Analysis on manifolds and the Laplace operator

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1 Introduction: Laplacian and the heat equation in $\mathbb{R}^n$

The Laplace operator. The Laplace operator Δ in $\mathbb{R}^n$ is defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

where $x_1, ..., x_n$ are the Cartesian coordinates.

Why do we consider the Laplace operator? One of the reason is that it has appeared in many equations derived from physical phenomena.

1. Laplace discovered in 18th century: if $U(x)$ is a potential of gravitational field then it satisfies the equation $\Delta U = 0$ in a free space. The gravitational potential of a particle at $0 \in \mathbb{R}^3$ is given by $U(x) = -\frac{C}{|x|}$. It is possible to verify that $\Delta \frac{1}{|x|} = 0$ in $\mathbb{R}^3 \setminus \{0\}$ so whence $\Delta U = 0$ follows. The potential of a body located in an open set $\Omega \subset \mathbb{R}^3$ is given by

$$U(x) = -\int_{\Omega} \frac{d\nu(y)}{|x-y|},$$

where ν is the mass density in Ω. It follows then that $\Delta U(x) = 0$ outside $\overline{\Omega}$.

2. Maxwell's equations (1860): each component $u = u(t, x)$ of an electromagnetic field satisfies the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

in a region free of charges and currents.

3. Fourier: the temperature $u(t,x)$ satisfies the heat equation
$$\frac{\partial u}{\partial t} = k\Delta u$$
is any region free of sources and sinks of the heat.

4. Einstein (1905): the probability density $u(t,x)$ that the Brownian particle starting at 0 hits x at time t, satisfies the diffusion equation
$$\frac{\partial u}{\partial t} = D\Delta u.$$

5. Schrödinger (1926): the wave function $\psi(t,x)$ of an elementary particle in a free space satisfies the equation
$$i\frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m}\Delta\psi.$$

The Laplace operator is tightly related to the metric properties of the Euclidean space. This can be seen from the following alternative definition of the Laplacian: if $f \in C^2(\mathbb{R}^n)$ then
$$\Delta f(x) = \lim_{r\to 0+} \frac{1}{c_n r^2}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy - f(x)\right)$$
for some $c_n > 0$.

Exercise 1.1 *Show that these two definitions of Laplacian are consistent.*

Exercise 1.2 *Show that*
$$\begin{aligned} \Delta\frac{1}{|x|^{n-2}} &= 0 \quad \textit{in } \mathbb{R}^n \setminus \{0\} \quad (n \neq 2), \\ \Delta \log|x| &= 0 \quad \textit{in } \mathbb{R}^2 \setminus \{0\}. \end{aligned}$$

The heat equation. Our main concern will be the heat equation
$$\frac{\partial u}{\partial t} = \Delta u$$
where $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. The following *Cauchy problem* is associated with this equation: given a function $f(x)$ on $\mathbb{R}^n$, find $u(t,x)$ such that
$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u|_{t=0} = f. \end{cases}$$
where function u is sought in the class $C^2(\mathbb{R}_+ \times \mathbb{R}^n) \cap C([0,\infty) \times \mathbb{R}^n)$. In general, solution to this problem is not unique, but as we will see below, it is unique in the class of bounded function. So, let us define *Bounded Cauchy Problem* (BCP) as a task to solve the above Cauchy problem with additional restriction $\sup|u| < \infty$.

Theorem 1.1 *If $f \in C_b(\mathbb{R}^n)$, then BCP has a solution $u(t,x)$ given by*
$$u(t,\cdot) = p_t * f,$$
where
$$p_t(x) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

The function p_t is called *Gauss-Weierstrass* function or the *heat kernel.* Here $u * v$ is the convolution defined by

$$u * v\,(x) = \int_{\mathbb{R}^n} u\,(x-y)\,v\,(y)\,dy.$$

Theorem 1.2 *For any $f \in C_b(\mathbb{R}^n)$, BCP has at most one solution.*

Proof. We will use the following properties of p_t:

1. $\displaystyle\int_{\mathbb{R}^n} p_t(x)dx = 1 \quad (t > 0),$
2. $\dfrac{\partial p_t}{\partial t} = \Delta p_t,$
3. semigroup property: $p_t * p_s = p_{t+s},$
4. $\displaystyle\int_{\{|x|<r\}} p_t(x)dx \to 1$ as $t \to 0$ for any $r > 0$.

Thus, $\{p_t\}_{t>0}$ is an approximation of identity (or Dirac family).
Proof of the property 1. We will use the facts

$$\int_{-\infty}^{\infty} \mathrm{e}^{-s^2} ds = \sqrt{\pi},$$
$$\int_{-\infty}^{\infty} \mathrm{e}^{-s^2/4t} ds = \sqrt{4\pi t}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} p_t(x)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{4t}\right) dx_1 \cdots dx_n \\
&= \frac{1}{(4\pi t)^{n/2}} \prod_{k=1}^{n} \int_{\mathbb{R}} \exp\left(-\frac{x_k^2}{4t}\right) dx_k \\
&= 1.
\end{aligned}$$

Proof of the property 4. We have

$$\begin{aligned}
\int_{\{|x|<r\}} p_t(x)dx &= 1 - \int_{\{|x|\geq r\}} p_t(x)dx \\
\int_{\{|x|\geq r\}} p_t(x)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\{|x|\geq r\}} \mathrm{e}^{-|x|^2/4t} dx \\
&= \frac{1}{\pi^{n/2}} \int_{\{|y|\geq r/\sqrt{4t}\}} \mathrm{e}^{-|y|^2} dy \qquad \left(\frac{x}{\sqrt{4t}} = y,\, dy = \frac{dx}{(4t)^{n/2}}\right) \\
&\to 0 \quad (t \to 0).
\end{aligned}$$

For the proof of 2 and 3, we use the Fourier transform:

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-ix\xi} u(x)dx, \quad \xi \in \mathbb{R}^n.$$

We have

$$\hat{p}_t(\xi) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t} - ix\xi\right) dx \tag{1.1}$$
$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{4t} - i(x_1\xi_1 + \cdots + x_n\xi_n)\right) dx_1 \cdots dx_n.$$

Consider

$$\int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t} - is\lambda\right) ds = \int_{-\infty}^{\infty} \exp\left(-\frac{s^2 + 4it\lambda s + (2it\lambda)^2 - (2it\lambda)^2}{4t}\right) ds$$
$$= \exp(-t\lambda^2) \int_{-\infty}^{\infty} \exp\left(-\frac{(s + 2i\lambda t)^2}{4t}\right) ds.$$

By the change $z = s + 2i\lambda t$, the last integral can be treated as a contour integral along the line $\mathrm{Im}z = 2i\lambda t$. Using the standard tools based on the Cauchy integral formula, one reduces the integral to $\mathrm{Im}z = 0$, whence

$$\int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t} - is\lambda\right) ds = \mathrm{e}^{-t\lambda^2} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4t}\right) ds$$
$$= \sqrt{4\pi t}\,\mathrm{e}^{-t\lambda^2}. \tag{1.2}$$

Hence, from (1.1) and (1.2), we obtain

$$\hat{p}_t(\xi) = \mathrm{e}^{-t(\xi_1^2 + \cdots + \xi_n^2)} = \mathrm{e}^{-t|\xi|^2}. \tag{1.3}$$

Proof of the property 3: It is obvious from (1.3) that $\hat{p}_{t+s} = \hat{p}_t \hat{p}_s$ whence by the property of convolution

$$p_{t+s} = p_t * p_s.$$

Proof of the property 2: We have

$$\widehat{\frac{\partial p_t}{\partial t}} = \frac{\partial}{\partial t}\hat{p}_t = \frac{\partial}{\partial t}\mathrm{e}^{-t|\xi|^2} = -|\xi|^2 \hat{p}_t$$

and

$$\widehat{\frac{\partial p_t}{\partial x_k}} = \int_{\mathbb{R}^n} \mathrm{e}^{-ix\xi} \frac{\partial}{\partial x_k} p_t(x) dx$$
$$= -\int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \mathrm{e}^{-ix\xi} p_t(x) dx$$
$$= i\xi_k \int_{\mathbb{R}^n} \mathrm{e}^{-ix\xi} p_t(x) dx$$
$$= i\xi_k \hat{p}_t,$$

$$\widehat{\frac{\partial^2 p_t}{\partial x_k^2}} = (i\xi_k)^2 \hat{p}_t = -\xi_k^2 \hat{p}_t$$

and therefore

$$\widehat{\Delta p_t} = -(\xi_1^2 + \cdots + \xi_n^2)\hat{p}_t = -|\xi|^2 \hat{p}_t = \widehat{\frac{\partial p_t}{\partial t}}.$$

Hence, we obtain

$$\frac{\partial p_t}{\partial t} = \Delta p_t.$$

Using the above properties of the Gauss-Weierstrass function, we can prove that the function $u(t,\cdot) = p_t * f$ solves BCP.
Boundedness:

$$
\begin{aligned}
|u| &\leq \int_{\mathbb{R}^n} p_t(x-y)|f(y)|dy \\
&\leq \sup|f| \int_{\mathbb{R}^n} p_t(x-y)dy \\
&= \sup|f|.
\end{aligned}
$$

The last equality follows from the property 1 of the heat kernel. Hence, $\sup|u| \leq \sup|f|$.

The heat equation:

$$
\frac{\partial u}{\partial t} - \Delta u = \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} - \Delta\right) p_t(x-y)f(y)dy = 0
$$

because of the property 2 of the heat kernel.
Smoothness: $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ follows from the fact that we can interchange the order of derivation and integration in $u(t,\cdot) = p_t * f$ any number of times.

Initial data: Let us show that

$$
u(t,x) \to f(x) \text{ as } t \to 0.
$$

We have

$$
\begin{aligned}
u(t,x) - f(x) &= \int_{\mathbb{R}^n} p_t(x-y)f(y)dy - \int_{\mathbb{R}^n} p_t(x-y)f(x)dy \\
&= \int_{\mathbb{R}^n} p_t(x-y)(f(y)-f(x))dy.
\end{aligned}
$$

By the continuity of f at x,

$$
\forall \varepsilon > 0, \quad \exists \delta > 0 \quad s.t. \quad |y-x| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon.
$$

Thus we have

$$
\begin{aligned}
&|\int_{\{|y-x|<\delta\}} p_t(x-y)(f(y)-f(x))dy| \leq \varepsilon \cdot 1 = \varepsilon, \\
&|\int_{\{|y-x|\geq\delta\}} p_t(x-y)(f(y)-f(x))dy| \leq 2\sup|f| \int_{\{|z|\geq\delta\}} p_t(z)dz,
\end{aligned}
$$

where the right hand side tends to 0 as $t \to 0$ by the property 4. Allowing x to vary in a compact set and using the uniform continuity of f, we obtain that $u(t,x) \to f(x)$ as $t \to 0$ locally uniformly. Therefore, if we extend $u(t,x)$ to $t = 0$ by $u(0,x) = f(x)$, we obtain $u \in C([0,\infty) \times \mathbb{R}^n)$ and $u|_{t=0} = f$.
■

Proof of Theorem 1.2. It suffices to prove that if u is a solution to BCP with $f = 0$ then $u \equiv 0$.
Compare u to the function $V(t,x) = |x|^2 + 2nt$, which is non-negative, satisfies the equation

$$
\frac{\partial V}{\partial t} = \Delta V
$$

(because $\frac{\partial V}{\partial t} = 2n = \Delta V$) and admits the estimate

$$V(x,t) \geq |x|^2 \to \infty \quad \text{as } x \to \infty.$$

Fix $\varepsilon > 0$ and compare u and εV in the cylinder $Q = B(0,R) \times [0,T]$. At the bottom of the cylinder (that is, at $t = 0$) we have $u = 0 \leq \varepsilon V$. At the side boundary of the cylinder (that is, at $|x| = R$) we have $u(x) \leq \sup |u|$ and $\varepsilon V(x) \geq \varepsilon R^2$. Choosing R so that $\varepsilon R^2 \geq \sup |u|$, we obtain that $u \leq \varepsilon V$ on the side boundary of Q.

Next, we apply the *parabolic comparison principle*: if u_1, u_2 are two solutions in Q of the heat equation such that $u_1 \leq u_2$ at $t = 0$ and at $|x| = R$, then $u_1 \leq u_2$ in Q. Hence, we conclude that $u(x) \leq \varepsilon V(x)$ in Q. Letting $R \to \infty$ and $T \to \infty$ we obtain $u(t,x) \leq \varepsilon V(t,x)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Letting $\varepsilon \to 0$, we obtain $u(t,x) \leq 0$. In the same way, we obtain $u \geq 0$. Therefore $u \equiv 0$. ■

2 Laplace operator on weighted manifolds

Smooth manifolds. Recall that a *smooth manifold* M is a Hausdorff topological space with a countable base endowed with a C^∞-*atlas*. The latter means that M is covered by *charts* (where a chart on M is an open set homeomorphic to an open subset of $\mathbb{R}^n$; here n is fixed and called the *dimension* of M), so that any chart features the *local coordinates* pulled back from the Cartesian coordinates in $\mathbb{R}^n$, and on the intersection of any two charts the change of coordinates is given by C^∞-functions. For example, if $x^1, ..., x^n$ are the coordinates in a chart U and $y^1, ..., y^n$ are the coordinates in a chart V then in the intersection $U \cap V$ the functions $x^i\left(y^1, ..., y^n\right)$ and $y^j\left(x^1, ..., x^n\right)$ are defined and C^∞-smooth.

If f is a (real valued) function on a manifold M then we write $f \in C^m(M)$ (or $f \in C^m$) if the restriction of f to any chart is a C^m-function of the local coordinates. Obviously, the set $C^m(M)$ is a linear space with respect to the usual addition of functions and multiplication by constant. Denote by $C_0^m(M)$ its subspace consisting of functions with compact support.

Tangent vectors. Let M be a smooth manifold of dimension n.

Definition. A mapping $\xi : C^\infty(M) \to \mathbb{R}$ is called an $\mathbb{R}$-*differentiation* at a point $x_0 \in M$ if

- ξ is linear
- ξ satisfies the Leibniz rule in the following form:

  $$\xi(fg) = \xi(f)\, g(x_0) + \xi(g)\, f(x_0),$$

  for all $f, g \in C^\infty$.

The set of all $\mathbb{R}$-differentiations at x_0 is denoted by $T_{x_0}M$. For any $\xi, \eta \in T_{x_0}M$ one defines the sum $\xi + \eta$ as the sum of two functions on C^∞, and similarly one defined $\lambda\xi$ for any $\lambda \in \mathbb{R}$. It is easy to check that both $\xi + \eta$ and $\lambda\xi$ are again $\mathbb{R}$-differentiations, so that $T_{x_0}M$ is a linear space over $\mathbb{R}$. The linear space $T_{x_0}M$ is called the *tangent space* of M at x_0, and its elements (that is, $\mathbb{R}$-differentiations) are also called *tangent vectors* at x_0.

Proposition 2.1 *The tangent space has the dimension n.*

The proof is left as an exercise.

In any local coordinate system $x^1, ..., x^n$, and operator of partial differentiation $\left.\frac{\partial}{\partial x^k}\right|_{x_0}$ is a tangent vector at x_0. It is easy to check that all n partial differentiations are linearly independent, whence it follows that they form a basis in $T_{x_0}M$ (in fact, the proof of Proposition 2.1 goes the other way around – one shows that these vectors form a basis whence the conclusion about the dimension follows). Hence, any tangent vector ξ at x_0 can be represented in the form

$$\xi = \xi^i \frac{\partial}{\partial x^i}$$

where we assume summation over repeated index i. The numbers ξ^i are called the *components* of ξ in the coordinate system $x^1, ..., x^n$. Using the components ξ^i, we can write, for any smooth function f,

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i}. \tag{2.1}$$

Normally, one uses the following notation for $\xi(f)$:

$$\xi(f) \equiv \frac{\partial f}{\partial \xi}$$

so that the identity (2.1) takes a familiar form

$$\frac{\partial f}{\partial \xi} = \xi^i \frac{\partial f}{\partial x^i}.$$

This explains why any $\mathbb{R}$-differentiation is called a tangent vector.

Riemannian metric. Let M be a smooth n-dimensional manifold. A *Riemannian metric* on M is a family $\{\mathbf{g}(x)\}_{x\in M}$ such that for any $x \in M$, $\mathbf{g}(x)$ is a symmetric, positive definite, bilinear form on the tangent space T_xM, smoothly depending on $x \in M$ (one can say also that $\mathbf{g}$ is a smooth $(0,2)$-tensor field on M). Using the Riemannian metric, one defines an inner product in any tangent space T_xM by

$$\langle \xi, \eta \rangle \equiv \mathbf{g}(x)(\xi, \eta).$$

In local coordinates $x^1, ..., x^n$, this takes the form

$$\langle \xi, \eta \rangle = g_{ij}(x)\, \xi^i \eta^j \tag{2.2}$$

where $(g_{ij}(x))_{i,j=1}^n$ is a symmetric positive definite $n \times n$ matrix, and the functions $g_{ij}(x)$ are called the components of the metric $\mathbf{g}$. The condition that $\mathbf{g}(x)$ smoothly depends on x means that all the components $g_{ij}(x)$ are C^∞-functions in the corresponding charts.

Often the metric $\mathbf{g}$ is written in the form

$$\mathbf{g} = g_{ij}\, dx^i dx^j \tag{2.3}$$

where dx^i is considered as a linear functional on T_xM (that is, a *covector*) defined by

$$dx^i(\xi) = \xi^i,$$

and $dx^i dx^j$ is in fact the tensor product of the covectors defined by

$$dx^i dx^j(\xi, \eta) = dx^i(\xi)\, dx^j(\eta).$$

With this explanation, it is clear that (2.3) is an equivalent form of (2.2).

Definition. A *Riemannian manifold* is a couple $(M, \mathbf{g})$ where $\mathbf{g}$ is a Riemannian metric on a smooth manifold M.

Riemannian measure. The Riemannian structure allows to introduce on M canonical measure ν which is called the *Riemannian volume.* Measure ν is defined in any chart $x^1, ..., x^n$ by

$$d\nu = \sqrt{D} dx^1 ... dx^n, \tag{2.4}$$

where $D(x) = \det(g_{ij}(x))$. That is, for any Borel set A in the given chart, we have

$$\nu(A) = \int_A \sqrt{D} dx^1 ... dx^n. \tag{2.5}$$

The next statement is left as an exercise.

Proposition 2.2 *If a set A lies in the intersection of two chart then $\nu(A)$ is the same for both charts.*

This allows to extend measure ν to *all* Borel sets on M, thus obtaining a Borel measure ν on M. In fact, ν is a Radon measure (that is, it is finite on compacts) because by (2.5) $\nu(A)$ is finite if A is a compact set lying inside a chart, and the same holds for any compact set $A \subset M$ because it can be covered by a finite number of charts.

Gradient. The *gradient* is an operator ∇ on $C^\infty(M)$ such that, for any $f \in C^\infty(M)$, ∇f is a vector field on M, that is, $\nabla f(x)$ is a tangent vector at $x \in M$; the latter is defined by the requirement that

$$\langle \nabla f(x), \xi \rangle = \xi(f) \quad \text{for any } \xi \in T_x M. \tag{2.6}$$

Indeed, the mapping $\xi \mapsto \xi(f)$ is a linear functional on $T_x M$, and by the Riesz representation theorem, there exists a unique vector $a \in T_x M$ such that this linear functional is given by $\langle a, \xi \rangle$; hence, we set $\nabla f(x) = a$. In the local coordinates $x^1, ..., x^n$, the defining identity (2.6) takes the form (all evaluated at the point x)

$$g_{ij}(\nabla f)^i \xi^j = \frac{\partial f}{\partial x^j} \xi^j,$$

which is equivalent to

$$g_{ij}(\nabla f)^i = \frac{\partial f}{\partial x^j}, \quad \text{for any } j = 1, ..., n. \tag{2.7}$$

The left hand side in (2.7) is the matrix product of the matrix $(g_{ij})_{i,=1}^n$ with the column $\left((\nabla f)^i\right)_{i=1}^n$. Denoting by (g^{ij}) the inverse matrix to (g_{ij}), we obtain from (2.7)

$$(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x^j}, \quad \text{for any } i = 1, ..., n.$$

Divergence. The *divergence* is an operator div on smooth vector fields on M such that for any smooth vector field v, $\operatorname{div} v$ is a smooth function on M. It is defined by means of the following statement.

Proposition 2.3 (The divergence theorem / the definition of the divergence) *For any C^∞-vector field $v(x)$ on M there exists a unique smooth function, denoted by $\operatorname{div} v$, such that the following identity holds*

$$\int_M \operatorname{div} v \, u \, d\nu = -\int_M \langle v, \nabla u \rangle d\nu, \tag{2.8}$$

for any $u \in C_0^\infty(M)$.

Proof. The uniqueness of div$v$ is simple: if there are two candidates for div$v$, say $(\mathrm{div}v)'$ and $(\mathrm{div}v)''$ then for all test functions u

$$\int (\mathrm{div}v)'\, u\, d\nu = \int (\mathrm{div}v)''\, u\, d\nu,$$

whence $(\mathrm{div}v)' = (\mathrm{div}v)''$ by the general properties of integration.

To prove the existence of div$v$, let us first show that div$v$ exists in any chart. Namely, if U is a chart on M with coordinates $x^1, ..., x^n$ and $D = \det(g_{ij})$ in this chart then, for any $u \in C_0^\infty(U)$,

$$\begin{aligned} \int_U \langle u, \nabla u\rangle d\nu &= \int_U g_{ij} v^j (\nabla u)^i \sqrt{D} dx \\ &= \int_U g_{ij} v^j g^{ik} \frac{\partial u}{\partial x^k} \sqrt{D} dx. \end{aligned}$$

Note that

$$g_{ij} g^{ik} = \delta_j^k := \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

because the matrices (g_{ij}) and (g^{ij}) are mutually inverse. Hence, we obtain, using the integration-by-parts formula

$$\begin{aligned} \int_U \langle u, \nabla u\rangle d\nu &= \int_U \delta_j^k v^j \frac{\partial u}{\partial x^k} \sqrt{D} dx \\ &= \int_U v^k \frac{\partial u}{\partial x^k} \sqrt{D} dx \\ &= -\int_U u \frac{\partial}{\partial x^k} \left(v^k \sqrt{D} \right) dx \\ &= -\int_U u \frac{1}{\sqrt{D}} \frac{\partial}{\partial x^k} \left(v^k \sqrt{D} \right) d\nu. \end{aligned}$$

Comparing with (2.8) we see that the divergence can be defined by

$$\mathrm{div} v = \frac{1}{\sqrt{D}} \frac{\partial}{\partial x^k} \left(\sqrt{D} v^k \right). \tag{2.9}$$

If U and V are two charts then (2.9) defines two divergences, in U and in V. However, by the uniqueness statement, the two definitions of the divergence coincide in $U \cap V$. This argument shows that the formula (2.9) defines div$v$ as a function on the entire manifold M, and the divergence defined in this way satisfies the identity (2.8) for all test functions u compactly supported in one of the charts. By the standard argument using the partition of the unity, this identity extends to all compactly supported u. ■

Laplace-Beltrami operator. Having defined gradient and divergence, we can now define the Laplace operator (called also the Laplace-Beltrami operator) on any Riemannian manifold by $\Delta = \mathrm{div} \circ \nabla$, or, for any smooth function u on M,

$$\Delta u = \mathrm{div}(\nabla u),$$

so that Δu is also a smooth function on M. In the local coordinates, we have

$$\Delta = \frac{1}{\sqrt{D}} \frac{\partial}{\partial x^i} \left(\sqrt{D} g^{ij} \frac{\partial}{\partial x^j} \right) \tag{2.10}$$

where $D = \det(g_{ij})$. The following statement easily follows from (2.8).

Corollary 2.4 (Green's formulas) *If u and v are smooth functions on a Riemannian manifold M and one of them has a compact support then*

$$\int_M u\Delta v\, d\nu = -\int_M \langle \nabla u, \nabla v \rangle d\nu = \int_M v\Delta u\, d\nu. \tag{2.11}$$

Weighted manifolds. Fix a smooth positive function m on a Riemannian manifold (M, g) and consider a new measure μ on M given by $d\mu = m d\nu$. The triple (M, g, μ) is called a *weighted manifold.* The notion of the gradient remains the same and does not depend on μ but the notion of divergence, which will be denoted by div_μ, changes to satisfy the identity

$$\int_M \mathrm{div}_\mu v\, u\, d\mu = -\int_M \langle v, \nabla u \rangle d\mu, \tag{2.12}$$

for all smooth vector fields v and functions $u \in C_0^\infty(M)$. In the same way as in the proof of Proposition 2.3, one proves that $\mathrm{div}_\mu v$ exists and unique, and is given in local coordinates by

$$\mathrm{div}_\mu v = \frac{1}{m\sqrt{D}} \frac{\partial}{\partial x^i}\left(m\sqrt{D} v^i\right),$$

where $D = \det(g_{ij})$. Respectively, we define the *weighted Laplace operator* $\Delta_\mu := \mathrm{div}_\mu \circ \nabla$, which in local coordinates is given by

$$\Delta_\mu = \frac{1}{m\sqrt{D}} \frac{\partial}{\partial x^i}\left(m\sqrt{D} g^{ij} \frac{\partial}{\partial x^j}\right). \tag{2.13}$$

The Green formula remains true, that is, if u and v are smooth functions on a Riemannian manifold M and one of them has a compact support then

$$\int_M u\Delta_\mu v\, d\mu = -\int_M \langle \nabla u, \nabla v \rangle d\mu = \int_M v\Delta_\mu u\, d\mu. \tag{2.14}$$

Example. Consider the weighted manifold $(\mathbb{R}, \mu)$ where $d\mu = m dx$. Then by (2.13)

$$\Delta_\mu f = \frac{1}{m}\frac{d}{dx}\left(m\frac{df}{dx}\right) = f'' + \frac{m'}{m} f'.$$

For example, if $m = e^{-x^2}$ then

$$\Delta_\mu f = f'' - 2xf'.$$

The corresponding eigenvalue problem $\Delta_\mu f + \lambda f = 0$ has eigenfunction, which are Hermite polynomials $H_k(x)$ defined by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2},$$

where $k = 0, 1, 2, ...$, because

$$H_k'' - 2xH_k' + 2kH_k = 0.$$

Doob transform. Let (M, g, μ) be a weighted manifold and let $\widetilde{\mu}$ be another measure on M defined by

$$d\widetilde{\mu} = h^2 d\mu$$

where h is a smooth positive function on M. The Laplace operator $\Delta_{\widetilde{\mu}}$ on $(M, g, \widetilde{\mu})$ is given by

$$\Delta_{\widetilde{\mu}} = \operatorname{div}_{\widetilde{\mu}} \circ \nabla = \frac{1}{h^2} \operatorname{div}_{\mu} \left(h^2 \nabla\right) = \Delta_\mu u + \langle \frac{\nabla h^2}{h^2}, \nabla \rangle = \Delta_\mu u + 2 \langle \frac{\nabla h}{h}, \nabla \rangle. \tag{2.15}$$

Lemma 2.5 *If $\Delta_\mu h = 0$ then*

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ \Delta_h \circ h,$$

that is, $\Delta_{\widetilde{\mu}}$ is obtained from Δ_μ by the Doob transform.

More generally, if $\Delta_\mu h + \Phi h = 0$ where Φ is a function on M then

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ (\Delta_h + \Phi) \circ h,$$

Proof. We have, using (2.15) and $\frac{\Delta_\mu h}{h} = -\Phi$

$$\begin{aligned} \frac{1}{h} \Delta_\mu (hu) &= \frac{1}{h} (h \Delta_\mu u + 2 \langle \nabla h, \nabla u \rangle + u \Delta_\mu h) \\ &= \Delta_\mu + 2 \langle \frac{\nabla h}{h}, \nabla u \rangle + u \frac{\Delta_\mu h}{h} \\ &= \Delta_{\widetilde{\mu}} u - \Phi u, \end{aligned}$$

whence

$$\Delta_{\widetilde{\mu}} u = \frac{1}{h} \Delta_\mu (hu) + \frac{1}{h} \Phi (hu) = \frac{1}{h} (\Delta_\mu + \Phi) (hu),$$

which was to be proved. $\blacksquare$

Note that the mapping $f \mapsto hf$ is an isometry of $L^2(M, \widetilde{\mu})$ and $L^2(M, \mu)$, because

$$\int f^2 d\widetilde{\mu} = \int (hf)^2 d\mu.$$

Hence, if A is an operator in $L^2(M, \mu)$ then $\frac{1}{h} \circ A \circ h$ is an operator in $L^2(M, \widetilde{\mu})$, which is unitarily equivalent to A. In particular, all properties of the operator $\Delta_\mu + \Phi$ in $L^2(M, \mu)$ are identical to those of $\Delta_{\widetilde{\mu}}$ in $L^2(M, \widetilde{\mu})$.

Example. Consider a weighted manifold (M, g, μ) where $M = \mathbb{R}^n \setminus \{o\}$, g is the Euclidean metric, and μ is the Lebesgue measure. Let

$$h(x) = |x|^\beta = r^\beta$$

where $r = |x|$ is the polar radius and $\beta \in \mathbb{R}$. We have by (2.22)

$$\Delta h = \frac{\partial^2 h}{\partial r^2} + \frac{n-1}{r} \frac{\partial h}{\partial r} = \beta (\beta + n - 2) r^{\beta - 2}$$

so that

$$\Delta h - \frac{b}{r^2} h = 0$$

where $b = \beta^2 + (n-2)\beta$. By Lemma 2.5, we conclude that for the measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$,

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ \left(\Delta - \frac{b}{r^2}\right) \circ h.$$

Model manifolds. Consider $\mathbb{R}^n$ $(n \geq 2)$ with the polar coordinates (r, θ) where $r > 0$ and $\theta \in \mathbb{S}^{n-1}$ (for any $x \in \mathbb{R}^n$, we have $r = |x|$ and $\theta = x/|x|$). The standard Euclidean metric g_{eucl} is given in the polar coordinates by

$$g_{eucl} = dr^2 + r^2 d\theta^2,$$

where $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ induced by the Euclidean metric from the ambient space $\mathbb{R}^n$.

A *model manifold* is the Riemannian manifold $(\mathbb{R}^n, g)$ where g is a Riemannian metric given in the polar coordinates by

$$ds^2 = dr^2 + \psi^2(r) d\theta^2, \tag{2.16}$$

where $\psi(r)$ is a positive smooth function on $\mathbb{R}_+$ (in fact, any smooth positive function $\psi(r)$ on $\mathbb{R}_+$ satisfying in addition $\psi(0) = 0$, $\psi'(0) = 1$, $\psi''(0) = 0$ gives rise to a metric that can be extended from $\mathbb{R}^n \setminus \{o\}$ to the whole $\mathbb{R}^n$).

Let $\theta^1, ..., \theta^{n-1}$ be any local coordinates on $\mathbb{S}^{n-1}$, and let

$$d\theta^2 = \gamma_{ij} d\theta^i d\theta^j$$

be the Riemannian metric of $\mathbb{S}^{n-1}$. Then (2.16) means that the matrix (g_{ij}) of g in the coordinates $r, \theta^1, ..., \theta^{n-1}$ has the form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \;\ldots\; 0 \\ 0 & \\ \ldots & \boxed{\psi^2(r)\,\gamma_{ij}} \\ 0 & \end{pmatrix}$$

and its inverse is given by

$$(g^{ij}) = \begin{pmatrix} 1 & 0 \;\ldots\; 0 \\ 0 & \\ \ldots & \boxed{\psi^{-2}(r)\,\gamma^{ij}} \\ 0 & \end{pmatrix}.$$

We have also

$$D = \det(g_{ij}) = \psi^{2(n-1)} \det(\gamma_{ij}),$$

so that the Riemannian measure ν on $(\mathbb{R}^n, g)$ is given by

$$d\nu = \sqrt{D} dr d\theta^1 ... d\theta^{n-1} = \psi^{n-1} \sqrt{\det(\gamma_{ij})} dr d\theta^1 ... d\theta^{n-1}.$$

Using (2.10), we obtain for the Laplace operator of $(\mathbb{R}^n, g)$

$$\Delta = \frac{1}{\sqrt{D}} \frac{\partial}{\partial r} \left(\sqrt{D} g^{11} \frac{\partial}{\partial r} \right) \tag{2.17}$$

$$+ \sum_{i=2}^{n} \frac{1}{\sqrt{D}} \frac{\partial}{\partial r} \left(\sqrt{D} g^{1i} \frac{\partial}{\partial \theta^{i-1}} \right) + \sum_{i=2}^{n} \frac{1}{\sqrt{D}} \frac{\partial}{\partial \theta^{i-1}} \left(\sqrt{D} g^{i1} \frac{\partial}{\partial r} \right) \tag{2.18}$$

$$+ \sum_{i,j=2}^{n} \frac{1}{\sqrt{D}} \frac{\partial}{\partial \theta^{i-1}} \left(\sqrt{D} g^{ij} \frac{\partial}{\partial \theta^{j-1}} \right). \tag{2.19}$$

All terms in (2.18) vanish because $g^{1i} = 0$ for $i \geq 2$. Since $g^{11} = 1$, the right hand side term in (2.17) is equal to

$$\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \log \sqrt{D} \frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \log \psi^{n-1} \frac{\partial}{\partial r}.$$

The term in (2.19) is equal to

$$\sum_{i,j=1}^{n-1}\frac{1}{\sqrt{D}}\frac{\partial}{\partial\theta^i}\left(\sqrt{D}\psi^{-2}\gamma^{ij}\frac{\partial}{\partial\theta^j}\right)=\sum_{i,j=1}^{n-1}\psi^{-2}\frac{1}{\sqrt{\det\left(\gamma_{ij}\right)}}\frac{\partial}{\partial\theta^i}\left(\sqrt{\det\left(\gamma_{ij}\right)}\gamma^{ij}\frac{\partial}{\partial\theta^j}\right)=\psi^{-2}\Delta_\theta$$

where Δ_θ is the Laplace operator on $\mathbb{S}^{n-1}$. Hence, we conclude that

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}\log\psi^{n-1}\frac{\partial}{\partial r}+\frac{1}{\psi^2\left(r\right)}\Delta_\theta\,. \tag{2.20}$$

Let B_r be the ball in $\mathbb{R}^n$ of radius r centered at o, that is

$$B_r=\left\{x\in\mathbb{R}^n:|x|<r\right\}.$$

Set $V\left(r\right)=\nu\left(B_r\right)$ and observe that

$$V\left(r\right)=\int_{B_r}d\nu=\int_0^r\int_{\mathbb{S}^{n-1}}\psi^{n-1}\left(r\right)\sqrt{\det\left(\gamma_{ij}\right)}dr d\theta^1...\theta^{n-1}=\int_0^r\psi^{n-1}\left(r\right)\omega_n dr,$$

where

$$\omega_n=\int_{\mathbb{S}^{n-1}}\sqrt{\det\left(\gamma_{ij}\right)}d\theta^1...\theta^{n-1}$$

is the area of the sphere $\mathbb{S}^{n-1}$. We will refer to

$$V\left(r\right)=\omega_n\int_0^r\psi^{n-1}\left(r\right)dr$$

as the *volume function* of the model manifold and to

$$S\left(r\right):=V'\left(r\right)=\omega_n\psi^{n-1}\left(r\right)$$

as the *boundary area function.* It follows from (2.20) that

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{S'}{S}\frac{\partial}{\partial r}+\frac{1}{\psi^2}\Delta_\theta\,. \tag{2.21}$$

Example. For $\mathbb{R}^n$ with the standard Euclidean metric we have $\psi(r)=r$ and hence $S\left(r\right)=\omega_n r^{n-1}$,whence by (2.21)

$$\Delta_{\mathbb{R}^n}=\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}\,. \tag{2.22}$$

The sphere $\mathbb{S}^n$ can also be considered as a model manifold although the range of the polar radius r has to be restricted to $(0,\pi)$. In this case, $\psi\left(r\right)=\sin r$ and $S\left(r\right)=\omega_n\sin^{n-1}r$ whence by (2.21)

$$\Delta_{\mathbb{S}^n}=\frac{\partial^2}{\partial r^2}+(n-1)\cot r\frac{\partial}{\partial r}+\frac{1}{\sin^2 r}\Delta_{\mathbb{S}^{n-1}}\,. \tag{2.23}$$

The hyperbolic space $\mathbb{H}^n$ can be regarded as a model manifold with $\psi\left(r\right)=\sinh r$. Hence, in this case $S\left(r\right)=\omega_n\sinh^{n-1}r$ and

$$\Delta_{\mathbb{H}^n}=\frac{\partial^2}{\partial r^2}+(n-1)\coth r\frac{\partial}{\partial r}+\frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}\,. \tag{2.24}$$

The formula (2.23) can be iterated in dimension to obtain a full expansion of $\Delta_{\mathbb{S}^n}$ in the polar coordinates. Indeed, we obviously have

$$\Delta_{\mathbb{S}^1}=\frac{d^2}{d\theta^2},$$

where θ is the angle on $\mathbb{S}^{n-1}$. If (ρ,θ) are the polar coordinates on $\mathbb{S}^2$ then by (2.23)

$$\Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial\rho^2} + \cot\rho\frac{\partial}{\partial\rho} + \frac{1}{\sin^2\rho}\frac{\partial^2}{\partial\theta^2}.$$

If (r,ρ,θ) are the spherical coordinates on $\mathbb{S}^3$ then we obtain

$$\begin{aligned}\Delta_{\mathbb{S}^3} &= \frac{\partial^2}{\partial r^2} + 2\cot r\frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\Delta_{\mathbb{S}^2} \\ &= \frac{\partial^2}{\partial r^2} + 2\cot r\frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\left(\frac{\partial^2}{\partial\rho^2} + \cot\rho\frac{\partial}{\partial\rho} + \frac{1}{\sin^2\rho}\frac{\partial^2}{\partial\theta^2}\right).\end{aligned}$$

Also, from (2.22) we obtain

$$\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

and

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\rho^2} + \cot\rho\frac{\partial}{\partial\rho} + \frac{1}{\sin^2\rho}\frac{\partial^2}{\partial\theta^2}\right).$$

Finally, one can similarly define a *weighted model manifold* (M,g,μ) as follows. Let (M,g) be a model manifold as above, and $d\mu = m d\nu$ where the density $m = m(r)$ depends only on the polar radius. In this case, we obtain

$$S(r) = \omega_n m(r)\psi^{n-1}(r)$$

and

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2}\Delta_\theta. \tag{2.25}$$

3 Laplacian in $L^2(M,\mu)$

Let (M,μ) be a weighted manifold (where we drop the Riemannian metric g from the notation). Initially the operator Δ_μ is defined on smooth functions, in particular, on the space $\mathcal{D} := C_0^\infty(M)$, and by (2.14) Δ_μ is a symmetric operator on $\mathcal{D}$ in the sense that

$$(\Delta_\mu u, v)_{L^2} = (u, \Delta_\mu v)_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ is the inner product in $L^2(M,\mu)$. Our aim is to consider Δ_μ as an operator in $L^2 = L^2(M,\mu)$. It is natural to ask whether the operator $\Delta_\mu|_{\mathcal{D}}$ has a self-adjoint extension in L^2 and whether is it essentially self-adjoint.

The Dirichlet Laplace operator. Let $\mathcal{D}'$ be the space of distributions on M. Every function $u \in L^2$ can be considered as a distribution by

$$(u,f) = (u,f)_{L^2}, \quad \text{for any } f \in \mathcal{D},$$

where $(\cdot,\cdot)$ denotes the action of a distribution on a test function.

The operator Δ_μ extends to $\mathcal{D}'$ by

$$(\Delta_\mu u, f) = (u, \Delta_\mu f)$$

for all $u \in \mathcal{D}'$ and $f \in \mathcal{D}$ (note that $\Delta_\mu f \in \mathcal{D}$). Similarly, one defines the gradient ∇ in the distributional sense by

$$(\nabla u, v) = -(u, \operatorname{div}_\mu v)$$

for all $u \in \mathcal{D}'$ and vector fields $v \in \overrightarrow{\mathcal{D}}$ (note that $\operatorname{div}_\mu v \in \mathcal{D}$).

Let $W^1 = W^1(M, \mu)$ be the space of all functions $u \in L^2$, whose distributional gradient ∇u is also in L^2.

Lemma 3.1 *Space W^1 is a Hilbert space with the inner product*

$$(u, v)_{W^1} = \int_M uv \, d\mu + \int_M \langle \nabla u, \nabla v \rangle \, d\mu.$$

Proof. We need only prove that any Cauchy sequence $\{u_n\}$ in W^1 converges in W^1. Indeed, the sequence $\{u_n\}$ is Cauchy in L^2 and hence converges in L^2 to a function u, and the sequence $\{\nabla u_n\}$ is Cauchy in $\overrightarrow{L^2}$ and hence converges to a vector field $v \in \overrightarrow{L^2}$. We are left to show that $v = \nabla u$. Note that for any $w \in \overrightarrow{\mathcal{D}}$,

$$(\nabla u_n, w) \to (v, w) \quad \text{as } n \to \infty,$$

whereas by the definition of the distributional gradient

$$(\nabla u_n, w) = -(u_n, \operatorname{div}_\mu w) \to -(u, \operatorname{div}_\mu w),$$

whence

$$(v, w) = -(u, \operatorname{div}_\mu w)$$

and hence $v = \nabla u$. ■

Let

$$W_0^1 = \text{closure of } \mathcal{D} \text{ in } W^1$$

and

$$W_0^2 = \left\{ f \in W_0^1 : \Delta_\mu f \in L^2 \right\},$$

where $\Delta_\mu f$ is understood in distributional sense. Both W_0^1 and W_0^2 are Hilbert spaces with the corresponding inner products. Since $\mathcal{D} \subset W_0^2$, the operator $\Delta_\mu|_{W_0^2}$ is an extension of $\Delta_\mu|_{\mathcal{D}}$.

It easily follows from definitions of W_0^1, W_0^2 that the Green formula

$$\int_M u \Delta_\mu v \, d\mu = -\int_M \langle \nabla u, \nabla v \rangle \, d\mu \tag{3.1}$$

holds for all $u \in W_0^1$ and $v \in W_0^2$. Indeed, if $u \in \mathcal{D}$ then by the definitions of the distributional Laplacian and gradient, we have

$$\int_M u \Delta_\mu v \, d\mu = (\Delta_\mu v, u) = (v, \Delta_\mu u) = (v, \operatorname{div}(\nabla u)) = -(\nabla v, \nabla u) = -\int_M \langle \nabla u, \nabla v \rangle d\mu.$$

For any $u \in W_0^1$ there is a sequence $\{u_n\} \subset \mathcal{D}$ approximating u in W^1. Applying (3.1) to u_n and passing to the limit we finish the proof.

Theorem 3.2 ([3], [5], [11], [13]) *The operator $\Delta_\mu|_{W_0^2}$ is a densely defined self-adjoint non-positive definite operator in L^2.*

Remark. Hence, $\Delta_\mu|_{W_0^2}$ is a self-adjoint extension of $\Delta_\mu|_{\mathcal{D}}$. It is possible to prove that $\Delta_\mu|_{W_0^2}$ is a *unique* self-adjoint extension of $\Delta_\mu|_{\mathcal{D}}$ with the domain in W_0^1. The operator $\Delta_\mu|_{W_0^2}$ is called the *Dirichlet* Laplace operator.

Proof. Consider the quadratic form

$$\mathcal{E}(u, v) = (\nabla u, \nabla v)_{\overrightarrow{L^2}}$$

with the domain W_0^1. This form is obviously symmetric and, by Lemma 3.1, it is closed in L^2. By the theory of quadratic forms, $\mathcal{E}$ has a self-adjoint generator H such that for all $u \in \operatorname{dom}(H)$ and $v \in \operatorname{dom}(\mathcal{E})$,

$$\mathcal{E}(u,v) = (Hu, v).$$

The domain of H is dense in W_0^1 and is defined by

$$\operatorname{dom}(H) = \left\{ u \in W_0^1 : f \mapsto \mathcal{E}(u,f) \text{ is a bounded linear functional of } f \in W_0^1 \text{ in } L^2 \right\}.$$

This condition means, by the Riesz representation theorem, that there exists a unique function $g \in L^2$ such that

$$\mathcal{E}(u,f) = (g,f)_{L^2} \quad \text{for all } f \in W_0^1. \tag{3.2}$$

Since $\mathcal{D}$ is dense in W_0^1, we can rewrite (3.2) as follows:

$$\mathcal{E}(u,f) = (g,f) \quad \text{for all } f \in \mathcal{D}. \tag{3.3}$$

Using the definitions of the distributional Laplacian and gradient, we obtain, for any $u \in \operatorname{dom}(H)$ and $f \in \mathcal{D}$,

$$\mathcal{E}(u,f) = (\nabla u, \nabla f) = -(u, \operatorname{div}_\mu \nabla f) = -(u, \Delta_\mu f) = -(\Delta_\mu u, f),$$

and, comparing with (3.3), we see that for the distribution $\Delta_\mu u$ and for any $f \in \mathcal{D}$,

$$-(\Delta_\mu u, f) = (g,f),$$

whence $-\Delta_\mu u = g$. In particular, this means that $\Delta_\mu u \in L^2$ and hence $u \in W_0^2$; furthermore, $Hu = g = -\Delta_\mu u$. Conversely, it is easy to see that $u \in W_0^2$ implies $u \in \operatorname{dom}(H)$. Hence, $W_0^2 = \operatorname{dom}(H)$ and $H = -\Delta_\mu|_{W_0^2}$, which finishes the proof of self-adjointness of $\Delta_\mu|_{W_0^2}$.

The operator H is non-negative definite because for all $f \in \operatorname{dom}(H)$

$$(Hf,f) = \mathcal{E}(f,f) \geq 0.$$

Therefore, $\Delta_\mu|_{W_0^2}$ is non-positive definite. ■

Essential self-adjointness. In general, $\Delta_\mu|_{\mathcal{D}}$ may have other self-adjoint extensions related to other boundary conditions. A densely defined operator in L^2 is said to be *essentially* self-adjoint if it admits a unique self-adjoint extension. Next we will state a theorem which guarantees essential self-adjointness of the operator $\Delta_\mu|_{\mathcal{D}}$.

For any smooth curve $\gamma(t) : [a,b] \mapsto M$ one defines the *length* as follows. If the image of γ lies in a chart U with coordinates $x^1, ..., x^n$ then set

$$\operatorname{length}(\gamma) = \int_a^b \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}\,dt$$

where γ^i is a components of γ and $\dot{\gamma}^i$ is its derivative in t. It is possible to prove that this definition does not depend on the choice of the chart. Therefore, it can be extended to arbitrary γ.

Assuming that (M,g) is a connected Riemannian manifold, define the *geodesic distance* $d(x,y)$ between points $x,y \in M$ by

$$d(x,y) = \inf_\gamma \operatorname{length}(\gamma)$$

where γ is any smooth curve connecting x and y. It is possible to prove that $d(x,y)$ is always a metric so that (M,d) is a metric space. Denote by $B(x,r)$ the *geodesic ball* in M of radius r centered at $x \in M$, that is

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

We state the following two theorems without proof.

Theorem 3.3 (Hopf-Rinow) [10] *For a connected Riemannian manifold (M, g), the following conditions are equivalent:*

1. *(M, d) is a complete metric space.*
2. *All geodesic balls in M are precompact sets.*
3. *(M, g) is geodesically complete, that is, any geodesic line extends to one with the length ∞.*

Theorem 3.4 (Gaffney [6]) *If the weighted manifold (M, g, μ) is geodesically complete then $\Delta_\mu|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M, \mu)$.*

Remark. In the proof, one uses bump functions in balls, for which one can control the gradient, and the precompactness of balls (which follow from the geodesic completeness by the Hopf-Rinov theorem) guarantees that these functions are in $\mathcal{D}$.

Corollary 3.5 *Let (M, g, μ) be a geodesically complete connected weighted manifold. Let Φ be a smooth function on M such that the equation*

$$\Delta_\mu h + \Phi h = 0$$

has a positive solution h on M. Then the operator $(\Delta_\mu + \Phi)|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M, \mu)$.

Proof. Consider measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$. Then by Lemma 2.5

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ (\Delta_h + \Phi) \circ h.$$

The mapping $f \mapsto hf$ is an isometry of $L^2(M, \widetilde{\mu})$ and $L^2(M, \mu)$, and the space $\mathcal{D}$ is preserved by this mapping. Therefore, the operator $(\Delta_\mu + \Phi)|_{\mathcal{D}}$ in $L^2(M, \mu)$ is unitary equivalent to $\Delta_{\widetilde{\mu}}|_{\mathcal{D}}$ in $L^2(M, \widetilde{\mu})$. Since the latter operator is essentially self-adjoint by Theorem 3.4, so is $(\Delta_\mu + \Phi)|_{\mathcal{D}}$. ■

Example. Consider a manifold $M = \mathbb{R}^n \setminus \{o\}$ with the Euclidean metric and Lebesgue measure μ. For the function $h(x) = |x|^\beta$ we have

$$\frac{\Delta h}{h} = \frac{\beta^2 + (n-2)\beta}{|x|^2}.$$

The function $\beta \mapsto \beta^2 + (n-2)\beta$ attains the minimum for $\beta = 1 - n/2$, and this minimum is equal to $-\left(\frac{n-2}{2}\right)^2$. Set

$$\Phi(x) = \left(\frac{n-2}{2}\right)^2 |x|^{-2}$$

so that $\Delta h + \Phi h = 0$. Consider the measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$. Since the operator $\Delta_{\widetilde{\mu}}|_{\mathcal{D}}$ is non-negative definite in $L^2(M, \widetilde{\mu})$, we conclude that the operator $(\Delta_\mu + \Phi)|_{\mathcal{D}}$ is also non-negative definite in $L^2(M, \mu)$. Therefore, for any $f \in \mathcal{D}(\mathbb{R}^n \setminus \{o\})$, we have

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \geq \int_M \Phi f^2 d\mu,$$

that is

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \geq \left(\frac{n-2}{2}\right)^2 \int_M \frac{f^2}{|x|^2} d\mu,$$

which is nothing else but the *Hardy inequality* in $\mathbb{R}^n$.

4 Heat semigroup

Let (M, g, μ) be a weighted manifold and H be the Dirichlet Laplace operator on M. Since H is self-adjoint and non-negative definite, its spectrum lies in $[0, +\infty)$ and hence any continuous bounded function ϕ on $[0, +\infty)$ gives rise to a self-adjoint operator $\phi(H)$. In particular, for any $t \geq 0$, the operator e^{-tH} is a bounded self-adjoint operator in $L^2(M, \mu)$. The family $\left\{e^{-tH}\right\}_{t\geq 0}$ is called the *heat semigroup* of H.

Theorem 4.1 ([3], [5], [11], [12], [13]) *For any $f \in L^2$, the function $(t, x) \mapsto e^{-tH} f(x)$ has a C^∞ version $u(t, x)$ $(t > 0, x \in M)$. The function $u(t, x)$ satisfies the heat equation*

$$\frac{\partial u}{\partial t} = \Delta_\mu u,$$

the initial condition

$$u(t, \cdot) \xrightarrow{L^2} f \quad \text{as } t \to 0+,$$

and the estimate

$$\operatorname{essinf} f \leq u(t, x) \leq \operatorname{esssup} f. \tag{4.1}$$

Approach to the proof. Using the functional calculus of self-adjoint operators, one easily shows that the function $e^{-tH} f(x)$ satisfies the heat equation in the L^2-sense, that is, as a path $t \mapsto e^{-tH} f$ in L^2:

$$\frac{d}{dt} e^{-tH} f = -He^{-tH} f = \Delta_\mu \left(e^{-tH} f\right).$$

Next, one shows that in fact the function $u(t, x) = e^{-tH} f(x)$ satisfies the heat equation in the distributional sense on $\mathbb{R}_+ \times M$, and then one applies the Weyl's lemma saying that a distributional solution to the heat equation is in fact a C^∞-smooth function. The proof of (4.1) is based on the parabolic comparison principle. ■

¿From now on, we will denote by $e^{-tH} f$ its C^∞-version.

Definition. A family $\{p_t(x, y)\}_{t>0}$ of measurable functions on $M \times M$ is called a *heat kernel* if for all $f \in L^2$

$$e^{-tH} f(x) = \int_M p_t(x, y) f(y)\, d\mu(y),$$

for all $t > 0$ and μ-a.a. $x \in M$.

A smooth function $u(t, x)$ on $\mathbb{R}_+ \times M$ is called a *fundamental solution* of the heat equation at a point $y \in M$ if the function $u(t, x)$ satisfies in $\mathbb{R}_+ \times M$ the heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u \tag{4.2}$$

and the Dirac condition

$$u(t, \cdot) \to \delta_y \quad \text{as } t \to 0+,$$

The latter is understood in the distributional sense as follows: for any $\varphi \in \mathcal{D} := C_0^\infty(M)$,

$$\int_M u(t, x) \varphi(x)\, d\mu(x) \to \varphi(y) \quad \text{as } t \to 0+. \tag{4.3}$$

A function $q_t(x, y)$ on $\mathbb{R}_+ \times M \times M$ is called a fundamental solution of the heat equation if, for any $y \in M$, the function $(t, x) \mapsto q_t(x, y)$ is a fundamental solution at y. We say that $q_t(x, y)$ is a *regular* fundamental solution if $q_t(x, y) > 0$ and, for all $t > 0$ and $x \in M$,

$$\int_M q_t(x, y)\, d\mu(y) \leq 1.$$

Theorem 4.2 ([2], [3], [5], [11], [13]) *On any weighted manifold, the heat semigroup* e^{-tH} *possesses a unique heat kernel* $p_t(x,y)$, *which is a* C^∞ *function on* $\mathbb{R}_+ \times M \times M$. *Furthermore, the function* $p_t(x,y)$ *satisfies the following additional properties:*

1. $p_t(x,y)$ *is a fundamental solution to the heat equation.*

2. $p_t(x,y) > 0$ *and, for all* $t > 0$ *and all* $x \in M$,
$$\int_M p_t(x,y)\, d\mu(y) \le 1, \tag{4.4}$$
that is, $p_t(x,y)$ *is a regular fundamental solution.*

3. *Symmetry:*
$$p_t(x,y) = p_t(y,x). \tag{4.5}$$

4. *The semigroup identity:*
$$p_{t+s}(x,y) = \int_M p_t(x,z)\, p_s(z,y)\, d\mu(z). \tag{4.6}$$

5. $p_t(x,y)$ *is the minimal positive fundamental solution to the heat equation.*

Approach to the proof. The difficult part is the existence of the heat kernel. The smoothness of $p_t(x,y)$ and the fact that $p_t(x,y)$ is a fundamental solution follow from Theorem 4.1. The positivity and (4.4) follow from (4.1): since $f \ge 0 \Longrightarrow e^{-tH} f \ge 0$, the heat kernel $p_t(x,y)$ must be non-negative (the positiveness requires a bit more of work); since $f \le 1 \Longrightarrow e^{-tH} f \le 1$, one obtained (4.4). The symmetry (4.5) follows from the self-adjointness of H, and (4.6) follows from $e^{-tH} e^{-sH} = e^{-(t+s)H}$. Finally, the minimality property of the heat kernel reflects the fact that H is the closure of $-\Delta_\mu$ with the minimal domain $\mathcal{D}$. ■

The existence of the heat kernel allows to extend the action of the semigroup e^{-tH} from L^2 to L^∞. Indeed, for any $f \in L^\infty$ the following function
$$u(t,x) = \int_M p_t(x,y)\, f(y)\, d\mu(y)$$
is finite by (4.4), and it is easy to show that it solves the heat equation in $\mathbb{R}_+ \times M$. Furthermore, if $f \in C_b(M)$ then $u(t,x)$ satisfies also the initial condition
$$u(t,x) \to f(x) \quad \text{as } t \to 0+$$
where the convergence is locally uniform. Hence, $u(t,x)$ solves a bounded Cauchy problem in the classical sense:
$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\mu u & \text{in } \mathbb{R}_+ \times M, \\ u|_{t=0} = f. \end{cases}$$
If in addition $f \ge 0$ then $u(t,x)$ is the smallest positive solution to the Cauchy problem.

Brownian motion and stochastic completeness. Once the heat kernel is defined, it can be used to construct the *Brownian motion* $\left(\{X_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ on M. The relation is given by the identity
$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)\, d\mu(y)$$

for any Borel set $A \subset M$. There is one difficulty, though: if

$$\int_M p_t(x,y)\,d\mu(y) < 1$$

then the full $\mathbb{P}_x$-probability that $X_t \in M$ is *smaller* than 1. Hence, to complete the probability measure, one needs to introduce an additional point, called the *cemetery*, which absorbs the Brownian particle with the $\mathbb{P}_x$-probability

$$1 - \int_M p_t(x,y)\,d\mu(y).$$

Definition. A manifold (M, g, μ) is called stochastically complete if for all $t > 0$ and $x \in M$,

$$\int_M p_t(x,y)\,d\mu \equiv 1.$$

An easy example of stochastically incomplete manifold is a bounded domain in $\mathbb{R}^n$. Let M be such a domain. Then the heat kernel $p_t(x,y)$ of M being the minimal positive fundamental solution, vanishes on ∂M (assuming that the boundary ∂M is smooth). Therefore, by the divergence theorem in $\mathbb{R}^n$,

$$\frac{d}{dt}\int_M p_t(x,y)\,dy = \int_M \frac{\partial}{\partial t} p_t(x,y)\,dy = \int_M \Delta_y p_t(x,y)\,dy = \int_{\partial M} \frac{\partial}{\partial \nu} p_t(x,y)\,d\sigma(y) < 0,$$

where ν is the outwards normal vector field on ∂M and σ is the area on ∂M. Hence, we conclude that

$$\int_M p_t(x,y)\,dy < 1.$$

There are examples of geodesically complete but stochastically incomplete manifolds (see [1] and [8]).

Lemma 4.3 *Let (M, g, μ) be a stochastically complete weighted manifold and let $q_t(x,y)$ be a regular fundamental solution to the heat equation. Then $q_t(x,y)$ is the heat kernel.*

Proof. By the minimality property of the heat kernel $p_t(x,y)$, we have $q_t \geq p_t$. Hence,

$$1 \geq \int_M q_t(x,y)\,d\mu(y) \geq \int_M p_t(x,y)\,d\mu(y) = 1,$$

where in the last part we have used the stochastic completeness of M. We conclude that all inequalities above are actually equalities, whence $q_t \equiv p_t$. ∎

Example. In $\mathbb{R}^n$ the function

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

is a regular fundamental solution as was proved in Theorems 1.1 and 1.2. The next theorem will imply that $\mathbb{R}^n$ is stochastically complete. Hence, Lemma 4.3 allows to conclude that q_t is the heat kernel in $\mathbb{R}^n$.

Theorem 4.4 ([4], [8], [9]) *The following conditions are equivalent.*

(1) *A manifold M is stochastically complete.*

(2) *Any bounded solution to the equation* $\Delta_\mu v - \lambda v = 0$ *on* M *is identical* 0*, where* λ *is a fixed positive constant.*

(3) *The bounded Cauchy problem on* $[0,T] \times M$ *has unique solution, where* T *is a fixed positive constant.*

Proof. Let us prove the following sequence of implications

$$\neg(1) \Longrightarrow \neg(2) \Longrightarrow \neg(3) \Longrightarrow \neg(1),$$

where $\neg$ means the negation of the statement.

Proof of $\neg(1) \Rightarrow \neg(2)$. So, we assume that M is stochastically incomplete and prove that there exists a non-zero bounded solution to the equation $\Delta_\mu v - \lambda v = 0$. Consider the function

$$u(t,x) = e^{-tH}1(x) = \int_M p_t(x,y)\, d\mu(y).$$

We have $0 < u \leq 1$ and by hypothesis $u \not\equiv 1$. Consider the function

$$w(x) = \int_0^\infty e^{-\lambda t} u(t,x)\, dt. \tag{4.7}$$

Obviously,

$$\Delta_\mu w = \int_0^\infty e^{-\lambda t} \Delta_\mu u\, dt = \int_0^\infty e^{-\lambda t} \frac{\partial u}{\partial t} dt = \left[e^{-\lambda t} u\right]_0^\infty + \lambda \int_0^\infty e^{-\lambda t} u dt = -1 + \lambda w, \tag{4.8}$$

where we have used that $u(t,x) \to 1$ as $t \to 0+$. From (4.7) we obtain

$$0 < w(x) \leq \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}. \tag{4.9}$$

Moreover, since there exists $x \in M$ and $t > 0$ such that $u(t,x) < 1$, for this x we have $w(x) < 1/\lambda$, that is $w \not\equiv 1/\lambda$. Finally, consider the function

$$v = 1 - \lambda w.$$

It follows from (4.8) that

$$\Delta_\mu v = -\lambda \Delta_\mu w = \lambda v$$

and from (4.9) that $0 \leq v \leq 1$ and $v \not\equiv 0$, which finishes the proof.

Proof of $\neg(2) \Rightarrow \neg(3)$. Let v be a bounded non-zero solution to $\Delta_\mu v - \lambda v = 0$, and consider the function

$$u(t,x) = e^{\lambda t} v(x).$$

Then

$$\Delta_\mu u = e^{\lambda t} \Delta_\mu v = \lambda e^{\lambda t} v = \frac{\partial u}{\partial t}.$$

Hence, u solves the heat equation on $\mathbb{R}_+ \times M$ with the initial condition $u(0,x) = v(x)$, and this solution is bounded on any strip $[0,T] \times M$. Compare $u(t,x)$ with another bounded solution to the same Cauchy problem: $w(t,x) = e^{-tH} v(x)$. By Theorem 4.1,

$$\sup |w(t,\cdot)| \leq \sup |v|$$

whereas we have

$$\sup |u(t,\cdot)| = e^{\lambda t} \sup |v| > \sup |v|.$$

Therefore, $u \not\equiv w$, and the bounded Cauchy problem has at least two different solutions.

Proof of $\neg(3) \Rightarrow \neg(1)$. Let $u(t,x)$ be a non-zero bounded solution to the Cauchy problem vanishing at $t=0$. Without loss of generality, we can assume that $0 < \sup u < 1$. Consider the function $w = 1-u$, which is bounded and for which we have $0 < \inf w < 1$. Function w is a positive solution to the bounded Cauchy problem with the initial function 1. Consider the function

$$v(t,x) = \int_M p_t(x,y)\, d\mu(y),$$

which solves the same problem. Since the latter is the minimal positive solution to this problem, we conclude $w(t,x) \geq v(t,x)$. Therefore, $\inf v < 1$ and hence M is stochastically incomplete. ∎

For example, $\mathbb{R}^n$ is stochastically complete because by Theorem 1.2 the bounded Cauchy problem in $\mathbb{R}^n$ has a unique solution. In the next section, we present a convenient criterion for the stochastic completeness.

5 Stochastic completeness and volume growth

Let (M,g,μ) be a connected weighted manifold. A convenient criterion of stochastic completeness of M is given in terms of the volume function of balls. Let $d(x,y)$ be the geodesic distance on M and let $B(x,r)$ denote the geodesic ball on M of radius r centered at x, that is

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Recall that a manifold M is geodesically complete if and only if all geodesic balls are precompact sets.

Define the *volume function* $V(x,r)$ of (M,μ) by

$$V(x,r) := \mu(B(x,r)),$$

and notice that $V(x,r)$ is finite on any geodesically complete manifold.

Theorem 5.1 ([7], [8]) *Let M be a geodesically complete manifold. If, for some point $x_0 \in M$,*

$$\int^\infty \frac{r\,dr}{\log V(x_0,r)} = \infty \tag{5.1}$$

then M is stochastically complete.

Condition (5.1) holds, in particular, if

$$V(x_0,r) \leq \exp\left(Cr^2\right) \tag{5.2}$$

for all r large enough or even if $V(x_0,r_k) \leq \exp\left(Cr_k^2\right)$, for a sequence $r_k \to \infty$ as $k \to \infty$.

By Theorem 4.4, in order to prove Theorem 5.1, it suffices to verify that the only bounded solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\mu u & \text{in } [0,T] \times M, \\ u|_{t=0} = 0, \end{cases} \tag{5.3}$$

is $u \equiv 0$ (for some fixed $T > 0$). The function $u(t,x)$ is assumed to be in the class

$$C^2((0,T) \times M) \cap C([0,T) \times M).$$

The assertion will follow from the following even more general fact.

Theorem 5.2 *Let M be a geodesically complete manifold, and let $u(x,t)$ be a solution to the Cauchy problem* (5.3)*. Assume that, for some $x_0 \in M$ and for all R large enough,*

$$\int_0^T \int_{B(x_0,R)} u^2(x,t)\, d\mu(x)\, dt \leq \exp\left(f(R)\right), \tag{5.4}$$

where $f(r)$ is a positive monotone increasing function on $(0,+\infty)$ such that

$$\int^{\infty} \frac{r dr}{f(r)} = \infty. \tag{5.5}$$

Then $u \equiv 0$ in $[0,T] \times M$.

Theorem 5.2 provides the uniqueness class (5.4) for the initial value problem. It can be regarded as a generalization of the classical uniqueness classes of Tichonov and Täcklind for the heat equation in $\mathbb{R}^n$. Indeed if, for example, the function $u(t,x)$ is a solution to the initial value problem (5.3) in $\mathbb{R}^n$ and belongs to the Tichonov class, that is

$$|u(t,x)| \leq C \exp\left(C\,|x|^2\right) \quad \text{for all } x \in \mathbb{R}^n \text{ and } 0 < t < 1,$$

then the conditions (5.4) and (5.5) are satisfied with $f(r) = (C+\varepsilon)r^2$, and we conclude by Theorem 5.2 that $u \equiv 0$.

Proof of Theorem 5.1. By Theorem 4.4, it suffices to verify that the only bounded solution to the Cauchy value problem (5.3) is $u \equiv 0$. Indeed, if

$$C := \sup |u| < \infty$$

then we have

$$\int_0^T \int_{B(x_0,R)} u^2(t,x) d\mu(x) \leq C^2 T V(x_0,R).$$

Set

$$f(r) := \log\left(C^2 T V(x_0,r)\right)$$

so that the hypothesis (5.4) is satisfied. Obviously, (5.5) is implied by (5.1). Hence, by Theorem 5.2 we obtain $u \equiv 0$. ■

Proof of Theorem 5.2. Denote for simplicity $B_r = B(x_0,r)$. We will prove the following inequality

$$\int_{B_R} u^2(t,\cdot) d\mu \leq \int_{B_{2R}} u^2(t-\delta,\cdot) d\mu + \frac{C}{R^2}\,, \tag{5.6}$$

assuming that R is large enough, $0 < \delta \leq t < T$, and in addition

$$\delta \leq c \frac{(2R)^2}{f(2R)}\,. \tag{5.7}$$

The values of the positive constants C and c will be specified later.

Assuming for a moment that we have already proved (5.6), let us show that $u \equiv 0$. Fix some (large) $R > 0$ and $t \in (0,T)$. For any $k = 0,1,2,...$ set $R_k = 2^k R$ and choose (so far arbitrarily) some δ_k to satisfy the condition

$$0 < \delta_k \leq c \frac{R_k^2}{f(R_k)}. \tag{5.8}$$

Then define the decreasing sequence $\{t_k\}$ inductively by $t_0 = t$ and $t_k = t_{k-1} - \delta_k$.

If $t_k \geq 0$ then the inequality (5.6) yields

$$\int_{B_{R_{k-1}}} u^2(t_{k-1}, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \frac{C}{R_{k-1}^2}. \tag{5.9}$$

Iterating it we obtain

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \sum_{i=1}^{k} \frac{C}{R_{i-1}^2}.$$

If it happens that $t_k = 0$ then by (5.3)

$$\int_{B_{R_k}} u^2(t_k, \cdot) d\mu = 0,$$

whence

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \frac{2C}{R^2}.$$

Letting $R \to \infty$, we conclude $u(\cdot, t) \equiv 0$.

Hence, to finish the proof, it suffices to construct, for any (large) R and for any $t \in (0, T)$, the sequence $\{t_k\}$ as above that vanishes at a finite k. In other words, this means that for some k

$$t = \delta_1 + \delta_2 + ... + \delta_k . \tag{5.10}$$

The only restriction on δ_k is the inequality (5.8). The hypothesis (5.5) implies that for any R

$$\sum_{k=1}^{\infty} \frac{R_k^2}{f(R_k)} = \infty.$$

Therefore, the sequence $\{\delta_k\}_{k=1}^{\infty}$ can be chosen to satisfy simultaneously (5.8) and

$$\sum_{k=1}^{\infty} \delta_k = \infty.$$

By diminishing some of δ_k, we can achieve (5.10) for any positive t.

Now let us turn to the proof of (5.6). Observe that if $\rho(x)$ is a Lipschitz function on M such that $|\nabla \rho| \leq 1$ then the function

$$\xi(t, x) := \frac{\rho^2(x)}{4(t-s)}$$

satisfies the inequality

$$\frac{\partial \xi}{\partial t} + |\nabla \xi|^2 \leq 0 \tag{5.11}$$

in any time interval that does not contain s.

Choose function $\eta \in C^\infty(M)$ to satisfy the following properties: $0 \leq \eta \leq 1$ on M, $\eta \equiv 1$ in $B_{\frac{3}{2}R}$, $\operatorname{supp}\eta \subset B_{2R}$, and $|\nabla \eta| \leq \frac{4}{R}$. By the assumption of the metric completeness of (M, d), all balls in M are precompact sets. Hence, we have $\eta \in C_0^\infty(M)$.

Fix some $0 < \delta < t$ and assume that $s \notin [t - \delta, t]$. Multiplying the equation (5.3) by $u\eta^2 e^\xi$ and integrating it over $[t - \delta, t] \times M$, we obtain

$$\int_{t-\delta}^{t} \int_M \frac{\partial u}{\partial t} u\eta^2 e^\xi d\mu dt = \int_{t-\delta}^{t} \int_M \Delta u \, u\eta^2 e^\xi d\mu dt. \tag{5.12}$$

The time part of the integral on the left hand side is equal to

$$\frac{1}{2}\int_{t-\delta}^{t}\frac{\partial(u^2)}{\partial t}\eta^2 e^{\xi}dt=\frac{1}{2}\,u^2\eta^2 e^{\xi}\Big|_{t-\delta}^{t}-\frac{1}{2}\int_{t-\delta}^{t}\frac{\partial\xi}{\partial t}u^2\eta^2 e^{\xi}dt.$$

The spacial part of the integral on the right hand side of (5.12) is estimated as follows

$$\begin{aligned}\int_M \Delta u\, u\eta^2 e^{\xi}d\mu &= -\int\langle\nabla u,\nabla(u\eta^2e^{\xi})\rangle\\ &= -\int|\nabla u|^2\eta^2e^{\xi}-\int\langle\nabla u,\nabla\xi\rangle u\eta^2e^{\xi}-2\int\langle\nabla u,\nabla\eta\rangle u\eta e^{\xi}\\ &\le -\int|\nabla u|^2\eta^2e^{\xi}+\int\left[\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\nabla\xi|^2u^2\right]\eta^2e^{\xi}\\ &\quad+\int\left[\frac{1}{2}|\nabla u|^2\eta^2+2|\nabla\eta|^2u^2\right]e^{\xi}\\ &= \frac{1}{2}\int|\nabla\xi|^2u^2\eta^2e^{\xi}+2\int|\nabla\eta|^2u^2e^{\xi}.\end{aligned}$$

Therefore, we obtain from (5.12):

$$\int_M u^2\eta^2e^{\xi}d\mu\Big|_{t-\delta}^{t}\le\int_{t-\delta}^{t}\int_M\left[\frac{\partial\xi}{\partial t}\eta^2+|\nabla\xi|^2\eta^2+4|\nabla\eta|^2\right]u^2e^{\xi}d\mu dt. \tag{5.13}$$

Due to inequality (5.11), the first and the second terms on the right hand side of (5.13) cancel, and we obtain

$$\int_M u^2\eta^2e^{\xi}d\mu\Big|_{t-\delta}^{t}\le 4\int_{t-\delta}^{t}\int_M|\nabla\eta|^2u^2e^{\xi}d\mu dt. \tag{5.14}$$

The function η satisfies the inequalities $0\le\eta\le1$ and $|\nabla\eta|\le\frac{4}{R}$. Taking into account that $\eta=1$ on $B_{\frac{3}{2}R}$ and $\eta=0$ outside B_{2R}, we can rewrite (5.14) as follows

$$\int_{B_R}u^2(t,\cdot)e^{\xi(t,\cdot)}d\mu\le\int_{B_{2R}}u^2(t-\delta,\cdot)e^{\xi(t-\delta,\cdot)}d\mu+\frac{64}{R^2}\int_{t-\delta}^{t}\int_{B_{2R}\setminus B_{3R/2}}u^2e^{\xi}d\mu dt. \tag{5.15}$$

Let us specify $\rho(x)$ and $\xi(x,t)$. Set $\rho(x)$ to be the distance function from the ball B_R i.e.

$$\rho(x)=(d(x,x_0)-R)_+\,.$$

Also, set $s=t+\delta$ so that for all $\theta\in[t-\delta,t]$

$$\xi(\theta,x)=-\frac{\rho^2(x)}{4(s-\theta)}\le-\frac{\rho^2(x)}{8\,\delta}\le0. \tag{5.16}$$

Consequently, we can drop the factor e^{ξ} on the left hand side of (5.15) because $\xi=0$ in B_R, and drop the factor e^{ξ} in the first integral on the right hand side of (5.15) because $\xi\le0$. Clearly, if $x\in B_{2R}\setminus B_{3R/2}$ then $\rho(x)\ge R/2$, which together with (5.16) implies that

$$\xi\le-\frac{R^2}{32\,\delta}\quad\text{in}\quad[t-\delta,t]\times B_{2R}\setminus B_{3R/2}.$$

Hence, we obtain from (5.15)

$$\int_{B_R} u^2(t,\cdot)d\mu \le \int_{B_{2R}} u^2(t-\delta,\cdot)d\mu + \frac{64}{R^2}\exp\left(-\frac{R^2}{32\,\delta}\right)\int_{t-\delta}^{t}\int_{B_{2R}} u^2 d\mu dt.$$

By (5.4) we have

$$\int_{t-\delta}^{t}\int_{B_{2R}} u^2 d\mu dt \le \exp\left(f(2R)\right)$$

whence

$$\int_{B_R} u^2(t,\cdot)d\mu \le \int_{B_{2R}} u^2(t-\delta,\cdot)d\mu + \frac{64}{R^2}\exp\left(-\frac{R^2}{32\,\delta} + f(2R)\right).$$

Therefore, if δ satisfies

$$\delta \le \frac{R^2}{32 f(2R)} \tag{5.17}$$

then we obtain (5.6) with $C = 64$ (and $c = \frac{1}{128}$ in (5.7)). ■

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