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Homogenization on Finitely Ramified Fractals

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Abstract.

Let X_t be a continuous time Markov chain on some finitely ramified fractal graph given by putting i.i.d. random resistors on each cell. We prove that under an assumption that a renormalization map of resistors has a non-degenerate fixed point, $\alpha^{-n} X_{\tau^n t}$ converges in law to a non-degenerate diffusion process on the fractal as $n \to \infty$, where α is a spatial scale and τ is a time scale of the fractal. Especially, when the fixed point of the renormalization map is unique, the diffusion is a constant time change of Brownian motion on the fractal. These results improve and extend our former results in [10].

§1. Introduction

In this paper, we consider the homogenization problem on uniform finitely ramified fractals, which is a class of finitely ramified fractals with a unique spatial scaling rate. We put random resistors on each cell of the fractal graph and set X_t be the corresponding continuous time Markov chain. Our aim is to show that $\alpha^{-n} X_{\tau^n t}$ converges in law to a non-degenerate diffusion process on the fractal as $n \to \infty$. Here α is the spatial scale and τ is a time scale of the fractal.

Homogenization of a diffusion process is interpreted as a limit theorem of a random process for changing scales. For $\mathbf{R}^d$ case, it is discussed that $\epsilon X_{t/\epsilon^2}$ converges to a constant time change of Brownian motion as $\epsilon \to 0$ under a condition that X_t has random diffusion coefficients or it moves in some random environment such as random scatterers. (See [8] for general references on homogenization of differential operators.) The martingale method has been well developed for this problem.

For the case of fractals, typical diffusions are sub-diffusive, in the sense that $E[|B_t|] \approx t^{1/d_w}$ as $t \to \infty$ for some $d_w > 2$. Those diffusions

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are not semi-martingales, thus we need a different approach. In [10], we developed a theory to be applied for homogenization problem on nested fractals, a class of finitely ramified fractals with good symmetries. In this paper, we inherits the basic approach of [10], which we now explain.

We first consider a Dirichlet form corresponding to the continuous time Markov chain with random resistors which are almost surely bounded from above and below by some non-degenerate resistor (this corresponds to the uniform ellipticity condition for the operator) and the distributions are *i.i.d.* for each cell. As a solution of a variation problem of the Dirichlet form, we induce a renormalization map F on the space of matrices, by which we can produce a new form that is a renormalization of the original one. We assume that there is a non-degenerate fixed point of the map. Then, what we should prove are the following:
1) Convergence of the iteration of the renormalization map to a fixed point and convergence of the forms.
2) Convergence of finite dimensional distributions and tightness.

In [10], we prove 1) under certain condition for an adjoint of a Fréchet derivative of the renormalization map at a fixed point (see Remark 3.6 1) for details). Unfortunately, the condition is not easy to check in general and there are examples (even for nested fractals) that the condition does not hold. On the other hand, the dynamics of the iteration of the renormalization map has been well studied recently for the finitely ramified fractals (see [16, 13, 15] etc.). In this paper, we apply the results to improve our former results in [10]. In Section 4, we will prove 1) under a very mild condition on the renormalization map (see Assumption 2.3). We note that the map F we study is on a infinite dimensional space since we have infinite number of resistors, whereas the renormalization map $\hat{F}$ studied in [16, 13, 15] is on a finite dimensional space. We thus make some efforts to show the stability of the map F from that of $\hat{F}$.

In general, non-degenerate fixed points of the renormalization map are not necessarily unique. When the uniqueness (up to constant multiples) is guaranteed, we can further show that the diffusion obtained as the limit is a constant time change of some special diffusion (which can be called Brownian motion) for the fractal. Especially, when we consider random resistors which are invariant under all reflection maps on a nested fractal, then the diffusion obtained is a constant time change of Brownian motion on the fractal.

For the proof of 2), uniform Harnack inequality and uniform heat kernel estimates of the Markov chains play important roles. Here we can adopt stability results of parabolic Harnack inequalities and heat kernel estimates which are actively studied recently for the fractal graph cases (see [2, 3, 4, 5]). In this paper, we skip the proof of 2) since we can apply the same argument as in [10], but note that we can shorten the proof by applying the results in [5].

The organization of the paper is as follows. In Section 2, we define uniform finitely ramified fractals (graphs), renormalization maps of resistors on them and briefly mention about Dirichlet forms and their heat kernel estimates on the fractals. In Section 3, we give our main theorem on homogenization. Section 4 is for the proof of 1) above. In Section 5, we state main propositions concerning 2) above. In Appendix, we give the proof of the stability results of the (finite dimensional) renormalization map $\hat{F}$ studied in [16, 13, 15].

§2. Uniform finitely ramified graphs and their Dirichlet forms

2.1. Uniform finitely ramified graphs

For $\alpha > 1$ and $I = \{1, 2, \cdots, N\}$, let $\{\Psi_i\}_{i\in I}$ be a family of α-*similitudes* on $\mathbf{R}^D$. An α-similitude is a map $\Psi_i \mathbf{x} = \alpha^{-1} U_i \mathbf{x} + \gamma_i,\ \mathbf{x} \in \mathbf{R}^D$ where U_i is a unitary map and $\gamma_i \in \mathbf{R}^D$. We will impose several assumption on this family. First, we assume
(H-0) $\{\Psi_i\}_{i\in I}$ satisfies the open set condition,
i.e., there is a non-empty, bounded open set W such that $\{\Psi_i(W)\}_{i\in I}$ are disjoint and $\cup_{i\in I}\Psi_i(W) \subset W$. As $\{\Psi_i\}_{i\in I}$ is a family of contraction maps, there exists a unique non-void compact set $\hat{K}$ such that $\hat{K} = \cup_{i\in I}\Psi_i(\hat{K})$. We assume
(H-1) $\hat{K}$ is connected.

Let Fix be the set of fixed points of the Ψ_i's, $i \in I$. A point $x \in Fix$ is called an *essential fixed point* if there exist $i, j \in I,\ i \neq j$ and $y \in Fix$ such that $\Psi_i(x) = \Psi_j(y)$. Let I_F be the set of $i \in I$ for which the fixed point of Ψ_i is an essential fixed point. We write $\hat{V}_0$ for the set of essential fixed points. Denote $\Psi_{i_1,\dots,i_n} = \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}$. We further assume the following finitely ramified property.
(H-2) If $\{i_1, \dots, i_n\}, \{j_1, \dots, j_n\}$ are distinct sequences, then

$$\Psi_{i_1,\dots,i_n}(\hat{K}) \bigcap \Psi_{j_1,\dots,j_n}(\hat{K}) = \Psi_{i_1,\dots,i_n}(\hat{V}_0) \bigcap \Psi_{j_1,\dots,j_n}(\hat{V}_0).$$

Definition 2.1. ([5]) *A (compact) uniform finitely ramified fractal (u.f.r. fractal for short) $\hat{K}$ is a set determined by α-similitudes $\{\Psi_i\}_{i\in I}$ satisfying the assumption (H-0), (H-1), (H-2) and that $\sharp\hat{V}_0 \geq 2$.*

If we further assume the following symmetry condition, then $\hat{K}$ is called a (compact) nested fractals introduced in [12].

(SYM) If $x, y \in \hat{V}_0$, then the reflection in the hyperplane $H_{xy} = \{z \in \mathbf{R}^D : |z-x| = |z-y|\}$ maps $\hat{V}_n$ to itself, where

$$\hat{V}_n = \cup_{i_1,\cdots,i_n \in I} \Psi_{i_1,\ldots,i_n}(\hat{V}_0). \tag{2.1}$$

Thus, u.f.r. fractals form a class of fractals which is wider than nested fractals, and is included in the class of p.c.f. self-similar sets ([9]).

For each $n \geq 0$ and $i_1, \cdots, i_n \in I$, we call a set of the form $\Psi_{i_1,\cdots,i_n}(\hat{V}_0)$ an n-cell and $\Psi_{i_1,\cdots,i_n}(\hat{K})$ an n-complex. For $x, y \in \hat{K}$, $\{x_0, \cdots, x_m\}$ is called a n-chain from x to y if $x_0 = x, x_m = y$, $x_j \in \hat{V}_n$ for $1 \leq j \leq m-1$ and x_i, x_{i+1} are in the same n-complex for $0 \leq i \leq m-1$. We then have the following topological properties of u.f.r. fractals.

Lemma 2.2.
1) Each element in $\hat{V}_0$ belongs to only one n-cell for each $n \geq 0$.
2) Any 1-cell contains at most one element of $\hat{V}_0$.
3) For each $x \in \hat{V}_1$ and $y \in \hat{V}_0$, there exists a 1-chain $\{x_0, \cdots, x_m\}$ from x to y such that $x_1, \cdots, x_{m-1} \notin \hat{V}_0$.

Proof. 1) and 2) can be proved in the same way as [11] (Lemma 2.8 and Proposition 2.9) and [12] (Proposition IV.13 and Corollary IV.14). (They discuss for nested fractals, but the symmetry assumption is not used there.) For 3), we first note that any 1-junction is not an element of $\hat{V}_0$ due to 1), where $x \in \hat{V}_1$ is called a 1-junction if there exist $i \neq j \in I$ such that $x \in \Psi_i(\hat{V}_0) \cap \Psi_j(\hat{V}_0)$. Using (H-1) and (H-2), we can choose a 1-chain $\{x_0, \cdots, x_m\}$ such that $x_1, \cdots, x_{m-1}$ are 1-junctions. Since 1-junction is not an element of $\hat{V}_0$, we obtain the result. Q.E.D.

Next we define unbounded u.f.r. fractals. We assume without loss of generality that $\Psi_1(\mathbf{x}) = \alpha_1^{-1}\mathbf{x}$ and $\mathbf{0}$ belongs to $\hat{V}_0$. Set $K = \cup_{n=1}^{\infty} \alpha^n \hat{K}$. Then, clearly $\Psi_1(K) = K$. We call K an unbounded uniform finitely ramified fractal. Let $V = V_0 = \cup_{n=0}^{\infty} \alpha^n \hat{V}_n$ and $V_n = \alpha^{-n} V$ for $n \in \mathbf{Z}$. (Note that this labelling is the opposite to the one given in [5]. As n gets bigger, the graph distance between each vertex of V_n gets smaller and $V_{n-1} \subset V_n$.) Then, $K = Cl(\cup_{n \in \mathbf{Z}} V_n)$. For $n \in \mathbf{Z}$, we define n-cells and n-complexes similarly as the compact fractals.

We now introduce uniform finitely ramified graphs. These will be graphs with vertices V and a collection of edges B. In order to define the edges, we first define $\hat{B}_0 := \{\{x, y\} : x \neq y \in \hat{V}_0\}$. Then inside each 0-cell we place a copy of $\hat{B}_0$ and we denote by B the set of all the edges determined in this way. We call the graph (V, B) a uniform finitely ramified (u.f.r.) graph. If we construct the graph starting from a nested fractal, then it will be called a nested fractal graph. Let

$$\begin{aligned} \Omega &= \{\omega \in I^{\mathbf{Z}} : \text{ there is an } n \in \mathbf{Z} \text{ such that } \omega_k = 1, k \geq n\}, \\ \Omega_+ &= \{\omega \in I^{\mathbf{N}} : \text{ there is an } n \in \mathbf{N} \text{ such that } \omega_k = 1, k \geq n\}. \end{aligned}$$

Then, there is a continuous map $\pi : \Omega \to \mathbf{R}^D$ such that $\pi(\omega) = \lim_{n\to\infty} \alpha^n \Psi_{\omega_n}(\Psi_{\omega_{n-1}}(\cdots(\Psi_{\omega_{-n}}(\mathbf{0}))\cdots))$. It is easy to see $K = \pi(\Omega)$. For any $\omega \in \Omega_+$ and $i \in I_F$, let $[\omega, i]$ denotes an element of Ω given by

$$[\omega, i](k) = \begin{cases} \omega_k, & k \geq 1 \\ i, & k \leq 0. \end{cases}$$

Then, $V = \{\pi([\omega, i]) : \omega \in \Omega_+, i \in I_F\}$.

2.2. Renormalization maps

Let $\mathcal{Q}$ be the set of $Q = \{Q_{ij} \in \mathbf{R} : i, j \in I_F \times I_F\}$ such that $Q_{ij} = Q_{ji}$ for any $i, j \in I_F$ and that $\sum_{j\in I_F} Q_{ij} = 0, i \in I_F$. $\mathcal{Q}$ is a vector space with an inner product $(\cdot,\cdot)_{\mathcal{Q}}$ given by

$$(Q, Q')_{\mathcal{Q}} = \sum_{j,k\in I_F} Q_{jk} Q'_{jk} = \text{Trace } Q^t Q', \qquad Q, Q' \in \mathcal{Q}.$$

For a set A, we denote $l(A) = \{f : A \to \mathbf{R}\}$. Let $\mathcal{Q}_+$ be the set of $Q \in \mathcal{Q}$ such that $\hat{S}_Q(\xi, \xi) \geq 0$ for any $\xi \in l(I_F)$, where

$$\hat{S}_Q(\xi, \xi) = -\sum_{i,j\in I_F} Q_{ij}\xi_i\xi_j = \frac{1}{2}\sum_{i,j\in I_F} Q_{ij}(\xi_i - \xi_j)^2.$$

Set $\|Q\|^2 = \sup_{\xi\in l(I_F)} \hat{S}_Q(\xi,\xi)/(\sum_{i\in I_F}\xi_i^2)$. Note that there exist $c_{2.1}$, $c_{2.2} > 0$ such that $c_{2.1}\|Q\|^2 \leq (Q, Q)_{\mathcal{Q}} \leq c_{2.2}\|Q\|^2$ for all $Q \in \mathcal{Q}_+$. We sometimes denote $\hat{S}_Q(\xi, \xi)$ as $\hat{S}_Q(\xi)$. Let $\mathcal{Q}_M$ be the set of $Q \in \mathcal{Q}$ such that $Q_{ij} \geq 0$ for any $i, j \in I_F$ with $i \neq j$. Also, let $\mathcal{Q}_{irr}$ be the set of $Q \in \mathcal{Q}_M$ such that $\hat{S}_Q(\xi, \xi) = 0$ if and only if ξ is constant. Note that $\mathcal{Q}_{irr} \subset \mathcal{Q}_M \subset \mathcal{Q}_+$.

Take $Q_* \in \text{Int}(\mathcal{Q}_M) := \{Q \in \mathcal{Q}_M : Q_{ij} > 0 \text{ for any } i \neq j \in I_F\}$ and let $\mathcal{X}_+ = \{X \in C(\Omega_+, \mathcal{Q}_+) :$ there exists $C_0 > 0$ such that $\hat{S}_{X(\omega)}(\xi, \xi) \leq C_0 \hat{S}_{Q_*}(\xi, \xi)$ for any $\omega \in \Omega_+$ and $\xi = (\xi_j)_{j\in I_F}\}$. Also, let $\mathcal{X}_M = \mathcal{X}_+ \cap C(\Omega_+, \mathcal{Q}_M)$, $\mathcal{X}_{irr} = \mathcal{X}_+ \cap C(\Omega_+, \mathcal{Q}_{irr})$ and $\text{Int}(\mathcal{X}_M) = \mathcal{X}_+ \cap C(\Omega_+, \text{Int}(\mathcal{Q}_M))$. Then $\mathcal{X}_+$ and $\mathcal{X}_M$ are convex cones. For any $X \in \mathcal{X}_+$, let S_X denote a non-negative definite bilinear form on $\mathbf{L}^2(V, d\nu_0)$ given by

$$S_X(u, u) = \frac{1}{2}\sum_{\omega\in\Omega_+} \hat{S}_{X(\omega)}(u(\pi([\omega,\cdot])), u(\pi([\omega,\cdot]))), \ u \in \mathbf{L}^2(V, d\nu_0).$$

Here ν_0 is a measure on V so that $\nu_0(\{x\}) = 1/N$ for all $x \in V$. If $X \in \mathcal{X}_M$, then S_X is a Dirichlet form on $\mathbf{L}^2(V, d\nu_0)$. So there is a Markov process which we denote by $\{P_X^x : x \in V\}$. We introduce an order relation $\leq$ in $\mathcal{X}_+$ as follows.

$$X \leq Y \quad \text{if} \quad S_X(u,u) \leq S_Y(u,u) \quad \text{for all} \quad u \in \mathbf{L}^2(V, d\nu_0).$$

The norm on $\mathcal{X}_+$ is given by $\|X\|^2 = \sup_{u \in \mathbf{L}^2(V,d\nu_0)} S_X(u,u)/\|u\|^2_{\mathbf{L}^2(V,d\nu_0)}$.

For any $X \in \mathcal{X}_+$, let $S_X^{\bar{F}} : \mathbf{L}^2(V, d\nu_0) \to [0, \infty)$ be given by

$$S_X^{\bar{F}}(u) = \inf\{S_X(v,v) : v \in \mathbf{L}^2(V, d\nu_0),\ v(\alpha x) = u(x),\ x \in V\}.$$

Let $S_X^{\bar{F}}(u,v) = \frac{1}{2}(S_X^{\bar{F}}(u+v) - S_X^{\bar{F}}(u) - S_X^{\bar{F}}(v)),\ u, v \in \mathbf{L}^2(V, d\nu_0)$. Then we see that $S_X^{\bar{F}}$ is a Dirichlet form on $\mathbf{L}^2(V, d\nu_0)$. Moreover, by the self-similarity of K, we see that there is a renormalization map $\bar{F} : \mathcal{X}_+ \to \mathcal{X}_+$ such that $S_X^{\bar{F}}(u) = S_{\bar{F}(X)}(u,u)$ for all $X \in \mathcal{X}_+$ and $u \in \mathbf{L}^2(V, d\nu_0)$. Let $\iota : \mathcal{Q}_+ \to \mathcal{X}_+$ be such that $\iota(Q)(\omega) = Q$ for all $\omega \in \Omega_+$ and $Q \in \mathcal{Q}_+$. We define a renormalization map $\tilde{F} : \mathcal{Q}_+ \to \mathcal{Q}_+$ as $\tilde{F}(Q) = \bar{F}(\iota(Q))(\omega)$ for $\omega \in \Omega_+$ (it is independent of the choice of $\omega \in \Omega_+$). Note that $\bar{F}, \tilde{F}$ is in general a non-linear map. By Schauder's fixed point theorem, we know that there exists $Q_* \in \mathcal{Q}_M$ (with $(Q_*)_{ij} > 0$ for some $i \neq j$) and $\rho_{Q_*} > 0$ such that $\tilde{F}(Q_*) = \rho_{Q_*}^{-1} Q_*$. Throughout this paper, we assume the following.

Assumption 2.3. *1) For each $Q \in \mathcal{Q}_{irr}$, there exists $l = l(Q) \in \mathbf{N}$ such that $\tilde{F}^n(Q) \in Int(\mathcal{Q}_M)$ for all $n \geq l$.*
2) There exists $Q_0 \in Int(\mathcal{Q}_M)$ and $\rho_{Q_0} > 0$ such that $\tilde{F}(Q_0) = \rho_{Q_0}^{-1} Q_0$.

Remark 2.4. *1) By Corollary* 6.20 *of* [1], *$\rho_{Q_0} > 0$ is uniquely determined, i.e., if $Q_1, Q_2 \in \mathcal{Q}_{irr}$ satisfies $\tilde{F}(Q_j) = \rho_{Q_j}^{-1} Q_j$ $(j = 1, 2)$ with $\rho_{Q_1}, \rho_{Q_2} > 0$, then $\rho_{Q_1} = \rho_{Q_2} = \rho_{Q_0}$. In the class of fractal graphs we consider, we can prove $\rho_{Q_0} > 1$ (see* [9] *etc.).*
2) A sufficient condition for Assumption 2.3 1) is the following.
(H-3) There exists $l \in \mathbf{N}$ such that for each $x, y \in \hat{V}_0$, there is a l-chain $\{x_0, \cdots, x_m\}$ from x to y such that for each $1 \leq i \leq m-2$, there is a l-cell containing x_i and x_{i+1} that does not contain any element of $\hat{V}_0$.

Indeed, if (H-3) holds, it is easy to show $\tilde{F}^n(Q) \in Int(\mathcal{Q}_M)$ for $n \geq l$, $Q \in \mathcal{Q}_{irr}$ by observing the corresponding Markov chain on $\hat{V}_n$.
3) Every nested fractals satisfy Assumption 2.3 1) and 2). Indeed, (H-3) can be shown for nested fractals using (SYM) and [11] *Lemma* 2.10 *(or* [12] *Proposition IV.*11*), so that 1) holds. 2) is proved in* [11] *Theorem* 3.10 *and in* [12] *Theorem V.*5.

Set $F = \rho_{Q_0}\bar{F} : \mathcal{X}_+ \to \mathcal{X}_+$ and $S_X^F(u) = \rho_{Q_0} S_X^{\bar{F}}(u)$ for $u \in \mathbf{L}^2(V, d\nu_0)$. Set $\hat{F} = \rho_{Q_0}\tilde{F}$ in the same way.

2.3. Dirichlet forms and heat kernel estimates

For $u, v \in l(\hat{V}_n)$, define

$$\hat{\mathcal{E}}^n_{Q_0}(u,v) = \rho^n_{Q_0} \sum_{i_1,\cdots,i_n \in I} \hat{S}_{Q_0}(u \circ \Psi_{i_1,\ldots,i_n}, v \circ \Psi_{i_1,\ldots,i_n}). \tag{2.2}$$

Let $\hat{\nu}$ be a normalized Hausdorff measure on $\hat{K}$. Then, the following is known (see for example, [6, 9]).

Theorem 2.5. *Let $Q_0 \in Int(\mathcal{Q}_M)$ be as in Assumption 2.3 2), i.e. $\hat{F}(Q_0) = Q_0$. Then, there is a local regular Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ in $\mathbf{L}^2(\hat{K}, d\hat{\nu})$ satisfying the following,*

$$\begin{aligned} \hat{\mathcal{F}} &= \{u \in C(\hat{K}, \mathbf{R}) : \sup_n \hat{\mathcal{E}}^n_{Q_0}(u,u) < \infty\}, \\ \hat{\mathcal{E}}(u,v) &= \lim_{n\to\infty} \hat{\mathcal{E}}^n_{Q_0}(u,v) \qquad \text{for } u, v \in \hat{\mathcal{F}}. \end{aligned}$$

For each $m \in \mathbf{N}$, let $K_m = \alpha^m \hat{K}$ and define $\sigma_m : C(K_m, \mathbf{R}) \to C(\hat{K}, \mathbf{R})$ by $\sigma_m u(x) = u(\alpha^m x)$ for $x \in \hat{K}$. Set $\mathcal{F}_{<m>} = \sigma_{-m}\hat{\mathcal{F}}$, $\mathcal{E}_{<m>}(u,v) = \rho_{Q_0}^{-m}\hat{\mathcal{E}}(\sigma_m u, \sigma_m v)$ for $u, v \in \mathcal{F}_{<m>}$. Let ν be a Hausdorff measure on K such that $\nu|_{\hat{K}} = \hat{\nu}$ and $N\nu = \nu \circ \Psi_1^{-1}$. Now let

$$\begin{aligned} \mathcal{F} &= \{u \in l(K) : u|_{K_m} \in \mathcal{F}_{<m>} \text{ for all } m \in \mathbf{N}, \\ &\qquad \lim_{m\to\infty} \mathcal{E}_{<m>}(u|_{K_m}, u|_{K_m}) < \infty\} \cap \mathbf{L}^2(K, d\nu) \\ \mathcal{E}(u,v) &= \lim_{m\to\infty} \mathcal{E}_{<m>}(u|_{K_m}, v|_{K_m}) \qquad \text{for } u, v \in \mathcal{F}. \end{aligned}$$

Then the following holds.

Theorem 2.6. *$(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $\mathbf{L}^2(K, d\nu)$. $\mathcal{F} \subset C(K, \mathbf{R})$ and this form has the following scaling property,*

$$\mathcal{E}(u,v) = \rho_{Q_0}\mathcal{E}(u \circ \Psi_1, v \circ \Psi_1) \qquad \text{for } u, v \in \mathcal{F}.$$

Finally, we will mention heat kernel estimates for Markov chains on u.f.r. graphs. For $X \in \mathrm{Int}(\mathcal{X}_M)$ and $x \neq y \in V$, define

$$R_X(x,y) = \big(\inf\{S_X(u,u) : u \in l(V), u(x) = 1, u(y) = 0\}\big)^{-1}.$$

Let $R_X(x,x) = 0$ for $x \in V$. Then, $R_X(\cdot,\cdot)$ is a metric which is called a resistance metric. By simple modifications of the proof of Corollary 4.12 in [5], the following holds (note that as we will mention later in Remark 3.6 4), Assumption 2.3 of [5] always holds under our Assumption 2.3).

Theorem 2.7. *For each $X \in Int(\mathcal{X}_M)$, let $p_k^X(\cdot,\cdot)$ be the heat kernel of the discrete time Markov chain which is induced from the continuous time Markov chain corresponding to $(S_X, \mathbf{L}^2(V, d\nu_0))$. Then, there exists $c_{2.3}, \cdots, c_{2.6} > 0$ (which depend on X) and $0 < \gamma_1 \le \gamma_2$ such that for each $x, y \in V$ and $k \ge d(x,y)$,*

$$\begin{aligned} p_k^X(x,y) &\le c_{2.3} k^{-\frac{S}{S+1}} \exp\left(-c_{2.4}\left(\frac{R_X(x,y)^{S+1}}{k}\right)^{\gamma_1}\right), \\ p_k^X(x,y) + p_{k+1}^X(x,y) &\ge c_{2.5} k^{-\frac{S}{S+1}} \exp\left(-c_{2.6}\left(\frac{R_X(x,y)^{S+1}}{k}\right)^{\gamma_2}\right), \end{aligned}$$

where $S = \log N / \log \rho_{Q_0}$ and $d(\cdot,\cdot)$ is a graph distance.

We note that similar heat kernel estimates for $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ and $(\mathcal{E}, \mathcal{F})$ (given in Theorem 2.5 and 2.6) can be also obtained (cf. [6]).

Let $\beta > 0$. We say $(S_X, \mathbf{L}^2(V, d\nu_0))$ satisfies $(PHI(\beta))$, a parabolic Harnack inequality of order β if whenever $u(n,x) \ge 0$ is defined on $[0,4N] \times \bar{B}(y,2r)$ and satisfies

$$u(n+1,x) - u(n,x) = \mathcal{L}u(n,x) \qquad (n,x) \in [0,4N] \times B(y,2r),$$

($\mathcal{L}$ is the corresponding difference operator), then

$$\max_{\substack{N \le n \le 2N \\ x \in B(y,r)}} u(n,x) \le c_{2.7} \min_{\substack{3N \le n \le 4N \\ x \in B(y,r)}} (u(n,x) + u(n+1,x)),$$

where $N \ge 2r$ and $c_{2.8} r^\beta \le N \le c_{2.9} r^\beta$ (cf. [2, 3, 4, 5]). By Theorem 2.7 and a standard argument, we can deduce the following.

Proposition 2.8.
$(S_X, \mathbf{L}^2(V, d\nu_0))$ satisfies $(PHI(S+1))$ w.r.t. the resistance metric.

§3. Homogenization

In this section, we will state our main theorem. First, we give some definition for later use. Let $\hat{V}_0 = \{a_i : i \in I_F\}$. For $Q^* \in \mathrm{Int}(\mathcal{Q}_M)$, we define a matrix $A_{k,Q^*} \in l(\hat{V}_0)$, $k \in I$ by

$$(A_{k,Q^*})_{ij} = P_{Q^*}^{\Psi_k(a_i)}\big(w^1(\tau_{\hat{V}_0}) = a_j\big), \tag{3.1}$$

where w^1 is a Markov chain on $\hat{V}_1$ whose transition probability is determined by Q^* and $\tau_{\hat{V}_0} = \inf\{n \ge 0 : w_n^1 \in \hat{V}_0\}$. Then, by Lemma 2.2 3), the following clearly holds for u.f.r. graphs.

Lemma 3.1. $0 < (A_{k,Q^*})_{ij} < 1$ *if $k \ne i$ and $(A_{k,Q^*})_{kj} = \delta_{kj}$.*

For any $X \in \mathcal{X}_+$ and any $Q_* \in \mathrm{Int}(\mathcal{Q}_M)$ with $\hat{F}(Q_*) = Q_*$, let $S_X^{H_{Q_*}} : \mathbf{L}^2(V, d\nu_0) \to [0, \infty)$ be given by

$$S_X^{H_{Q_*}}(u) = \rho_{Q_0} S_X(v, v) \qquad \text{for} \quad u \in \mathbf{L}^2(V, d\nu_0),$$

where $v \in \mathbf{L}^2(V, d\nu_0)$ satisfies $v(\alpha x) = u(x)$, $x \in V$, and v is Q_*-harmonic on $V \setminus (\alpha^{-1} V)$, i.e.,

$$v(\pi([\omega \cdot i, j])) = \sum_{k \in I_F} (A_{i,Q_*})_{jk} u(\pi([\omega, k])) \qquad \text{for} \quad i \in I, j \in I_F.$$

Here $\omega \cdot i \in \Omega_+$ is given by $(\omega \cdot i)_n = \omega_{n-1}$, $n \geq 2$ and $(\omega \cdot i)_1 = i$. In the same way as we did for $S_X^{\bar{F}}$, we can define a Dirichlet form $S_X^{H_{Q_*}}(\cdot, \cdot)$ on $\mathbf{L}^2(V, d\nu_0)$. It is easy to see that $S_X^{H_{Q_*}}(u) = S_{H_{Q_*}(X)}(u, u)$ where $H_{Q_*}(X)(\omega) = \rho_{Q_0} \sum_{k \in I} {}^t A_{k,Q_*} X(\omega \cdot k) A_{k,Q_*}$ for all $X \in \mathcal{X}_+$ and $u \in \mathbf{L}^2(V, d\nu_0)$. We define a map $\hat{H}_{Q_*} : \mathcal{Q}_+ \to \mathcal{Q}_+$ as $\hat{H}_{Q_*}(Q) = H_{Q_*}(\iota(Q))(\omega)$ for $\omega \in \Omega_+$ (it is independent of the choice of $\omega \in \Omega_+$).

Definition 3.2.
Let μ be a probability measure on $\mathcal{X}_M$ satisfying the following.
1) $\{X(\omega) : \omega \in \Omega_+\}$ are independently identically distributed $\mathcal{Q}_M$-valued random variables under μ.
2) $\mu(\{X \in \mathcal{X}_M : X(\omega) \in \mathcal{Q}_{C_1 Q_0, C_2 Q_0}$ for all $\omega \in \Omega_+\}) = 1$ for some $C_1, C_2 > 0$, where $\mathcal{Q}_{C_1 Q_0, C_2 Q_0} := \{Q \in \mathcal{Q}_M : C_1 Q_0 \leq Q \leq C_2 Q_0\}$.

The following properties are easy, but important.

Proposition 3.3. *Let $Q_0, Q_* \in Int(\mathcal{Q}_M)$ be as above.*
1) $F : \mathcal{X}_M \to \mathcal{X}_M$ and $H_{Q_*} : \mathcal{X}_M \to \mathcal{X}_M$ are continuous maps.
2) $F(\iota(Q_0)) = \iota(Q_0)$, $F(\iota(Q_*)) = H_{Q_*}(\iota(Q_*)) = \iota(Q_*)$.
3) If $X, Y \in \mathcal{X}_+$ and $X \leq Y$, then $F(X) \leq F(Y)$ and $H_{Q_*}(X) \leq H_{Q_*}(Y)$.
4) $F(X) \leq H_{Q_*}(X)$ for all $X \in \mathcal{X}_+$.
5) For any $X, Y \in \mathcal{X}_+$ and $a, b \geq 0$, $F(aX + bY) \geq aF(X) + bF(Y)$ and $H_{Q_*}(aX + bY) = aH_{Q_*}(X) + bH_{Q_*}(Y)$.
6) $E_\mu[F(X)] \leq F(E^\mu[X])$ for all $X \in \mathcal{X}_+$.

Note that the same results hold for $\hat{F}$ and $\hat{H}_{Q_*}$.

Let F^n be the n-th iteration of F. Then we have the following key theorem.

Theorem 3.4. *Under Assumption 2.3, there exists $Q_\mu \in Int(\mathcal{Q}_M)$ such that for all $\omega \in \Omega_+$,*

$$Q_\mu = \lim_{n \to \infty} F^n(X)(\omega) \qquad \text{in} \quad \mathbf{L}^1(\mathcal{Q}_M, \mu). \tag{3.2}$$

Since $\hat{F}(Q_\mu) = Q_\mu \in \mathrm{Int}(\mathcal{Q}_M)$, we can construct a local regular Dirichlet form on K using Q_μ (see Theorem 2.6). We denote the corresponding diffusion as $(X_\mu, \{P_\mu^x\}_{x\in K})$. We now state our main theorem.

Theorem 3.5. *Let μ be the probability measure on $\mathcal{X}_M$ as in Definition* 3.2 *and let $\tau_{Q_0} := \rho_{Q_0} N$. Under Assumption* 2.3*, the following holds.*

$$E^{P_X^{x_n}}[f(\alpha^{-n} w(\tau_{Q_0}^n \cdot))] \to E^{P_\mu^{x_\infty}}[f(w(\cdot))] \quad \text{as } n \to \infty$$

in probability under μ, for any $f \in C_b(D([0,\infty), K) \to \mathbf{R})$ and any sequence $\{x_n\}_{n=1}^\infty \subset V$ with $\alpha^{-n} x_n \to x_\infty \in K$ as $n \to \infty$. Here the expectations are taken over $\omega \in D([0,\infty), K)$.

Further, if there is a convex cone $\mathcal{X}_{sub} \subset \mathcal{X}_{irr}$ such that the following holds; a) $F(\mathcal{X}_{sub}) \subset \mathcal{X}_{sub}$, b) there exists a unique (up to constant multiples) $Q \in \mathcal{X}_{sub} \cap Int(\mathcal{X}_M)$ which satisfies $F(Q) = Q$, c) the support of μ is in $\mathcal{X}_{sub}$. Then P_μ is a constant time change of the diffusion constructed from Q on K.

Remark 3.6. *1) In* [10]*, similar statement is given under the assumption that there exists $Q_1 \in \mathcal{Q}_{irr}$ such that $\hat{H}_{Q_0}^*(Q_1) = Q_1$ where $\hat{H}_{Q_0}^*$ is an adjoint operator of $\hat{H}_{Q_0}$ in $\mathcal{Q}$ (*[10] *Assumption* 3.1*). In general, it is not easy to check this assumption and there is a example in nested fractals that this does not hold.*
2) Let $\mathcal{I}_I$ be the set of all bijective maps σ on I such that $\sigma(I_F) = I_F$, and let G be a subgroup of $\mathcal{I}_I$. Then, as in [10] *Section* 7*, we can obtain similar results for random resistors on $\mathcal{X}_M^G$, a subcone of $\mathcal{X}_M$ which consists of G-invariant elements, if a), b), c) in Theorem 3.5 holds for $\mathcal{X}_M^G$. Especially, we can prove the following; For nested fractals, let G_0 be a subgroup of $\mathcal{I}_I$ generated by all the reflection maps and suppose that the support of μ is in $\mathcal{X}_M^{G_0}$. Then P_μ in Theorem 3.5 is a constant time change of Brownian motion on the nested fractal. (This is because, it is known that a non-degenerate fixed point for G_0-invariant resistors on nested fractals is unique up to constant multiples; see* [16, 13, 15]*.)*
3) Note that non-degenerate fixed points of $\hat{F}$ is not necessarily unique even for nested fractals. In [1] *Example* 6.13*, one parameter family of non-degenerate fixed points on the Vicsek set are given. The homogenization problem for this particular fractal is studied in* [7].
4) By Theorem 3.4 (or Proposition 4.1), we see that Assumption 2.3 *in* [5] *always holds under our Assumption 2.3.*

§4. Convergence of Dirichlet forms

In this section, we will prove Theorem 3.4 and show a convergence of the corresponding forms (Proposition 4.4).

The next proposition is a restricted version of the result by Peirone ([15]), whose original ideas come from Sabot ([16]). For completeness, we give the proof in Appendix A.

Proposition 4.1. *Under Assumption* 2.3*, for each* $M \in \mathcal{Q}_{irr}$*, there exists* $Q_M \in Int(\mathcal{Q}_M)$ *such that*

$$Q_M = \lim_{n\to\infty} \hat{F}^n(M) \tag{4.1}$$

For the proof of Theorem 3.4, we use two lemmas in [10]. Let $H^n_{Q_*}$ be the n-th iteration of H_{Q_*}.

Lemma 4.2. *Assume that* $Q_* \in Int(\mathcal{Q}_M)$ *satisfies* $\hat{F}(Q_*) = Q_*$*. Then, there exist* $c_{4.1} > 0$ *and* $0 < \epsilon < 1$ *such that*

$$E^\mu[\|H^n_{Q_*}(X)(\omega) - H^n_{Q_*}(E^\mu[X])(\omega)\|^2] \le c_{4.1}(1-\epsilon)^n, \ \forall \omega \in \Omega_+, n \ge 1.$$

In particular,

$$\lim_{n\to\infty} \|H^n_{Q_*}(X)(\omega) - H^n_{Q_*}(E^\mu[X])(\omega)\| = 0, \ \mu\text{-}a.e.\ X, \ \forall\omega \in \Omega_+.$$

Proof. By the linearity of H_{Q_*}, $E^\mu[H_{Q_*}(X)(\omega)] = H_{Q_*}(E^\mu[X])(\omega)$. Then the proof is basically the same as that of Lemma 4.1 in [10]. Q.E.D.

Lemma 4.3. ([10], Lemma 4.2) *Let* $\{Y_n\}_{n=1}^\infty$ *be random variables such that* $\sup_n E[Y_n^2] < \infty$*. Let* $Y = \limsup_{n\to\infty} Y_n$ *and assume that* $\lim_{n\to\infty} E[Y_n] = E[Y]$*. Then* $\lim_{n\to\infty} E[|Y - Y_n|] = 0$.

Proof of Theorem 3.4. Let $R_m = E^\mu[F^m(X)(\omega)]$ (R_m is independent of ω). By Proposition 4.1, for each $m \in \mathbf{N}$, there exists $Q_m \in \mathrm{Int}(\mathcal{Q}_M)$ such that $\lim_{n\to\infty} \hat{F}^n(R_m) = Q_m$ and $\hat{F}(Q_m) = Q_m$. On the other hand, by Proposition 3.3 6) we see

$$\hat{F}^n(R_m) \ge R_{n+m} \qquad \forall m, n \in \mathbf{N} \cup \{0\}, \tag{4.2}$$

so that $Q_m \ge Q_{n+m}$. Denote the limit of $\{Q_m\}$ as Q_+, then $\hat{F}(Q_+) = Q_+$. In particular, $Q_+ \in \mathrm{Int}(\mathcal{Q}_M)$ due to Assumption 2.3 1). For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{N}$ such that

$$(1+\epsilon)Q_+ \ge R_m \qquad \forall m \ge N_\epsilon. \tag{4.3}$$

Indeed, if this does not hold, then because $\mathcal{Q}_{(1+\epsilon)Q_+,C_2Q_0}$ is compact, there exists a subsequence $\{l_j\}$ such that $R_{l_j} \geq (1+\epsilon)Q_+$ and $\lim_{j\to\infty} R_{l_j} =: \bar{R}$ exists. On the other hand, by (4.2), we have $\hat{F}^{l_{j'}-l_j}(R_{l_j}) \geq R_{l_{j'}}$ for all $j' \geq j$ so that $Q_+ \geq \bar{R}$, which is a contradiction. By definition of $\{Q_m\}$, for each m and $\epsilon > 0$, there exists $L_{m,\epsilon}$ such that $(1-\epsilon)Q_m \leq \hat{F}^n(R_m)$ for all $n \geq L_{m,\epsilon}$. Combining these facts and noting $\hat{H}^n_{Q_+}(R_m) \geq \hat{F}^n(R_m)$, we have

$$(1-\epsilon)Q_+ \leq \hat{H}^n_{Q_+}(R_m) \leq (1+\epsilon)Q_+ \qquad \forall n \geq L_{m,\epsilon}, m \geq N_\epsilon. \tag{4.4}$$

On the other hand, by Lemma 4.2, we have

$$\begin{aligned} &\lim_{n\to\infty} \|H^n_{Q_+}(F^m(X))(\omega) - H^n_{Q_+}(\iota(R_m))(\omega)\| \\ =\ &\lim_{n\to\infty} \|H^n_{Q_+}(F^m(X))(\omega) - \hat{H}^n_{Q_+}(R_m)\| = 0 \end{aligned} \tag{4.5}$$

μ-a.e. X and for all $\omega \in \Omega_+$. Since $H^n_{Q_+}(F^m(X))(\omega) \geq F^{n+m}(X)(\omega)$, we see that the following holds for some $N'_{\epsilon,\omega} \in \mathbf{N}$,

$$(1+\epsilon)Q_+ \geq F^m(X)(\omega) \qquad \mu\text{-a.e. } X,\ \forall \omega \in \Omega_+, m \geq N'_{\epsilon,\omega}. \tag{4.6}$$

We now consider more about $\hat{H}_{Q_+}$. It is easy to see

$$\sup_n |||\hat{H}^n_{Q_+}||| := \sup_n \sup_{Q\in\mathcal{Q}_M, \|Q\|=1} \|\hat{H}^n_{Q_+}(Q)\| < \infty$$

(see Lemma 4.3 in [10]). Using this, we see that the size of each Jordan cell corresponding to the largest eigenvalue of $\hat{H}_{Q_+}$ is 1. We thus obtain that there exists an orthogonal projection $P_0 : \mathcal{Q}_M \to \mathcal{Q}_M$ so that for each $k \in \mathbf{N}$, there exists $n_k \in \mathbf{N}$ such that

$$|||\hat{H}^{n_k}_{Q_+} - P_0||| \leq 2^{-k}. \tag{4.7}$$

By (4.4) and (4.7), we have $R_m \geq P_0 R_m \geq (1-\epsilon)Q_+$ for all $m \geq N_\epsilon$. Together with (4.3), we have

$$\lim_{n\to\infty} R_n = Q_+. \tag{4.8}$$

Now, by Fatou's lemma and (4.8),

$$E^\mu\big[\limsup_{n\to\infty} \hat{S}_{F^n(X)(\omega)}(u,u)\big] \geq \limsup_{n\to\infty} \hat{S}_{R_n}(u,u) = \hat{S}_{Q_+}(u,u), \tag{4.9}$$

for all $\omega \in \Omega_+$, $u \in l(V_\omega)$, where $V_\omega := \{\pi([\omega, i]) : i \in I_F\}$ is a 0-cell whose address is ω. By (4.6) and (4.9), we have

$$\limsup_{n\to\infty} \hat{S}_{F^n(X)(\omega)}(u,u) = \hat{S}_{Q_+}(u,u),$$

μ-a.e. X and for all $\omega \in \Omega_+$, $u \in l(V_\omega)$. Applying Lemma 4.3 with $Y_n = \hat{S}_{F^n(X)(\omega)}(u,u)$ and $Y = \hat{S}_{Q_+}(u,u)$ (Y is non-random), we have

$$\lim_{n\to\infty} E^\mu\left[|\hat{S}_{F^n(X)(\omega)}(u,u) - \hat{S}_{Q_+}(u,u)|\right] = 0 \qquad \forall \omega \in \Omega_+,\ u \in l(V_\omega).$$

Since $l(V_\omega)$ is finite dimensional, we obtain (3.2) where $Q_\mu = Q_+$. Q.E.D.

Using Theorem 3.4, we can prove the convergence of forms.

Proposition 4.4. *For all* $u \in \mathbf{L}^2(V, d\nu_0)$,

$$\lim_{n\to\infty} E^\mu[|S_{F^n(X)}(u,u) - S_{\iota(Q_\mu)}(u,u)|] \to 0.$$

Proof. When the support of u is in one 0-cell whose address is ω, then the result is clear by Theorem 3.4. When u is compactly supported, we can decompose the form into finite number of forms on 0-cells, so the result still holds. It is then a routine work to show the result for all $u \in \mathbf{L}^2(V, d\nu_0)$. Q.E.D.

We note here that there are several errors in [10] Section 4, where the proof of the theorem corresponding to our Theorem 3.4 is given. They can be fixed, but since we have given a proof of the improved theorem, we omit mentioning where the errors are and how to fix them.

§5. Proof of Theorem 3.5

Now that we obtain Proposition 4.4, the proof of Theorem 3.5 is basically the same as the proof of Theorem 3.6 in [10]. Here we will just state key propositions and briefly comments how to prove them. For detailed arguments, we refer to Section 5 in [10].

As before, define $K_m = \alpha^m \hat{K}$. We consider processes killed at $\alpha^m \hat{V}_0 \setminus \{0\}$. Set $\mathcal{F}_{n,m} = \{u \in l(V_n) : u|_{K\setminus K_m} = 0\}$. For $u, v \in \mathcal{F}_{n,m}$ and $X \in \mathcal{X}_M$, we set $\mathcal{E}_X^{n,m}(u,v) = \rho_{Q_0}^n S_X(u \circ \Psi_1^n, v \circ \Psi_1^n)$. Then, $(\mathcal{E}_X^{n,m}, \mathcal{F}_{n,m})$ is a regular Dirichlet form. We denote the corresponding process $(Y^{X,n,m}, \{P_{X,n,m}^x\}_{x\in V_n\cap K_m})$ and the corresponding generator $L^{(X,n,m)}$. Also, let $(Y^{\mu,m}, \{P_{\mu,m}^x\}_{x\in K_m})$ denote the process corresponding to $(\mathcal{E}_{Q_\mu}, \mathcal{F}_m)$, where $\mathcal{E}_{Q_\mu}$ is a form constructed from Q_μ in Theorem

3.4 and $\mathcal{F}_m = \{f \in \mathcal{F} : f|_{K\setminus K_m} = 0\}$. $L^{(m)}$ is the corresponding generator.

The first key proposition is the following convergence of finite dimensional distribution. This can be obtained by using (H-2), Theorem 2.7 and Proposition 4.4. See Proposition 5.7 in [10] for the proof.

Proposition 5.1. *Let $\{x_n\}_{n=1}^{\infty} \subset V$ be a sequence such that $\alpha^{-n}x_n \to x_\infty \in K_m$. Then, for all $0 < t_1 < \cdots < t_k$,*

$$E^{P_{X,n,m}^{\alpha^{-n}x_n}}[f_1(\omega_{t_1})\cdots f_k(\omega_{t_k})] \to E^{P_{\mu,m}^{x_\infty}}[f_1(\omega_{t_1})\cdots f_k(\omega_{t_k})],$$

in probability under μ, for any $f_1, \cdots, f_k \in C(K_m, \mathbf{R})$.

For a process Z on K let $T_0^r(Z) = \inf\{t \geq 0 : Z(t) \in V_r\}$ and define inductively $T_i^r(Z) = \inf\{t > T_{i-1}^r(Z) : Z(t) \in V_r \setminus Z(T_{i-1}^r(Z))\}$ for $i \in \mathbf{N}$. Then the following holds.

Lemma 5.2. *Let $\{x_n\}_n \subset V$ be as in Proposition 5.1. Then there exist $\gamma, c_{5.1}, c_{5.2} > 0$ such that the following holds for $s \geq 0$, μ-a.e. X.*

$$\limsup_{n\to\infty} \sup_{i\geq 0} P_{X,n,m}^{\alpha^{-n}x_n}(T_{i+1}^r(Y^{X,n,m}) - T_i^r(Y^{X,n,m}) \leq s) \leq c_{5.1}e^{-c_{5.2}(\tau_{Q_0}^r s)^{-\gamma}}.$$

To show this, uniform (elliptic) Harnack inequality for $L^{(X,n,m)}$ is important (see [10] Lemma 5.8). In our case, we can obtain it easily by using Proposition 2.8. See [10] Lemma 5.10 for the detailed proof (thanks to Proposition 2.8, the proof can be shortened).

Using Lemma 5.2, the following tightness is deduced by a standard argument (see Proposition 5.11 in [10]).

Proposition 5.3. *Let $\{x_n\}_n \subset V$ be as in Proposition 5.1. Then $\{P_{X,n,m}^{\alpha^{-n}x_n};\ n \geq 1\}$ is tight (pre-compact) in $D([0,\infty), K_m)$ for μ-a.e. X.*

By Proposition 5.1 and Proposition 5.3, we have the killed process version of Theorem 3.5. Using Lemma 5.2 again, it is easy to deduce the full version of Theorem 3.5.

§Appendix A. Proof of Proposition 4.1

We will prepare several results for the proof. First, we define Hilbert's projective metric on $\mathcal{Q}_+$ (cf. [14]). For $X, Y \in \mathcal{Q}_+$, let

$$h_+(X,Y) = \inf\{\alpha > 0 : X \leq \alpha Y\},\ h_-(X,Y) = \sup\{\alpha > 0 : \alpha Y \leq X\}.$$

Clearly, $h_-(X,Y) \leq h_+(X,Y)$. Define

$$h(X,Y) = \log \frac{h_+(X,Y)}{h_-(X,Y)}.$$

Note that $h(aX, bY) = h(X, Y)$ for all $a, b > 0$ and $h(X, Y) = 0$ if and only if $X = aY$ for some $a > 0$, so that $h(\cdot, \cdot)$ is not a metric. But it is a metric on $\{X \in \mathcal{Q}_+ : \|X\| = 1\}$. Using Proposition 3.3, it is easy to prove the following (cf. [13] Section 3, [15] Remark 3.2 and [16] Proposition 3.3).

Proposition A.1.
1) $h_+(\hat{F}(X), \hat{F}(Y)) \le h_+(X, Y)$ *and* $h_-(\hat{F}(X), \hat{F}(Y)) \ge h_-(X, Y)$ *for all* $X, Y \in \mathcal{Q}_+$. *In particular,* $h(\hat{F}(X), \hat{F}(Y)) \le h(X, Y)$.
2) If $Q_* \in Int(\mathcal{Q}_M)$ *satisfies* $\hat{F}(Q_*) = Q_*$, *then for each* $n \in \mathbf{N} \cup \{0\}$ *and each* $X \in \mathcal{Q}_{irr}$, $h_-(X, Q_*)Q_* \le \hat{F}^n(X) \le h_+(X, Q_*)Q_*$.

For $X, Y \in \mathcal{Q}_+$, let $A^{\pm}(X, Y) = \{u \in l(\hat{V}_0) : u$ is non-constant, $\hat{S}_X(u, u) = h_{\pm}(X, Y)\hat{S}_Y(u, u)\}$. Also, for each $Q \in \mathcal{Q}_{irr}$ and $u \in l(\hat{V}_0)$, define $\mathcal{H}_{n,Q}(u)$ as a unique function on $\hat{V}_n$ so that

$$\hat{S}_{\hat{F}^n(Q)}(u, u) = \hat{\mathcal{E}}^n_Q(\mathcal{H}_{n,Q}(u), \mathcal{H}_{n,Q}(u)),$$

where $\hat{\mathcal{E}}^n_Q(\cdot, \cdot)$ is defined in (2.2). In other word, $\mathcal{H}_{n,Q}(u) \in l(\hat{V}_n)$ is a Q-harmonic extension of $u \in l(\hat{V}_0)$. By definition, $A_{j,Q}(u) = \mathcal{H}_{1,Q}(u) \circ \Psi_j$. Thus the following holds for all $m \ge 0$ and $l \ge n \ge 0$.

$$\begin{aligned} \text{(A.1)} \qquad & \mathcal{H}_{m+n, \hat{F}^{l-n}(Q)}(u) \circ \Psi_{i_1, \cdots, i_m, j_1, \cdots, j_n} \\ = \ & A_{j_n, \hat{F}^{l-n}(Q)} \circ \cdots \circ A_{j_1, \hat{F}^{l-1}(Q)}(\mathcal{H}_{m, \hat{F}^l(Q)}(u) \circ \Psi_{i_1, \cdots, i_m}). \end{aligned}$$

We have the following (cf. [15] Proposition 3.3, [16] Lemma 5.8).

Lemma A.2. *For* $X, Y \in \mathcal{Q}_{irr}$, *define* $h_{\pm,n} = h_{\pm}(\hat{F}^n(X), \hat{F}^n(Y))$. *Then. for each* $0 \le m \le n$,

$$\text{(A.2)} \qquad h_{+,n} \le h_{+,m} \le h_{+,0}, \qquad h_{-,n} \ge h_{-,m} \ge h_{-,0}.$$

$$\text{(A.3)} \qquad \textit{There exists } \lim_{n \to \infty} h_{\pm,n} \in (0, \infty).$$

Further, if $h_{\pm,n} = h_{\pm,0}$, *then for all* $u \in A^{\pm}(\hat{F}^n(X), \hat{F}^n(Y))$, *we have*

$$\begin{aligned} \text{(A.4)} \qquad & \mathcal{H}_{n-m, \hat{F}^m(X)}(u) \circ \Psi_{i_1, \ldots, i_{n-m}} \\ = \ & \mathcal{H}_{n-m, \hat{F}^m(Y)}(u) \circ \Psi_{i_1, \ldots, i_{n-m}} \in A^{\pm}(\hat{F}^m(X), \hat{F}^m(Y)). \end{aligned}$$

Proof. (A.2) is from Proposition A.1 1). (A.3) is a simple consequence of (A.2) and the fact $h_-(X, Y) \le h_+(X, Y)$. Next, if $h_{\pm,n} = h_{\pm,0}$

and $u \in A^-(\hat{F}^n(X), \hat{F}^n(Y))$, then we have

$$\begin{aligned} h_{-,0}\hat{S}_{\hat{F}^n(Y)}(u) &= \hat{S}_{\hat{F}^n(X)}(u) = \hat{\mathcal{E}}^{n-m}_{\hat{F}^m(X)}(\mathcal{H}_{n-m,\hat{F}^m(X)}(u)) \\ &\geq h_{-,m}\hat{\mathcal{E}}^{n-m}_{\hat{F}^m(Y)}(\mathcal{H}_{n-m,\hat{F}^m(X)}(u)) \\ &\geq h_{-,m}\hat{\mathcal{E}}^{n-m}_{\hat{F}^m(Y)}(\mathcal{H}_{n-m,\hat{F}^m(Y)}(u)) \\ &= h_{-,m}\hat{S}_{\hat{F}^n(Y)}(u) \geq h_{-,0}\hat{S}_{\hat{F}^n(Y)}(u). \end{aligned}$$

Thus all the inequalities above are in fact equalities. By the uniqueness of the harmonic extension, we obtain the $(-)$-version of (A.4). $(+)$-version of (A.4) can be proved similarly. Q.E.D.

We next mention a convergence result on positive matrices (cf. [15] Proposition 3.5).

Lemma A.3. *Let B be a finite set. Suppose $A_1, \cdots, A_n, \cdots, A_\infty$ are positive matrices from $l(B)$ to itself and suppose there exists a subsequence $\{\sigma(n)\}_n$ such that*

$$\lim_{n\to\infty}(A_{\sigma(n)})_{ij} = (A_\infty)_{ij} \qquad \textit{for all } i,j \in B. \tag{A.5}$$

Then, for each family of non-negative non-zero vectors $\{v_n\}_n \subset l(B)$,

$$\lim_{n\to\infty}\frac{A_1 \circ \cdots \circ A_n v_n}{\|A_1 \circ \cdots \circ A_n v_n\|} \tag{A.6}$$

exists and it is a positive vector.

Proof. For $p, r \geq 0$, define $T_{p,r} = A_{p+1} \circ \cdots \circ A_{p+r}$. By (A.5) and Theorem 3.6 in [17], $\lim_{r\to\infty}(T_{p,r})_{ij}/(\sum_{s\in B}(T_{p,r})_{is})$ exists for each $i, j \in B$, $p \geq 0$ and it is independent of i and p (in [17], such a property is called strongly ergodic). Using Lemma 3.3 in [17], we obtain the result. Q.E.D.

We now give a key lemma (cf. [15] Lemma 3.6, [16] Section 5.3).

Lemma A.4. *Let $X \in \mathcal{Q}_{irr}$. Then, $\hat{F}(X) = X$ if and only if*

$$h(\hat{F}^n(X), \hat{F}^{n+1}(X)) = h(X, \hat{F}(X)) \qquad \textit{for all } n \in \mathbf{N}. \tag{A.7}$$

Proof. We will assume (A.7) and prove $\hat{F}(X) = X$ (since the other direction is clear). First, note that there exist $j \in I_F$ and a subsequence $\{s(n)\}_n$ so that we can take $u_{\pm,n} \in A^{\pm}(\hat{F}^{s(n)}(X), \hat{F}^{s(n)+1}(X))$ with

$u_{\pm,n}(a_j) = \min_x u_{\pm,n}(x) = 0$ (a_j is a fixed point of Ψ_j). Then by Lemma A.2,

$$\begin{aligned}\text{(A.8)} \qquad u_{\pm,n,m} &:= \mathcal{H}_{s(n)-m,\hat{F}^m(X)}(u_{\pm,n}) \circ \Psi_j^{s(n)-m} \\ &= \mathcal{H}_{s(n)-m,\hat{F}^{m+1}(X)}(u_{\pm,n}) \circ \Psi_j^{s(n)-m} \in A^{\pm}(\hat{F}^m(X), \hat{F}^{m+1}(X)),\end{aligned}$$

for all $m \le s(n)$ where Ψ_j^n is a n-th iteration of Ψ_j. By (A.1), we have

$$\begin{aligned}\text{(A.9)} \qquad u_{\pm,n,m} &= A_{j,\hat{F}^m(X)} \circ \cdots \circ A_{j,\hat{F}^{s(n)-1}(X)}(u_{\pm,n}) \\ &= A_{j,\hat{F}^{m+1}(X)} \circ \cdots \circ A_{j,\hat{F}^{s(n)}(X)}(u_{\pm,n}).\end{aligned}$$

Now choose N_0 large enough so that $\hat{F}^m(X) \in \mathrm{Int}(\mathcal{Q}_M)$ for all $m \ge N_0$ (we use Assumption 2.3 1) here). Using Lemma 3.1, we see that $\{(A_{j,\hat{F}^m(X)})|_B\}_{m \ge N_0}$ are positive matrices from $l(B)$ to itself where $B := \hat{V}_0 \setminus \{a_j\}$. Since all the elements of the matrices are less than 1 (due to Lemma 3.1) and $\sharp B < \infty$, we see that (A.5) holds. We can thus apply Lemma A.3 and obtain that

$$\lim_{n\to\infty} \frac{(u_{\pm,n,m})|_B}{\|(u_{\pm,n,m})|_B\|}$$

exists for $m \ge N_0$. By (A.9), this limit is independent of $m \ge N_0$, we thus denote it as $u_{\pm}$.

We now regard $u_{\pm}$ as a function on $\hat{V}_0$. Then, by the choice of $u_{\pm,n}$, $u_{\pm}(a_j) = 0$ so that $u_{\pm}$ is non-constant. By (A.8), we see that $u_{\pm} \in A^{\pm}(\hat{F}^m(X), \hat{F}^{m+1}(X))$ for all $m \ge N_0$. Thus, $\hat{S}_{\hat{F}^m(X)}(u_{\pm}) = h_{\pm}(\hat{F}^m(X), \hat{F}^{m+1}(X))\hat{S}_{\hat{F}^{m+1}(X)}(u_{\pm})$. On the other hand, by (A.7), $h_+(\hat{F}^m(X), \hat{F}^{m+1}(X))/h_-(\hat{F}^m(X), \hat{F}^{m+1}(X)) \ge 1$ is independent of $m \ge 0$ which we denote by β. Then, we obtain

$$\frac{\hat{S}_{\hat{F}^{m+1}(X)}(u_+)}{\hat{S}_{\hat{F}^{m+1}(X)}(u_-)} = \beta^{-1}\frac{\hat{S}_{\hat{F}^m(X)}(u_+)}{\hat{S}_{\hat{F}^m(X)}(u_-)} = \cdots = \beta^{-(m+1-N_0)}\frac{\hat{S}_{\hat{F}^{N_0}(X)}(u_+)}{\hat{S}_{\hat{F}^{N_0}(X)}(u_-)},$$

for all $m \ge N_0$. If $\beta > 1$, it contradicts to Proposition A.1 2). So, $\beta = 1$ which means $h(X, \hat{F}(X)) = 0$ (by taking $m = 0$). Thus, $\hat{F}(X) = cX$ for some $c > 0$. Using Remark 2.4 1), we have $c = 1$. Q.E.D.

Proof of Proposition 4.1. First, since $c_1 Q_0 \le \hat{F}^n(M) \le c_2 Q_0$ for all $n \in \mathbf{N}$, there exists a subsequence (which could depend on M) $\{\sigma(n)\}_n$ and $Q_M \in \mathcal{Q}_{irr}$ such that

$$\text{(A.10)} \qquad \lim_{n\to\infty} \hat{F}^{\sigma(n)}(M) = Q_M.$$

On the other hand, by Lemma A.2, the following limit exists,

$$h_{\pm} := \lim_{n\to\infty} h_{\pm}(\hat{F}^n(M), \hat{F}^{n+1}(M)) \in (0,\infty).$$

Thus,

$$\begin{aligned} & h(\hat{F}^m(Q_M), \hat{F}^{m+1}(Q_M)) \\ = \quad & \lim_{n\to\infty} h(\hat{F}^{m+\sigma(n)}(M), \hat{F}^{m+1+\sigma(n)}(M)) = \log \frac{h_+}{h_-}, \end{aligned}$$

for all $m \in \mathbf{N}$. By Lemma A.4, this implies $\hat{F}(Q_M) = Q_M$. In particular, $Q_M \in \mathrm{Int}(\mathcal{Q}_M)$ due to Assumption 2.3 1). Using Lemma A.2 again, $\lim_{n\to\infty} h_{\pm}(Q_M, \hat{F}^n(M))$ exists and the limit is 1 due to (A.10). By Proposition A.1 2), this implies (4.1). Q.E.D.

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