(z), z) \simeq 1$. This estimate along with (4.11) and (4.10) yields that

$$I_1 \leqslant cI_{1,1,2} + cI_{1,1} \leqslant c_1 I_{1,1} \leqslant c_2 \phi(t^{-1}) \int_{\Phi(z)}^{2/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))}\, dr.$$

(1-b) Suppose that $\Phi(z)\phi(t^{-1}) \geqslant 1$. Then also by (1.12), (4.1) and (1.17), we have

$$\begin{aligned} I_{1,1} =& \phi(t^{-1}) \int_0^{2/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_0^* m(r,z))\, dr \\ =& \phi(t^{-1}) \sum_{n=0}^{\infty} \int_{2/(2^{n+1}\phi(t^{-1}))}^{2/(2^n\phi(t^{-1}))} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_0^* m(r,z))\, dr \\ \leqslant& \phi(t^{-1}) \sum_{n=0}^{\infty} \frac{2/(2^{n+1}\phi(t^{-1}))}{V(\Phi^{-1}(2/(2^{n+1}\phi(t^{-1}))))} \exp(-c_0^* m(2/(2^n\phi(t^{-1})), z)) \\ \leqslant& \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \sum_{n=0}^{\infty} 2^{n((d_2/\alpha_1)-1)} \exp\left(-c_1 m(1/\phi(t^{-1}), z) 2^{n/(\alpha_2-1)}\right) \\ \leqslant& \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\left(-c_2 m(1/\phi(t^{-1}), z)\right), \end{aligned}$$

which implies that

$$I_1 \leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\left(-c_0 m(1/\phi(t^{-1}), z)\right).$$

(2-a) Suppose that $\Phi(z)\phi(t^{-1}) \leqslant 1$. Then by the argument of (4.12),

$$\begin{aligned} I_{2,1} &= \int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \bar{q}^*(r,z)\, dr \\ &\leqslant c \int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \frac{1}{V(\Phi^{-1}(r))}\, dr \leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}, \end{aligned}$$

hence

$$I_2 \leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}.$$

(2-b) We now consider the case that $\Phi(z)\phi(t^{-1}) \geqslant 1$, which is more complex and difficult than the previous case.

To get the estimate for $I_{2,1}$, we need to consider the following two functions inside the exponential terms of $\bar{q}^*(r,z)$ and the estimates of $\mathbb{P}(S_r \leqslant t)$ respectively:

$$G_1(r) = t(\phi')^{-1}(t/r) \quad \text{and} \quad G_2(r) = m(r,z) \tag{5.7}$$

for all $r > 0$ and fixed $z, t > 0$. Note that, by (3.3), (1.17) and the facts that ϕ' and $m(\cdot, z)$ are non-increasing on $(0,\infty)$, $G_1(r)$ is a non-decreasing function on $(0,\infty)$ such that $G_1(0) = 0$ and $G_1(\infty) = \infty$, and $G_2(r)$ is a non-increasing function on $(0,\infty)$ such that $G_2(0) = \infty$ and $G_2(\infty) = 0$. Thus, there is a unique $r_0 = r_0(z,t) \in (0,\infty)$ such that $G_1(r_0) = G_2(r_0)$, $G_1(r) \geqslant G_2(r)$ when $r \geqslant r_0$, and $G_1(r) \leqslant G_2(r)$ when $r \leqslant r_0$.

On the other hand, when $\Phi(z)\phi(t^{-1}) \geqslant 1$, by (3.1), (1.18) and the fact that $m(\cdot, z)$ is non-increasing on $(0,\infty)$,

$$\begin{aligned} G_1(1/\phi(t^{-1})) &= t(\phi')^{-1}(t\phi(t^{-1})) \simeq t(\phi')^{-1}(\phi'(t^{-1})) = 1 \\ &\leqslant c_1 m(\Phi(z), z) \leqslant c_1 m(1/\phi(t^{-1}), z) = c_1 G_2(1/\phi(t^{-1})) \end{aligned}$$

and

$$\begin{aligned} G_1(\Phi(z)) \geqslant & G_1(1/\phi(t^{-1})) \simeq t(\phi')^{-1}(\phi'(t^{-1})) = 1 \\ \geqslant & c_2 m(\Phi(z), z) = c_2 G_2(\Phi(z)) \end{aligned}$$

where constants c_1, c_2 are independent of t and z. Hence there are constants $c_3, c_4 > 0$ independent of t and z such that

$$\frac{2}{\phi(t^{-1})} \leqslant c_3 r_0 \leqslant c_4 \Phi(z). \tag{5.8}$$

Combining all the estimates above, we find that

$$\begin{aligned} I_{2,1} &= \int_{2/\phi(t^{-1})}^{\infty} \exp(-c_0 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \bar{q}^*(r,z)\, dr \\ &\leqslant \frac{1}{V(\Phi^{-1}(2/\phi(t^{-1})))} \int_{2/\phi(t^{-1})}^{\infty} \frac{1}{r} \cdot \exp(-c_0 t(\phi')^{-1}(t/r)) \cdot \exp(-c_0^* m(r,z))\, dr \\ &\leqslant \frac{c_5}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{2/\phi(t^{-1})}^{c_3 r_0} \frac{1}{r} \cdot \exp(-c_0^* m(r,z))\, dr \\ &\quad + \frac{c_5}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{c_3 r_0}^{\infty} \frac{1}{r} \cdot \exp(-c_0 t(\phi')^{-1}(t/r))\, dr \\ &=: \frac{c_5}{V(\Phi^{-1}(1/\phi(t^{-1})))} I_{2,1,1} + \frac{c_5}{V(\Phi^{-1}(1/\phi(t^{-1})))} I_{2,1,2}. \end{aligned}$$

According to (1.17),

$$\begin{aligned} I_{2,1,1} &\leqslant \int_0^{c_3 r_0} \frac{1}{r} \cdot \exp(-c_0^* m(r,z))\, dr = \sum_{n=0}^{\infty} \int_{c_3 r_0/(2^{n+1})}^{c_3 r_0/(2^n)} \frac{1}{r} \cdot \exp(-c_0^* m(r,z))\, dr \\ &\leqslant c_6 \sum_{n=0}^{\infty} \exp(-c_0^* m(c_3 r_0/2^n, z)) \leqslant c_6 \sum_{n=0}^{\infty} \exp\left(-c_7 m(r_0,z) 2^{n/(\alpha_2-1)}\right) \\ &\leqslant c_6 \exp(-c_8 G_2(r_0)). \end{aligned}$$

On the other hand, by (3.2),

$$\begin{aligned} I_{2,1,2} &= \sum_{n=0}^{\infty} \int_{2^n c_3 r_0}^{2^{n+1} c_3 r_0} \frac{1}{r} \cdot \exp(-c_0 t (\phi')^{-1}(t/r))\, dr \\ &\leqslant c_9 \sum_{n=0}^{\infty} \exp(-c_0 t (\phi')^{-1}(t/(2^n c_3 r_0))) \leqslant c_9 \sum_{n=0}^{\infty} \exp(-c_{10} t (\phi')^{-1}(t/r_0) 2^{n(1-\beta_2)}) \\ &\leqslant c_9 \exp(-c_{11} t (\phi')^{-1}(t/r_0)) = c_9 \exp(-c_{11} G_1(r_0)) = c_9 \exp(-c_{11} G_2(r_0)). \end{aligned}$$

Putting these estimates together, we have

$$I_{2,1} \leqslant \frac{c_{12}}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp(-c_{13} G_2(r_0)).$$

Since $G_2(r_0) = G_1(r_0) \leqslant c_{14} G_1(1/\phi(t^{-1}))$ (thanks to (5.8)), we obtain

$$I_2 \leqslant \frac{c_{14}}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp(-c_{15} G_2(r_0)). \tag{5.9}$$

Next, we rewrite the exponential term in the right hand side of (5.9). By the fact that $m(r_0, z) = G_2(r_0) = G_1(r_0) = t(\phi')^{-1}(t/r_0)$ and the definition of $m(r_0, z)$, we have

$$\frac{r_0}{t(\phi')^{-1}(t/r_0)} \simeq \Phi\left(\frac{z}{t(\phi')^{-1}(t/r_0)}\right).$$

Let $s_0 = (\phi')^{-1}(t/r_0)$. Then $t/r_0 = \phi'(s_0)$ and, by (3.1),

$$\frac{1}{\phi((ts_0)/t)} = \frac{1}{\phi(s_0)} \simeq \frac{1}{\phi'(s_0) s_0} \simeq \Phi\left(\frac{z}{ts_0}\right). \tag{5.10}$$

Thus, $G_2(r_0) = G_1(r_0) = t(\phi')^{-1}(t/r_0) = t(\phi')^{-1}(t/(t/\phi'(s_0))) = ts_0$. This together with (5.9) and (5.10) yields that

$$I_2 \leqslant \frac{c_{16}}{V(x, \Phi^{-1}(1/\phi(t^{-1})))} \exp(-c_{17} n(t,z)),$$

where $n = n(t,z)$ satisfies

$$\frac{1}{\phi(n/t)} \simeq \Phi\left(\frac{z}{n}\right).$$

Combining all the estimates above, we get the desired upper bounded estimates for $p(t,x,y)$.

Proof of the lower bound of $p(t,x,y)$**.** **(1)** Suppose that $\Phi(z)\phi(t^{-1}) \leqslant 1$. In this case, the proof is almost the same as the jump case except that one uses Lemma

5.1 in place of Lemma 4.1. Nevertheless for reader's convenience, we present a proof here. By (5.2) and (5.3), we can find a constant $c_1 > 1$ such that when $c_1\Phi(z) \leqslant r$,

$$\frac{\partial \bar{q}(r,z)}{\partial r} \leqslant -\frac{c_2}{r}\bar{q}(r,z); \tag{5.11}$$

and when $0 < r \leqslant c_1\Phi(z)$,

$$\frac{\partial \bar{q}(r,z)}{\partial r} \leqslant \frac{c_3}{r}\bar{q}^*(r,z). \tag{5.12}$$

Using the fact that

$$\int_{r_0}^{\infty} \bar{q}(r,z)\, d_r\mathbb{P}(S_r \leqslant t) \leqslant 0, \quad r_0 > 0,$$

following the arguments of (4.7) and (4.8), and applying (5.11) and (5.12), we find that

$$\begin{aligned} p(t,x,y) &\geqslant c_4\phi(t^{-1}) \int_{c_1\Phi(z)}^{2c_1/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))}\, dr \\ &\quad - c_5\phi(t^{-1}) \int_0^{c_1\Phi(z)} \frac{1}{V(\Phi^{-1}(r))} \exp(-c_6 m(r,z))\, dr \\ &=: I_{1,1} - I_{1,2}. \end{aligned}$$

According to (5.6),

$$I_{1,2} \leqslant \frac{c_7\phi(t^{-1})\Phi(z)}{V(z)},$$

and so

$$p(t,x,y) \geqslant c_4 \int_{c_1\Phi(z)\phi(t^{-1})}^{2c_1} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds - \frac{c_7\phi(t^{-1})\Phi(z)}{V(z)}.$$

Therefore, combining this estimates and Lemma 3.4, we obtain

$$p(t,x,y) \geqslant c_8 \int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds.$$

See the end of part **(1)** in the proof of the lower bound estimates of $p(t,x,y)$ in Subsection 4.2.

(2) Suppose that $\Phi(z)\phi(t^{-1}) \geqslant 1$. By (5.4) and (5.2), we can find a constant $0 < c_1 < 1$ such that when $r \leqslant c_1\Phi(z)$,

$$\frac{\partial \bar{q}(r,z)}{\partial r} \geqslant \frac{c_2}{r}\bar{q}(r,z), \tag{5.13}$$

while for $r \geqslant c_1\Phi(z)$,

$$\frac{\partial \bar{q}(r,z)}{\partial r} \geqslant -\frac{c_3}{r}\bar{q}^*(r,z) \tag{5.14}$$

Using (3.1), (5.13) and (5.14), following the arguments of (4.7) and (4.9), and noting that $\int_a^{c_1\Phi(z)} = \int_a^\infty - \int_{c_1\Phi(z)}^\infty$, we find that for any $a \in (0, c_1\Phi(z)]$,

$$\begin{aligned} p(t,x,y) \geqslant& c_4 \int_a^{c_1\Phi(z)} \frac{1}{V(\Phi^{-1}(r))} \cdot \frac{1}{r} \cdot \exp(-c_5 t(\phi')^{-1}(t/r)) \cdot \exp(-c_5 m(r,z))\, dr \\ &- c_6 \int_{c_1\Phi(z)}^{\infty} \frac{1}{V(\Phi^{-1}(r))} \cdot \frac{1}{r} \cdot \exp(-c_7 t(\phi')^{-1}(t/r)) \cdot \exp(-c_7 m(r,z))\, dr \end{aligned}$$

$$\begin{aligned}
\geqslant& c_4\int_a^\infty \frac{1}{V(\Phi^{-1}(r))}\cdot\frac{1}{r}\cdot\exp(-c_5 t(\phi')^{-1}(t/r))\cdot\exp(-c_5 m(r,z))\,dr\\
&-(c_4+c_6)\int_{c_1\Phi(z)}^\infty \frac{1}{V(\Phi^{-1}(r))}\cdot\frac{1}{r}\cdot\exp(-c_7 t(\phi')^{-1}(t/r))\,dr\\
=:& c_4 I_{2,1}(a)-(c_4+c_6)I_{2,2}.
\end{aligned}$$

By (1.12), (4.1) and (3.2),

$$\begin{aligned}
I_{2,2}&=\sum_{n=0}^\infty\int_{c_1 2^n\Phi(z)}^{c_1 2^{n+1}\Phi(z)}\frac{1}{V(\Phi^{-1}(r))}\cdot\frac{1}{r}\cdot\exp(-c_7 t(\phi')^{-1}(t/r))\,dr\\
&\leqslant\sum_{n=0}^\infty\frac{1}{V(\Phi^{-1}(c_1 2^n\Phi(z)))}\exp(-c_7 t(\phi')^{-1}(t/(c_1 2^n\Phi(z))))\\
&\leqslant\frac{c_8}{V(z)}\sum_{n=0}^\infty 2^{-nd_1/\alpha_2}\exp(-c_9 2^{n(1-\beta_2)}t(\phi')^{-1}(t/\Phi(z)))\\
&\leqslant\frac{c_{10}}{V(z)}\exp(-c_{11}t(\phi')^{-1}(t/\Phi(z))).
\end{aligned}$$

Similar argument as above yields that

$$\begin{aligned}
&\int_a^\infty\frac{1}{V(\Phi^{-1}(r))}\cdot\frac{1}{r}\cdot\exp(-2c_5 t(\phi')^{-1}(t/r))\,dr\\
&=\sum_{n=0}^\infty\int_{a2^n}^{a2^{n+1}}\frac{1}{V(\Phi^{-1}(r))}\cdot\frac{1}{r}\cdot\exp(-2c_5 t(\phi')^{-1}(t/r))\,dr\\
&\geqslant\frac12\sum_{n=0}^\infty\frac{1}{V(\Phi^{-1}(a2^{n+1}))}\exp(-2c_5 t(\phi')^{-1}(t/(a2^{n+1})))\\
&\geqslant\frac{c_{12}}{V(\Phi^{-1}(a))}\sum_{n=0}^\infty 2^{-nd_2/\alpha_1}\exp(-c_{13}2^{n(1-\beta_1)}t(\phi')^{-1}(t/a))\\
&\geqslant\frac{c_{14}}{V(\Phi^{-1}(a))}\exp(-c_{15}t(\phi')^{-1}(t/a)),
\end{aligned}\tag{5.15}$$

where all the constants c_k's are independent of a.

Without loss of generality we assume $c_{11}<2c_5<c_{15}$, and let

$$G_1^*(r)=c_{11}t(\phi')^{-1}(t/r)\quad\text{and}\quad G_2^*(r)=2c_{15}m(r,z)\tag{5.16}$$

for all $r>0$ and fixed $z,t>0$. We let $r_0=r_0(t,z)>0$ be the unique constant such that $G_1^*(r_0)=G_2^*(r_0)$. Note that $G_1^*(r)\geqslant G_2^*(r)$ for all $r\geqslant r_0$. In particular,

$$t(\phi')^{-1}(t/r)\geqslant m(r,z),\quad r\geqslant r_0,$$

and so

$$\exp(-c_5 t(\phi')^{-1}(t/r))\cdot\exp(-c_5 m(r,z))\geqslant\exp(-2c_5 t(\phi')^{-1}(t/r)),\quad r\geqslant r_0.\tag{5.17}$$

By (3.1) and (3.3), we can choose $C_0>1$ large such that

$$\begin{aligned}
G_1^*(c_1C_0/\phi(t^{-1}))=&c_{11}t(\phi')^{-1}(t\phi(t^{-1})/(c_1C_0))\\
\geqslant&c_{11}t(\phi')^{-1}(c_{17}\phi'(t^{-1})/(c_1C_0))\\
>&(4c_{16}c_{15})\vee(2\log(2c_{10}(c_4+c_6)/(c_4c_{14}))),
\end{aligned}\tag{5.18}$$

where in the last inequality $c_{16} > 0$ satisfies that $m(c_1\Phi(z), z) \leqslant c_{16}$ (due to (1.17) and (1.18)). Then since G_1^* is non-decreasing, if $\Phi(z)\phi(t^{-1}) \geqslant C_0$

$$G_1^*(c_1\Phi(z)) \geqslant G_1^*(c_1C_0/\phi(t^{-1})) \geqslant 4c_{16}c_{15} \geqslant 4c_{15}m(c_1\Phi(z), z) = 2G_2^*(c_1\Phi(z)),$$

which, in particular, implies that $c_1\Phi(z) \geqslant r_0$. Thus, we can take $a = r_0$ in (5.15) and find that for $\Phi(z)\phi(t^{-1}) \geqslant C_0$,

$$I_{2,1}(r_0) \geqslant \frac{c_{14}}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_2^*(r_0)) = \frac{c_{14}}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_1^*(r_0)).$$

Therefore, combining all the inequalities above, we obtain that for $\Phi(z)\phi(t^{-1}) \geqslant C_0$,

$$p(t, x, y) \geqslant \frac{c_4c_{14}}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_1^*(r_0)) - \frac{c_{10}(c_4 + c_6)}{V(z)}\exp(-G_1^*(\Phi(z))).$$

By the fact that G_1^* is non-decreasing and (5.18), for $\Phi(z)\phi(t^{-1}) \geqslant C_0$,

$$\exp(2^{-1}G_1^*(c_1\Phi(z))) \geqslant \exp(2^{-1}G_1^*(c_1C_0/\phi(t^{-1}))) \geqslant 2c_{10}(c_4 + c_6)/(c_4c_{14}),$$

so that, using again the fact that G_1^* is a non-decreasing function, we have

$$\begin{aligned}\frac{c_4c_{14}}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_1^*(r_0)) &\geqslant \frac{c_4c_{14}}{V(z)}\exp(-2^{-1}G_1^*(c_1\Phi(z)))\\ &= \frac{c_4c_{14}}{V(z)}\exp(2^{-1}G_1^*(c_1\Phi(z)))\exp(-G_1^*(\Phi(z)))\\ &\geqslant \frac{2c_{10}(c_4 + c_6)}{V(z)}\exp(-G_1^*(\Phi(z))).\end{aligned}$$

Thus for $\Phi(z)\phi(t^{-1}) \geqslant C_0$

$$p(t, x, y) \geqslant \frac{2^{-1}c_4c_{14}}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_1^*(r_0)). \tag{5.19}$$

By (3.1),

$$G_1^*\left(1/\phi(t^{-1})\right) = c_{11}t(\phi')^{-1}(t\phi(t^{-1})) \simeq 1.$$

Using this, (1.12), (3.3) and (5.8), we have

$$\begin{aligned}&\frac{1}{V(\Phi^{-1}(r_0))}\exp(-2^{-1}G_1^*(r_0))\\ &\geqslant \frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))}\exp(-G_1^*(r_0))\left[\frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(r_0))}\exp\left(\frac{c_{18}G_1^*(r_0)}{G_1^*(1/\phi(t^{-1}))}\right)\right]\\ &\geqslant \frac{c_{19}}{V(\Phi^{-1}(1/\phi(t^{-1})))}\exp(-G_1^*(r_0))\left[(r_0\phi(t^{-1}))^{-d_2}\exp(c_{20}(r_0\phi(t^{-1}))^{1/(1-\beta_1)})\right]\\ &\geqslant \frac{c_{21}}{V(\Phi^{-1}((1/\phi(t^{-1})))}\exp(-G_1^*(r_0)),\end{aligned}$$

where in the last inequality we used the fact that $\inf_{r>0} r^{-d}\exp(c_{20}r^{1/(1-\beta)}) > 0$.

Combining the inequality above with (5.19), we obtain that for $\Phi(z)\phi(t^{-1}) \geqslant C_0$

$$p(t, x, y) \geqslant \frac{c_{22}}{V(\Phi^{-1}(1/\phi(t^{-1}))))}\exp(-G_1^*(r_0)).$$

Furthermore, by the same argument as that for the expression of $G_1(r_0)$ at the end of part **(2-b)** in the proof of upper bound for $p(t, x, y)$, we arrive at that for $\Phi(z)\phi(t^{-1}) \geqslant C_0$

$$p(t, x, y) \geqslant \frac{c_{22}}{V(\Phi^{-1}(1/\phi(t^{-1}))))}\exp(-c_{23}n(t, z)),$$

where

$$\frac{1}{\phi(n/t)} \simeq \Phi\left(\frac{z}{n}\right).$$

Combining this with Lemma 3.4, we finish the proof. □

At the end of the section, we give a corollary of Theorems 1.6 and 1.8, which is concerned with explicit forms for the estimate (1.14).

Corollary 5.2. *For non-local and local Dirichlet forms for which heat kernels are given by* (1.13) *and* (1.15)*, the following two statements hold for* $p(t, x, y)$ *given by* (1.5)*.*

(i) *If* $d_2 < \alpha_1$ *and* $\Phi(d(x, y))\phi(t^{-1}) \leqslant 1$*, then*

$$p(t, x, y) \simeq \frac{1}{V(x, \Phi^{-1}(1/\phi(t^{-1})))}.$$

(ii) *If* $d_1 > \alpha_2$ *and* $\Phi(d(x, y))\phi(t^{-1}) \leqslant 1$*, then*

$$p(t, x, y) \simeq \frac{\Phi(d(x, y))\phi(t^{-1})}{V(x, d(x, y))}.$$

(iii) *If* $d_1 = d_2 = \alpha_1 = \alpha_2$ *and* $\Phi(d(x, y))\phi(t^{-1}) \leqslant 1$*, then*

$$p(t, x, y) \simeq \frac{1}{V(x, \Phi^{-1}(1/\phi(t^{-1})))} \log\left(\frac{2}{\Phi(d(x, y))\phi(t^{-1})}\right).$$

Proof. We use the same notation as in the proof of Theorem 1.6. We have already observed in (4.10) and (4.11) that, when $\Phi(z)\phi(t^{-1}) \leqslant 1$,

$$\begin{aligned}\int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds =& \phi(t^{-1}) \int_{\Phi(z)}^{2/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))}\, dr \\ \geqslant& \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))}\end{aligned}$$

and

$$\int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds = \phi(t^{-1}) \int_{\Phi(z)}^{2/\phi(t^{-1})} \frac{1}{V(\Phi^{-1}(r))}\, dr \geqslant \frac{c\Phi(z)\phi(t^{-1})}{V(z)}.$$

(i) Suppose that $d_2 < \alpha_1$. Then for $\Phi(z)\phi(t^{-1}) \leqslant 1$,

$$\begin{aligned}&\int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds \\ &= \frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{\Phi(z)\phi(t^{-1})}^{2} \frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(s/\phi(t^{-1})))}\, ds \\ &\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{\Phi(z)\phi(t^{-1})}^{2} s^{-d_2/\alpha_1}\, ds \\ &\leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{0}^{2} s^{-d_2/\alpha_1}\, ds \leqslant \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))},\end{aligned} \tag{5.20}$$

where in the first inequality we used (1.12) and (4.1).

(ii) Suppose that $d_1 > \alpha_2$. Then using (1.12) and (4.1) again, we have that for $\Phi(z)\phi(t^{-1}) \leqslant 1$,

$$
\begin{aligned}
&\int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds \\
&= \frac{1}{V(z)} \int_{\Phi(z)\phi(t^{-1})}^{2} \frac{V(\Phi^{-1}(\Phi(z)))}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds \leqslant \frac{c}{V(z)} \int_{\Phi(z)\phi(t^{-1})}^{2} \left(\frac{\Phi(z)}{s/\phi(t^{-1})}\right)^{d_1/\alpha_2} ds \\
&\leqslant \frac{c}{V(z)} (\Phi(z)\phi(t^{-1}))^{d_1/\alpha_2} \int_{\Phi(z)\phi(t^{-1})}^{\infty} s^{-d_1/\alpha_2}\,ds \leqslant \frac{c\Phi(z)\phi(t^{-1})}{V(z)}.
\end{aligned}
$$

(iii) When $d_1 = d_2 = \alpha_1 = \alpha_2$, by the argument of (5.20), we have for $\Phi(z)\phi(t^{-1}) \leqslant 1$,

$$
\begin{aligned}
&\int_{\Phi(z)\phi(t^{-1})}^{2} \frac{1}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds \\
&= \frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{\Phi(z)\phi(t^{-1})}^{2} \frac{V(\Phi^{-1}(1/\phi(t^{-1})))}{V(\Phi^{-1}(s/\phi(t^{-1})))}\,ds \\
&\simeq \frac{1}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{\Phi(z)\phi(t^{-1})}^{2} s^{-1}\,ds = \frac{c}{V(\Phi^{-1}(1/\phi(t^{-1})))} \log\left(\frac{2}{\phi(t^{-1})\Phi(z)}\right).
\end{aligned}
$$

Therefore, the desired assertion now follows from all the estimates above. □

Proof of Theorem 1.3. The conclusion for the case that $d(x,y)\phi(t^{-1}) \leqslant 1$ immediately follows from Theorem 1.6 (i), Theorem 1.8 (i) and Corollary 5.2. When $d(x,y)\phi(t^{-1}) \geqslant 1$, the assertion for pure jump type Dirichlet form $(\mathcal{E},\mathcal{F})$ is a direct consequence of Theorem 1.6 (ii); for the case that $(\mathcal{E},\mathcal{F})$ is local, according to Theorem 1.8 (ii), $n := n(t, d(x,y))$ is now determined by

$$
\frac{1}{\phi(n/t)} \simeq \left(\frac{d(x,y)}{n}\right)^{\alpha}, \quad t > 0, x, y \in M.
$$

This is,

$$
\bar{\phi}_\alpha(n/t) = \frac{(n/t)^\alpha}{\phi(n/t)} \simeq \left(\frac{d(x,y)}{t}\right)^{\alpha}, \quad t > 0, x, y \in M.
$$

Then we can prove the desired assertion. □

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