# Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets

By

Ben M. HAMBLY, Takashi KUMAGAI[1], Shigeo KUSUOKA
and
Xian Yin ZHOU

**Abstract**

We consider homogeneous random Sierpinski carpets, a class of infinitely ramified random fractals which have spatial symmetry but which do not have exact self-similarity. For a fixed environment we construct "natural" diffusion processes on the fractal and obtain upper and lower estimates of the transition density for the process that are up to constants best possible. By considering the random case, when the environment is stationary and ergodic, we deduce estimates of Aronson type.

# 1 Introduction

The Sierpinski carpet is a fractal subset of $\mathbf{R}^2$ defined as the fixed point of a family of eight contraction maps. We can equivalently construct the fractal by taking $[0,1]^2$, dividing it into nine equal squares of side length $1/3$, and removing the central square. This procedure is then repeated for each of the eight remaining squares and iterated infinitely. The carpet is the resulting fractal and has Hausdorff dimension $d_f = \log 8/\log 3$. A fundamental geometrical property of this set is its infinite ramification, in that any connected subset of the fractal can only be disconnected from the rest by removing a set of dimension 1. This makes analysis on this set much more difficult than for the case of the Sierpinski gasket, (the set formed from dividing a triangle into four equal area triangles with repeated removal of the central, downward pointing triangle) which is a finitely ramified set in that removal of only a finite number of points is required to disconnect a subset of the fractal.

The previous work on infinitely ramified fractals has concentrated on generalised Sierpinski carpets with exact self-similarity. In a series of papers [3], [4], [5], [6], the existence and properties of a Brownian motion, an

[1]Research partly supported by JSPS Program.

isotropic diffusion process, on the two dimensional carpet were determined. This process was defined as the weak limit of a sequence of reflected Brownian motions on a sequence of subsets of $\mathbf{R}^2$ converging to the fractal. Using this probabilistic approach it is possible to examine the Laplacian and the heat kernel on the fractal as these are respectively, the infinitesimal generator and transition density of the Brownian motion. The key to proving the existence of these objects lies in establishing a Harnack inequality, which is accomplished via a straightforward coupling argument in two dimensions. In [8], this work was extended to higher dimensional carpets, using a more complicated coupling argument to prove the necessary Harnack inequality. We will be concerned here with a class of Sierpinski carpets in any dimension but with the added feature of scale irregularity.

There have now been many results on finitely ramified fractals and in this setting some non-self-similar sets have been explored, [11],[18],[19]. There are two natural 'random' fractals that have been considered. Firstly one with spatial homogeneity but scale irregularity and secondly, one without spatial symmetry. For these fractals there are greater oscillations in the heat kernel than that observed in the exactly self-similar case. We will consider a class of infinitely ramified fractals which are scale irregular, thus extending the work on homogeneous random fractals initiated in [18], [11]. We do not consider random recursive fractals [19], as it is essential to our approach via the Harnack inequality, that there is spatial homogeneity for the fractals in our class.

We construct a simple example of the fractals that we will consider in this paper. Firstly define a family of two dimensional carpets, which we will call $\mathrm{SC}(n)$ for $n \geq 3$, where the side length of the carpet is divided by $n$ and a central square of side $(n-2)/n$ is removed. This gives a family of carpets of Hausdorff dimension $d_f = \log 4(n-1)/\log n$, the first member, with $n = 3$ is the original Sierpinski carpet and it, along with the case $n = 4$, is shown in Figure 1.

In order to construct a carpet with scale irregularity we take a sequence $\{\xi_n\}_{n=1}^{\infty}$ where $\xi_n \in \{3,4\}, \quad \forall n$, called the environment sequence. We then apply the affine transformations corresponding to either type 3 or 4 according to the sequence. In this way we produce a fractal which has spatial symmetry but not scale invariance. An example of the first stages in the construction of a version of the type of carpet that we consider is shown in Figure 2.

The recent work of [8] has shown how to construct processes on infinitely ramified fractals in dimensions greater than two. The technique used was to scale a sequence of reflecting Brownian motions to obtain the Brownian

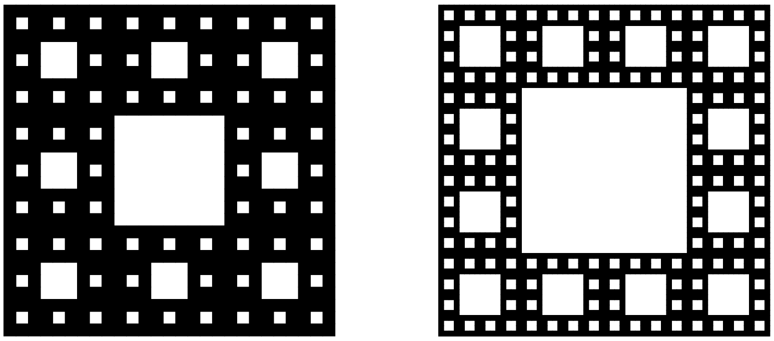

Figure 1: Two Sierpinski carpets from the family SC($n$)

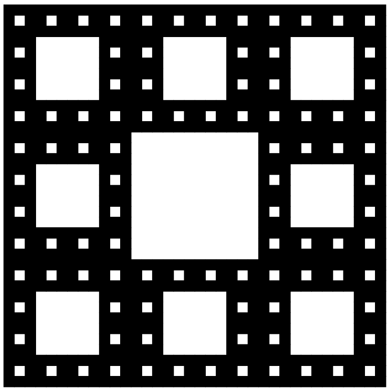

Figure 2: The first stages in the construction of a scale irregular carpet

motion as a limit. We will take a slightly different approach, using a combination of ideas from Barlow-Bass and Dirichlet form techniques. The Dirichlet form methods have been widely applied to finitely ramified fractals and, for infinitely ramified fractals with spectral dimension less than two, they were developed in [22]. We will consider scale irregular carpets in all dimensions and obtain a construction and estimate transition densities using this Dirichlet form based approach. We fix an environment sequence and consider firstly the problem of constructing the diffusion. The main technical point requires an easy (due to the spatial symmetry) extension of the Harnack inequality proved in [8] and this will allow us to construct the process itself via Dirichlet forms, extending [22].

The approach of [22] is to define a series of graph approximations to the fractal and consider the sequence of Poincaré constants generated by the Dirichlet forms on the graph approximations. We will prove that this sequence can be used to normalize the Dirichlet forms to obtain the existence of a limiting Dirichlet form which is local and regular, and hence there is an associated continuous strong Markov process.

Our second result is to obtain bounds on the transition density for the diffusion. We introduce a shortest path counting function which allows us to obtain estimates for the transition density that are best possible up to constants. For the case of the Sierpinski gasket, scale irregularity has been investigated in detail in [11]. The gaskets were used as it is possible to get very precise information about the effect of the scale irregularity because of the simplicity of the structure. The results we obtain here are less precise as we cannot find a nice representation of the spectral dimension. It would be possible to apply the techniques used here to construct diffusions on spatially homogeneous random nested fractals, a class of finitely ramified fractals introduced by Lindstrøm [23], satisfying a very strong symmetry assumption.

Finally we consider the case when the environment is generated by a stationary and ergodic sequence of random variables. Our estimates for the transition density can then be compared with the usual form for such estimates for diffusion processes on fractals. Using subadditivity we can deduce the existence of a resistance scale factor and corresponding exponent, which we use to find expressions for both the spectral and walk dimension. The bounds on the transition density can be expressed in terms of these dimensional exponents but we are only able to control the oscillations in the density by introducing an $\epsilon$. Even if the types of carpet are chosen according to an i.i.d. environment sequence, we cannot describe the precise nature of this oscillation as in [11].

The structure of the paper is as follows. In Section 2 we introduce the fractals and define our notation. In Section 3 and 4 we construct the diffusion process on the carpet. In Section 5 we introduce the shortest path metric and use it in Section 6 to describe the transition density. Section 7 discusses the relationship with the usual form for transition density estimates and the asymptotics of the spectral counting function when the environment is stationary and ergodic. In this paper, $c_i$ $(i \in \mathbf{N})$ will be used as a positive finite constant whose value remains fixed within each proof, while $c_{n.i}$ $(i \in \mathbf{N})$ denotes a fixed constant appearing in Section $n$.

This research was started by three of the authors who considered the resistance and the construction of the process on homogeneous random fractals [21]. After the fourth author passed away, the first author and second author completed the research, obtaining heat kernel estimates for the processes on these fractals using ideas from work which had appeared after the original research was undertaken. This work is dedicated to the fourth author Dr. X. Y. Zhou, our sincere friend who loved stochastic processes.

# 2 Homogeneous random Sierpinski carpets

Let $V$ be a finite set. For each $v \in V$, take $l_v, m_v \in \mathbf{N}$, with $l_v \geq 2$, and let

$$\Psi^{(v)} = \{\psi_1^v, \ldots, \psi_{m_v}^v\}, \quad v \in V,$$

be a family of contractive affine transformations on $\mathbf{R}^d$. That is, each $\psi_i^v$ is a composition of a linear transformation and a translation and maps $[0,1]^d$ to $\Pi_{r=1}^d [k_r^{v,i}/l_v, (k_r^{v,i}+1)/l_v]$ (for $0 \leq \exists k_r^{v,i} \leq l_v - 1, k_r^{v,i} \in \mathbf{N}$) and $\psi_i^v \neq \psi_j^v$ for $i \neq j$. Set $F_0 = [0,1]^d$, $I_v = \{1, \cdots, m_v\}$ for $v \in V$. For $\xi_1, \cdots, \xi_n \in V$, define $F_n^{(\xi_1,\cdots,\xi_n)}$ inductively as

$$F_n^{(\xi_1,\cdots,\xi_n)} \equiv \bigcup_{\substack{w_j \in I_{\xi_j} \\ 1 \leq j \leq n}} \psi_{w_1,\cdots,w_n}^{\xi_1,\cdots,\xi_n}(F_0) = \bigcup_{l \in I_{\xi_1}} \psi_l^{\xi_1}(F_{n-1}^{(\xi_2,\cdots,\xi_n)}).$$

Here, and in the following, we denote $\psi_{w_1,\cdots,w_n}^{\xi_1,\cdots,\xi_n} = \psi_{w_1}^{\xi_1} \circ \cdots \circ \psi_{w_n}^{\xi_n}$.

Let $\Xi = V^{\mathbf{N}}$; we call $\xi \in \Xi$ an *environment*. We will occasionally need a left shift $\theta$ on $\Xi$: if $\xi = (\xi_1, \xi_2, \ldots)$ then $\theta\xi = (\xi_2, \xi_3, \ldots)$. The fractal $E^{(\xi)}$ associated with the environment sequence $\xi$ is defined by

$$E^{(\xi)} = \bigcap_n F_n^{(\xi_1,\cdots,\xi_n)}. \tag{2.1}$$

We call $E^{(\xi)}$ a *homogeneous random Sierpinski carpet* if it satisfies the following.

**Assumption 2.1**
(A1) *(Symmetry) For all $v \in V$, $F_1^v$ is preserved by all the isometries of the unit cube $F_0$.*
(A2) *(Connectedness) For all $\xi \in \Xi$, $E^{(\xi)}$ is connected and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.*
(A3) *(Non-diagonality) Let $B$ be a cube in $F_0$ with length $2/l_v$ and with vertices on $l_v^{-1}\mathbf{Z}$. Then if Int$(F_1^v \cap B)$ is non-empty, it is connected.*

Note that these conditions correspond to the hypotheses $(H1) \sim (H3)$ of [6] and [8]. In this paper, we do not make the assumption $(H4)$ of those papers. This is a condition that $\partial F_0$ is contained in $F_1^v$ and so we allow the boundaries between cells to be fractal (see Figure 3 for an example). Also, note that $E^{(\xi)}$ is not in general self-similar, but the family $\{E^{(\xi)}, \xi \in \Xi\}$ does satisfy the equation $E^{(\xi)} = \cup_{l \in I_{\xi_1}} \psi_l^{\xi_1}(E^{(\theta\xi)})$.

At this point we fix an environment sequence $\xi$, and, except where clarity requires it, will drop $\xi$ from our notation. We define the *word space* $I$ associated with $E$ by

$$I = \bigotimes_{i=1}^{\infty} I_{\xi_i} = \{(w_1, w_2, \dots) : 1 \le w_i \le m_{\xi_i}\}.$$

For $w \in I$ write $w|n = (w_1, \dots, w_n)$, and

$$\psi_{w|n} = \psi_{w_1,\dots,w_n}^{\xi_1,\dots,\xi_n} = \psi_{w_1}^{\xi_1} \circ \dots \circ \psi_{w_n}^{\xi_n}.$$

We write $I_n = \{(w_1, \dots, w_n) : 1 \le w_i \le m_{\xi_i}, 1 \le i \le n\}$ for the set of words of length $n$. Define $M_{\xi|n} = m_{\xi_1} \cdots m_{\xi_n}$ and let $\mu$ ($\mu^{(\xi)}$ to be exact) be the unique measure on $E$ such that $\mu(\psi_{w|n}(E)) = M_{\xi|n}^{-1}$ for all $w \in I$, $n \ge 0$. We call sets of the form $\psi_{w|n}(E)$ *$n$-complexes.*

We construct a sequence of basic graphs which approximate the random carpet. The suffix $n$ will be used to denote $\xi_1, \cdots, \xi_n$ but we will sometimes denote it by $\xi|n$ if some confusion may occur. For $x, y \in I_n$, we write $x \overset{n}{\sim} y$ if the Hausdorff dimension of the set $\psi_x(F_0) \cap \psi_y(F_0)$ equals $d-1$. We use this to define a connectivity matrix for our carpet by defining $q_{xy}^{(n)}$, $x, y \in I_n$ by $q_{xy}^{(n)} = 1$ if $x \overset{n}{\sim} y$ and $q_{xy}^{(n)} = 0$ otherwise. We will consider the finite graph $(I_n, \{q_{xy}^{(n)}\}_{x,y \in I_n})$. Note that by Assumption 2.1, this graph is connected.

For $A \subset I_n, B \subset I_{\theta^n \xi|m}$, we write $A \cdot B \equiv \{x \cdot y \in I_{n+m} : x \in A, y \in B\}$. Let $\mathcal{B}_{n,m} = \{\{x \cdot I_{\theta^n \xi|m}\}; x \in I_{n-m}\}$, $0 \le m \le n$. For $k = m, \cdots, n$ and $B \in \mathcal{B}_{n,m}$, $B_{n,k}(B)$ denotes the set in $\mathcal{B}_{n,k}$ which contains $B$. For each $n \ge 1$, $\partial I_n$ denotes the set of points $x \in I_n$ such that $\psi_x([0,1]^d) \cap \partial [0,1]^d \ne \phi$.

# 3 Basic estimates for constructing the processes

We now construct a sequence of symmetric bilinear forms on the graph approximations to the random carpet. We follow quite closely the development of [22].

For any subset $A \subset I_n$, let $\mathcal{E}_{n,A}$ be a symmetric bilinear form in $C(I_n;\mathbf{R})$ defined by

$$\mathcal{E}_{n,A}(u,v) = \sum_{x,y\in A} q_{xy}^{(n)}(u(x)-u(y))(v(x)-v(y)), \qquad u,v \in C(I_n;\mathbf{R}).$$

We denote $\mathcal{E}_{n,I_n}$ by $\mathcal{E}_n$. Let $< u >_A = |A|^{-1}\sum_{x\in A} u(x)$ for any finite set $A$ and $u \in C(A;\mathbf{R})$, where $|A|$ denotes the cardinality of the set $A$.

We now introduce the following Poincaré constant and effective resistance,

$$\begin{aligned} \lambda_n &= \sup\{\sum_{x\in I_n}(u(x) - < u >_{I_n})^2 : u \in C(I_n;\mathbf{R}), \mathcal{E}_n(u,u) = 1\}, \\ R_n(A,B) &= \min\{\mathcal{E}_n(u,u) : u \in C(I_n;\mathbf{R}), u|_A = 0, u|_B = 1\}^{-1}, \end{aligned}$$

for $A, B \subset I_n$ with $A \cap B = \phi$. Let

$$\begin{aligned} K_n^{(1)} &= K_{\xi_1,\cdots,\xi_n}^{(1)} = [0,(l_{\xi_1}\cdots l_{\xi_n})^{-1}] \times [0,1]^{d-1} \cap F_n^{(\xi_1,\cdots,\xi_n)}, \\ K_n^{(2)} &= K_{\xi_1,\cdots,\xi_n}^{(2)} = [1-(l_{\xi_1}\cdots l_{\xi_n})^{-1},1] \times [0,1]^{d-1} \cap F_n^{(\xi_1,\cdots,\xi_n)}, \end{aligned}$$

and define

$$R_n = R_n(K_n^{(1)}, K_n^{(2)}), \tag{3.1}$$

$$\begin{aligned} B_n = \min\Big\{ & s \in \mathbf{N} : \exists x_1,\cdots,x_s \in I_n, \psi_{x_1}([0,1]^d) \subset K_n^{(1)}, \\ & \psi_{x_s}([0,1]^d) \subset K_n^{(2)}, x_i \overset{n}{\sim} x_{i+1},\ 1 \le \forall i \le s-1 \Big\}. \end{aligned} \tag{3.2}$$

The quantity $B_n$ is the number of steps in the shortest path across the fractal in the $n$-stage approximation. We consider $B_n$ in more detail in Section 5.

## 3.1 Resistance estimates, Poincaré constants and the Harnack inequality

We have the following submultiplicativity result for the resistance $R_n$ defined in (3.1).

**Proposition 3.1** *There exist constants* $c_{3.1}, c_{3.2} > 0$ *such that for each* $n, m \in \mathbf{N}$,

$$c_{3.1} R_n R_{\theta^n \xi|m} \leq R_{n+m} \leq c_{3.2} R_n R_{\theta^n \xi|m}. \tag{3.3}$$

This can be proved in the same way as [25] (see [5] for the case $d = 2$) using a subadditivity argument. As this is a lengthy proof and the basic idea is the same, the details are omitted.

Now, let $L^{(n)}$ be a linear operator in $C(I_n; \mathbf{R})$ given by

$$\sum_{x \in I_n} L^{(n)} u(x) v(x) = -\mathcal{E}_n(u, v), \qquad u, v \in C(I_n; \mathbf{R}).$$

We let $P_t^{(n)} = \exp(tL^{(n)})$, $t \geq 0$ so that $\{P_t^{(n)}\}_{t \geq 0}$ is a symmetric Markov semigroup and denote by $W_t^n$ the continuous time Markov chain on $I_n$ which corresponds to $\mathcal{E}_n$. We say that a subset $G$ of $I_l, l \geq 1$ is $l$-connected, if for any $x, y \in G$, there exists an $n \geq 1$ and a sequence $z_0, \cdots, z_n \in G$ such that $z_0 = x, z_n = y$ and $z_{i-1} \overset{l}{\sim} z_i$ for $1 \leq i \leq n$. The hitting times for the Markov chain are written as $\tau_A = \inf\{t \geq 0 : W_t^n \in A\}$.

We define the sequence of the first Dirichlet eigenvalue of the discrete Laplacian on $I_n$ as

$$\lambda_n^D = \sup \left\{ \sum_{x \in I_n} u(x)^2 : u \in C(I^n; \mathbf{R}), u(y) = 0, y \in \partial I_n, \mathcal{E}_n(u, u) = 1 \right\}.$$

An alternative expression for this quantity is

$$(\lambda_n^D)^{-1} = \inf \left\{ \frac{\mathcal{E}_n(u, u)}{\sum_{x \in I_n} u(x)^2} : u \in C(I^n; \mathbf{R}), u(y) = 0, y \in \partial I_n \right\}. \tag{3.4}$$

We also need another version of the Poincaré constant $\sigma_m$. This is defined by first setting for any $B, B' \in \mathcal{B}_{n,m}$,

$$\sigma_{n,m}(B, B') = \sup\{M_m(< u >_B - < u >_{B'})^2; u \in C(I^n; \mathbf{R}), \mathcal{E}_{n, B \cup B'}(u, u) = 1\},$$

and then

$$\sigma_m = \sup\{\sigma_{n,m}(B, B'); n \geq m \vee 1, B, B' \in \mathcal{B}_{n,m}, B \overset{n}{\sim} B'\}.$$

We now give the relationships between the various scaling constants.

**Lemma 3.2** *There exist constants $c_{3.3}, c_{3.4}, c_{3.5}$ such that*

$$\sup_x E^x(\tau_{\partial I_n}) \geq c_{3.3}\lambda_n^D \geq c_{3.4}\sigma_n \geq c_{3.5}\lambda_n.$$

*Proof.* For the left inequality we use ideas in [4]. Define a Green function $G_n$ as $G_n f(x) = E^x \int_0^{\tau_{\partial I_n}} f(W_t^n)dt$ for $f \in C(I^n; \mathbf{R})$. Then, by definition, $\mathcal{E}_n(G_n f, g) = (f, g)$ for all $f, g : I_n \to \mathbf{R}$ such that $g|_{\partial I_n} = 0$, where we set $(f, g) = \sum_{x \in I_n} f(x)g(x)$. Also from (3.4) there exists a $v_n$ such that $\mathcal{E}_n(v_n, v_n) = (\lambda_n^D)^{-1}(v_n, v_n)$ and indeed we see that $\mathcal{E}_n(v_n, g) = (\lambda_n^D)^{-1}(v_n, g)$ for all $g$ with $g|_{\partial I_n} = 0$ as $v_n$ is the first Dirichlet eigenfunction. Thus

$$(g, v_n) = \mathcal{E}_n(G_n g, v_n) = (\lambda_n^D)^{-1}(G_n g, v_n) = (\lambda_n^D)^{-1}(g, G_n v_n),$$

for all $g$ with $g|_{\partial I_n} = 0$ (the last equality is from the self-adjointness of $G_n$), and hence $G_n v_n = \lambda_n^D v_n$. We also see that if $h_n(x) = E^x \tau_{\partial I_n}$, the mean crossing time, then $h_n = G_n 1$ and

$$\mathcal{E}_n(h_n, h_n) = \mathcal{E}_n(G_n 1, G_n 1) = (1, G_n 1) = \|h_n\|_1.$$

Normalizing $v_n$ so that $\sup_x v_n(x) = 1$ and $v_n(x_0) = 1$, then

$$1 = v_n(x_0) = \frac{G_n v_n(x_0)}{\lambda_n^D} \leq \frac{G_n 1(x_0)}{\lambda_n^D} = \frac{h_n(x_0)}{\lambda_n^D} = \frac{E^{x_0}\tau_{\partial I_n}}{\lambda_n^D},$$

as desired.

In [22] the quantity

$$\lambda_n^{(D)} = \sup\{M_n < u >_{I_n}^2 : u(y) = 0, y \in \partial I_n, \mathcal{E}_n(u, u) = 1\},$$

is defined. Using the fact that $\|u\|_2 \geq \|u\|_1$, we see that $\lambda_n^{(D)} \leq \lambda_n^D$.

For the middle inequality we compare $\lambda_n^{(D)}$ and $\sigma_n$. This is done for the usual Sierpinski carpet (SC(3) in the introduction) by [22], where it is Assumption B-1. It is proved in [22] Proposition 8.1, that there exists $C \in (0, \infty)$ and $k \geq 0$ such that $\sigma_n \leq C\lambda_{n+k}^{(D)}$ for all $n \geq 1$. The proof given in [22] depends upon the symmetry assumption and hence will extend to higher dimensions. We give a brief discussion.

By definition

$$\sigma_m = \sup\{\sigma_{n,m}(B, B') : n \geq m \vee 1, B, B' \in \mathcal{B}_{n,m}, B \overset{n}{\sim} B'\},$$

and we can take $B, B'$ where the supremum is attained and let $u$ be the function such that $\mathcal{E}_n(u, u) = 1$ and $M_m(< u >_B - < u >_{B'})^2 = \sigma_m$. The function $u$ must be symmetric and so we can take a reflection $S : B' \to B$ such that $u(x) \geq 0, u(S(x)) = -u(x), x \in B'$. We now define a function $v_0(x) \in C(I_m)$ by

$$v_0(x) = u(b'.x), \ \ (b'.I_m = B'),$$

and observe that

$$\mathcal{E}_m(v_0, v_0) \leq \mathcal{E}_{n,B\cup B'}(u \vee 0, u \vee 0) \leq 1,$$

as well as $< v_0 >^2_{I_m} = \sigma_m/4M_m$. Now define functions $v_i \in C(I_m)$, for each cell $i$ neighbouring $b'$, by using suitable reflections of $v_0$ which ensure that the boundary values on adjacent cells are equal. We then define a function $v \in C(I_{m+2})$ by setting

$$\begin{array}{rcl} v(a.x) & = & v_0(x), \ \ a \notin \partial I_2, \\ v(b.x) & = & v_b(x), \ \ q_{ba} > 0. \end{array}$$

Thus we have a function which has the properties that

$$v|_{\partial I_m} = 0, \ \ < v >^2_{I_{m+2}} = \frac{\sigma_m}{4M_{m+2}}.$$

Thus, by our choice of $v$, we have ensured that there is no gain in the Dirichlet form as we add up the pieces on the cell and all its neighbours, and hence

$$\mathcal{E}_{m+2}(v, v) \leq (2d+1)\mathcal{E}_m(v_0, v_0) \leq 2d+1.$$

With a final adjustment of $v$ to scale out the $2d+1$ factor, we have a function $v$ such that

$$\lambda^{(D)}_{n+2} \geq M_{m+2} < v >^2_{I_{m+2}} \geq \sigma_m/4,$$

as desired.

The third inequality is [22] (4.3). This is proved by showing that $\lambda_{n+m} \leq \lambda_m + C\lambda_n\sigma_m$ and observing that $\lambda_n \to \infty$. The first part follows from definitions and is exactly the same as [22] Proposition 2.13(1) using [22] Lemma 2.12. The increasing nature of the sequence $\lambda_n$ is proved from the definitions. We can consider the function

$$f_n(x) = (\text{number of the steps from } 0 \text{ to } x \text{ in } I_n)/\sqrt{M_n},$$

so that $\mathcal{E}_n(f_n, f_n) \leq C$ and calculate $\lambda_n \geq \|f_n - < f_n > \|^2_2 \geq C'B^2_n$ for some constants $C, C' > 0$. Thus we can find $n$ such that $\lambda_m \leq \lambda_{n+m}/2$ and hence $\lambda_m \leq \lambda_{n+m} \leq 2C\lambda_n\sigma_m = c_1\sigma_m$ and we have the result. ∎

The most difficult part in the construction of the diffusion process on the Sierpinski carpet was to show the continuity of harmonic functions via a Harnack inequality. This was first shown by Barlow-Bass [3] using coupling arguments, valid only for the 2-dimensional case. After that, Kusuoka-Zhou [22] obtained the Harnack inequality under mild conditions, but the argument was restricted to the case when the domain of the Dirichlet form was contained in the continuous functions. Recently, Barlow-Bass [7], [8] obtained the inequality for the higher dimensional carpets using a coupling result for reflecting Brownian motion on the pre-carpet. Their arguments rely strongly on the spatial symmetry of the carpets, and, as our random fractals still have that symmetry, we can apply their arguments directly. We now translate the theorems in [8], [9] into our setting.

**Theorem 3.3** *(Knight move)*
*For any $l \geq 1$, any $l$-connected non-void subset $G_0$ of $I_l$ and any non-void subset $G_1$ of $I_l$, if dist $(\cup_{x\in G_0}\psi_x(E), \cup_{x\in G_1}\psi_x(E)) > 0$, then*

$$\inf\{P_x^{(l+n)}(\tau_{z\cdot I_n} < \tau_{G_1\cdot I_n}); z \in G_0, x \in G_0\cdot I_n, n \geq 1, \theta^l\xi \in \Xi\} > 0.$$

Note that the infimum is uniformly positive regardless of the environments as we have a finite family of contraction maps ($|V| < \infty$).

Using this fact in an essential way, one can obtain the coupling result i.e., in our case, that there are (not independent) random walks on the graphical approximations to the carpet which couple with positive probability before they exit some region. We do not state the result here but will state the coupling result for the limiting process later (Theorem 4.10). From the coupling result we can deduce the following uniform Harnack inequality for our approximating Markov chains.

**Theorem 3.4** *(Uniform Harnack inequality)*
*There exists $\delta > 0$ which is independent of the choice of $\xi \in \Xi$ such that the following holds. For $G_0, G_1$ as in Theorem 3.3, if $n \geq 1$, $u \in C(I_{l+n}; [0,\infty))$ and $L^{(l+n)}u|_{I_{l+n}\setminus G_1\cdot I_n} = 0$, then*

$$\delta \max_{x\in G_0\cdot I_n} u(x) \leq \min_{x\in G_0\cdot I_n} u(x).$$

We learned the following lemma for electric networks, which is an extension of the theorem in [13], [28], from M.T. Barlow.

**Lemma 3.5** *Let $(V, E)$ be a connected graph and $X_n$ be a simple random walk on $V$. For each $A, B \subset V$, $A \cap B = \phi$, there exists a probability measure $\Pi_A$ on $A$ such that*

$$\sum_{x \in A} E^x \tau_B \Pi_A(x) = R(A, B) \sum_{y \in V} f_{A,B}(y), \tag{3.5}$$

*where $f_{A,B}(x) = P^x(X_n \text{ hits } A \text{ before } B)$.*

*Proof.* As $f_{A,B}$ is non-negative and harmonic on $V \setminus (A \cup B)$ and $f|_B = 0$, there exists $\Pi'_A : A \to \mathbf{R}_+$ such that

$$f_{A,B}(x) = \sum_{y \in A} g_B(x, y) \Pi'_A(y),$$

where $g_B(x, y)$ is a Green function for $V$ killed on $B$, which is the average number of times for the random walk starting at $x$ to visit $y$ before arriving at $B$. By Ohm's law, $1 = R(A, B)\Pi'_A(A)$. On the other hand, $E^x \tau_B = \sum_{y \in V} g_B(x, y)$. Thus,

$$\begin{aligned} \sum_{x \in A} E^x \tau_B \Pi'_A(x) &= \sum_{x \in A} \sum_{y \in V} g_B(x, y) \Pi'_A(x) \\ &= \sum_{y \in V} \sum_{x \in A} g_B(y, x) \Pi'_A(x) = \sum_{y \in V} f_{A,B}(y). \end{aligned}$$

We thus obtain (3.5). ∎

For $A \subset I_n$ and for $m \le n$, define

$$\begin{aligned} (3.6)\quad D^0_m(A) &= \{m\text{-complex which contains } A\}, \\ (3.7)\quad D^1_m(A) &= D^0_m(A) \cup \{B : B \text{ is a } m\text{-complex}, D^0_m(A) \cap B \neq \emptyset\}. \end{aligned}$$

**Proposition 3.6**
*1) For $G_0, G_1$ as in Theorem 3.3 and for $l \in \mathbf{N}$, there exists $c_{3.6} = c_{3.6}(l) > 0$ so that*

$$c_{3.6} R_{\theta^l \xi|k} \le R_{l+k}(G_0 \cdot I_k, G_1 \cdot I_k), \qquad \textit{for all } n \in \mathbf{N}. \tag{3.8}$$

*2) There exist $c_{3.7}, c_{3.8} > 0$ so that*

$$c_{3.7} R_n M_n \le E^x \tau_{\partial I_n}, \qquad \textit{for all } x \in (D^1_2(\partial I_n))^c, \tag{3.9}$$

$$E^x \tau_{\partial I_n} \le c_{3.8} R_n M_n, \qquad \textit{for all } x \in I_n. \tag{3.10}$$

*Proof.* We first remark on a fundamental property of resistance. Resistance increases if we cut bonds in the network and it decreases if we short vertices. Using such a shorting argument, one can easily obtain (3.8).

Now, set $A_n = (D_2^1(\partial I_n))^c$. Note that there exist $c_1, c_2 > 0$ such that $c_1 M_n \le |A_n| \le c_2 M_n$ for large $n$. From Lemma 3.5, we have

$$\sum_{x \in A_n} E^x \tau_{\partial I_n} \Pi_{A_n}(x) = R(A_n, \partial I_n) \sum_{y \in I_n} f_{A_n, \partial I_n}(y).$$

As $0 \le f \le 1$ and $f|_{A_n} = 1$, $c_3 M_n \le \sum_{y \in I_n} f_{A_n, \partial I_n}(y) \le c_4 M_n$. Further, by using cutting and shorting arguments again, we have $c_5 R_{\theta^1 \xi|n-1} \le R(A_n, \partial I_n) \le R_{\theta^1 \xi|n-1}$. Using these facts and the Harnack inequality for $E^x \tau_{\partial I_n}$ (which can be proved in the same way as [3] Proposition 4.2 using Theorem 3.4), we have (3.9) and (3.10) for $x \in A_n$. Now, using Theorem 3.3, we can show that $\sup_{x \in I_n} E^x \tau_{\partial I_n} \le c_6 \sup_{x \in A_n} E^x \tau_{\partial I_n}$ for some $c_6 > 0$ in the same way as [3] (4.5). This proves (3.10) for $x \in I_n$. ∎

As a corollary we have the following control on the scaling constants. Let $T_n = R_n M_n$ denote the $n$-th level time scale factor.

**Corollary 3.7** *There exist constants $c_{3.9}, c_{3.10}, c_{3.11}$ such that*

$$T_n \ge c_{3.9} \lambda_n^D \ge c_{3.10} \sigma_n \ge c_{3.11} \lambda_n.$$

## 3.2 Hitting time estimates and tightness of the processes

We now use the Harnack inequality to obtain some hitting time estimates for the sequence of Markov chains. We consider the scaled Markov chain $W_t^{(m)} = W_{T_m t}^m$ and write $S_{D_r^i(x)}(W^{(m)}) = \inf\{t \ge 0 : W_t^{(m)} \notin D_r^i(x)\}$ $(i = 0, 1)$, and $S_B$ for the exit time from any set $B$.

**Lemma 3.8** *There exist constants $c_{3.12}, c_{3.13}$ such that for each $m > r$,*

$$\begin{aligned} c_{3.12} T_r^{-1} &\le E^z S_{D_r^1(x)}(W^{(m)}), && \forall z \in D_r^0(x), \\ E^z S_{D_r^1(x)}(W^{(m)}) &\le c_{3.13} T_r^{-1}, && \forall z \in D_r^1(x). \end{aligned}$$

*Proof.* As $S_{D_l^1(x)}(W^{(m)})$, $l \geq n$ is a decreasing sequence, we deduce

$$S_{D_l^1} = \sum_{i=l}^{\infty} (S_{D_i^1(x)}(W^{(m)}) - S_{D_{i+1}^1(x)}(W^{(m)})). \tag{3.11}$$

From Proposition 3.6 we have $E(S_{D_i^1} - S_{D_{i+1}^1}) \leq \gamma(\xi_{i+1}) T_{i+1}^{-1}$, where $\gamma(v)$ is a constant determined by the type of the carpet, $v \in V$, used.

Let $c_1 = \max_{v \in V} \gamma(v)$. From (3.11) we have, for all $y \in D_l^1(x)$,

$$E^y S_{D_l^1(x)}(W^{(m)}) \leq c_1 \sum_{i=l}^{\infty} T_{i+1}^{-1} \leq c_2 T_l^{-1}. \tag{3.12}$$

Lower bounds can be obtained in the same way using Proposition 3.6. ∎

Since $S_{D_l^1(x)}(W^{(m)}) \leq t + 1_{(S_{D_l^1} > t)}(S_{D_l^1} - t)$ we have, from (3.12),

$$\begin{aligned} E^z S_{D_l^1} &\leq t + E^z(1_{(S_{D_l^1} > t)} E^{X_t}(S_{D_l^1})) \\ &\leq t + P^z(S_{D_l^1} > t) c_2 T_l^{-1}. \end{aligned}$$

So $P^z(S_{D_l^1} \leq t) \leq c_2^{-1} T_l t + (1 - c_2^{-1})$ for each $z \in D_l^0(x)$, and we deduce there exist $c_3 > 0$, $c_4 \in (0,1)$ such that

$$P^z(S_{D_l^1(x)}(W^{(m)}) \leq t) \leq c_3 T_l t + c_4, \quad t \geq 0. \tag{3.13}$$

We can improve this to an exponential estimate on $P^z(S_{D_l^1(x)}(W^{(m)}) \leq t)$. In order to do this we define the following function of time and space,

$$k = k(n,l) = \inf\{j \geq 0 : \frac{T_{n+j}}{B_{n+j}} \geq \frac{T_l}{B_n}\}. \tag{3.14}$$

The function $k(n,l)$ was defined in [11] and a version of it used in [20]. Its properties will be as in those papers. First, the following inequalities are clear: $2 \leq b_v \leq b^*$, $t_* \leq t_v \leq t^*$, $2 \leq b_v \leq t_v/b_v \leq t^*/2$, where $b^* \equiv \max_v b_v, t_* \equiv \min_v t_v$, $t^* \equiv \max_v t_v$ ([8] Proposition 5.1, here we define $b_v = B_1, t_v = T_1$ if $\xi_1 = v$). Summarising, we have

1. If $n \geq l$ then $T_n/B_n \geq T_l/B_n$, and so $k(n,l) = 0$.
2. If $n < l$ then $k(n,l) > l - n$ and we can show that there exists a constant $c_5 > 1$ such that

$$l - n < k(n,l) \leq c_5(l-n) \quad \text{when} \quad n < l. \tag{3.15}$$

Note also

$$l \le n + k(n,l) \le c_5 l \quad \text{ if } n < l. \tag{3.16}$$

Using the bounds on $t_v/b_v$ above, Proposition 3.1, and Proposition 5.1 (which can be proved independently), there exist $c_6, c_7 > 0$ such that for $i \ge 0$,

$$c_6 2^{i+1} \frac{T_{n+j}}{B_{n+j}} \le \frac{T_{n+1+j+i}}{B_{n+1+j+i}} \le c_7 (t^*/2)^{i+1} \frac{T_{n+j}}{B_{n+j}}.$$

From this, it follows that

$$|k(n+1,l) - k(n,l)| \le c_8, \text{ for all } n, l. \tag{3.17}$$

So, we have,

$$|\log\Big(\frac{B_{n'+k(n',l)}}{B_{n'}}\Big) - \log\Big(\frac{B_{n+k(n,l)}}{B_n}\Big)| \le (1+c_8)|n'-n| \log b^*. \tag{3.18}$$

As in [11] we can define the approximate walk and spectral dimensions,

$$d_w(n) = \frac{\log T_n}{\log B_n}, \qquad d_s(n) = \frac{2\log M_n}{\log T_n}. \tag{3.19}$$

**Lemma 3.9** *Let $0 < t < 1$, $0 < r < 1$, and let $l, n$ satisfy*

$$T_l^{-1} \le t < T_{l-1}^{-1}, \quad B_n^{-1} \le r < B_{n-1}^{-1}.$$

*Then writing $k = k(n,l)$, there exist constants $c_{3.14}, c_{3.15}$ such that*

$$\begin{aligned} \tfrac{1}{2} \exp\Big(c_{3.14} \frac{B_{n+k}}{B_n}\Big) &\le \exp\Big(\Big(\frac{r^{d_w(n+k)}}{t}\Big)^{1/(d_w(n+k)-1)}\Big) \\ &\le \exp\Big(c_{3.15} \frac{B_{n+k}}{B_n}\Big). \end{aligned} \tag{3.20}$$

*Proof.* As in [11] Lemma 4.2. ∎

**Lemma 3.10** *There exist constants $c_{3.16}, c_{3.17}$ such that if $k = k(n,l)$ then for all $x \in E$, and $n, l \le m$,*

$$P^x(S_{D_n^1(x)}(W^{(m)}) \le T_l^{-1}) \le c_{3.16} \exp(-c_{3.17} B_{n+k}/B_n). \tag{3.21}$$

*Proof.* If $j \geq 0$, then for the process $X$ to cross one $n$-complex it must cross at least $N = B_{n+j}/B_n$, $(n+j)$-complexes. So, there exists $0 < c < 1$ such that

$$S_{D_n^1(x)}(W^{(m)}) \geq \sum_{i=1}^{cB_{j+n}/B_n} V_i,$$

where $V_i$ are i.i.d. and have distribution $S_{D_{n+j}^1(x)}(W^{(m)})$. Lemma 1.1 of [3] states that if $P(V_i < s) \leq p_0 + \alpha s$, where $p_0 \in (0,1)$ and $\alpha > 0$, then

$$\log P(\sum_1^{cN} V_i \leq t) \leq 2(\alpha c N t/p_0)^{1/2} - cN \log(1/p_0). \tag{3.22}$$

Thus, using (3.13) and (3.22), we have

$$\log P(S_{D_n^1(x)}(W^{(m)}) \leq T_l^{-1}) \leq c_1(B_{n+j}/B_n)^{1/2}[(T_{n+j}/T_l)^{1/2} - c_2(B_{n+j}/B_n)^{1/2}]. \tag{3.23}$$

Given $k = k(n,l)$ as above, there exists $c_3$ and $k_0$ such that $k - c_3 \leq k_0 \leq k$, and

$$(T_{n+k_0}/T_l)^{1/2} < \tfrac{1}{2} c_2 (B_{n+k_0}/B_n)^{1/2}.$$

Provided $k_0 \geq 1$ we deduce

$$\log P(S_{D_n^1(x)}(W^{(m)}) \leq T_l^{-1}) \leq -\tfrac{1}{2} c_1 c_2 B_{n+k_0}/B_n \leq -c_{3.17} B_{n+k}/B_n.$$

Choosing $c_{3.16}$ large enough we have $1 < c_{3.16} \exp(-c_{3.17} B_{n+k}/B_n)$ whenever $k < c_3 + 1$, so that (3.21) holds in all cases. ∎

Let $\{P_x^{(n)}; x \in I_n\}$ be a Markov process on $I_n$, whose generator is $L^{(n)}$. Then, as a corollary to this lemma, we have the following tightness of the processes.

**Proposition 3.11**

$$\lim_{T\to 0} \limsup_{m\to\infty} \sup\{|B|^{-1} \sum_{x \in B} P_x^{(n)}[W^n_{T_n t} \in I_n \backslash B];\ t \in (0,T], B \in \mathcal{B}_{n,m}, n \geq m\} = 0$$

Note that this corresponds to Proposition 4.9 of [22]. As was shown in that paper, the Harnack inequality is not necessary for the proof of tightness. Here we obtain the sharper estimate (3.21), using the Harnack inequality, as we will need this estimate later for deriving detailed heat kernel bounds.

We proceed following [22] Section 4. For each $n \geq 1$, let $\tilde{P}_n : \mathbf{L}^1(E, d\mu) \to C(I_n; \mathbf{R})$ and $\iota_n : C(I_n; \mathbf{R}) \to \mathbf{L}^\infty(E, d\mu)$ be given by

$$\tilde{P}_n f(x) = \mu(\psi_x(E))^{-1} \int_{\psi_x(E)} f(x)\mu(dx), \; x \in I_n, f \in \mathbf{L}^1(E, d\mu),$$

$$\iota_n u(y) = u(x), \quad \text{if } y \in \psi_x(E), \quad x \in I_n, u \in C(I_n; \mathbf{R}).$$

We want to construct a process on the random carpet and use the projection and injection operators to transfer the Markov chains on the approximating graphs onto the fractal itself. Let $Q_t^{(n)} = \iota_n \circ P_{T_n t}^{(n)} \circ \tilde{P}_n, \; t > 0, n \geq 1$. Then $\{Q_t^{(n)}\}_{t>0}$ is a semigroup of symmetric Markov operators in $\mathbf{L}^2(E, d\mu)$. Let $P_n = \iota_n \circ \tilde{P}_n, \; n \geq 1$. We denote $\|\cdot\|_2 = \|\cdot\|_{\mathbf{L}^2(E,d\mu)}$. Then, as in Lemma 4.10 of [22], we have the following.

**Lemma 3.12**
*(1)* $\|(I - P_m)\iota_n u\|_2^2 \leq \lambda_{n-m} M_n^{-1} \mathcal{E}_n(u, u), \; u \in C(I_n; \mathbf{R}), 1 \leq m \leq n.$
*(2) There is a constant* $c_{3.18} > 0$ *such that*

$$\|(I - P_m)Q_t^{(n)}\|_{\mathbf{L}^2 \to \mathbf{L}^2} \leq c_{3.18} t^{-1/2} \lambda_{n-m}/\lambda_n, \; t > 0, 1 \leq m \leq n.$$

*(3)* $\limsup_{t\to 0} \limsup_{n\to\infty} \{\|f - Q_t^{(n)} f\|_2; n \geq 1\} = 0$ *for any* $f \in C(E; \mathbf{R})$.

We now construct the paths of our Markov chains on the carpet. Let us take $x_0 \in E$ and fix it. Let $Q^{(n)}$ be the probability law of $\{\psi_{w(T_n t)}(x_0), t \in \mathbf{Q}_+\}$ under $M_n^{-1} \sum_{x \in I_n} P_x^{(n)}(dw)$ where $\mathbf{Q}_+ \equiv \mathbf{Q} \cap [0, \infty)$. Then, $Q^{(n)}, \; n \geq 1$ are probability measures in $E^{\mathbf{Q}_+}$. As $E^{\mathbf{Q}_+}$ is compact, we see that $\{Q^{(n)}; n \geq 1\}$ is tight. Using Lemma 3.12, we can prove the following in the same way as [22] Theorem 4.5.

**Theorem 3.13** *For each cluster point $\tilde{Q}$ of $\{Q^{(n)}\}$, there is a strongly continuous symmetric Markov semigroup $\{Q_t\}_{t\geq 0}$ in $\mathbf{L}^2(E, d\mu)$ such that*

$$\begin{aligned} &E^{\tilde{Q}}[f_0(w(t_0))f_1(w(t_1)) \cdots f_n(w(t_n))] \\ &= (Q_{t_n - t_{n-1}}(f_{n-1}(Q_{t_{n-1}-t_{n-2}}(f_{n-2}(\cdots(Q_{t_1-t_0}f_0)\cdots)))), f_n)_{\mathbf{L}^2} \end{aligned}$$

*for any $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \in \mathbf{Q}_+$ and $f_0, \cdots, f_n \in C(E; \mathbf{R})$. Moreover, for any $f \in \mathbf{L}^2(e; d\mu)$ with $\int_E f d\mu = 0$, we have*

$$\|Q_t f\|_2 \leq e^{-t}\|f\|_2, \; t \geq 0.$$

# 4 Dirichlet forms

This section gives a construction of the limiting Dirichlet form on a random Sierpinski carpet. We follow the results of [22] Section 5. Let $\mathcal{E}^{(n)}$, $n \geq 1$ be a Dirichlet form in $\mathbf{L}^2(E, d\mu)$ given by

$$\mathcal{E}^{(n)}(f,g) = R_n \mathcal{E}_n(\tilde{P}_n f, \tilde{P}_n g), \;\; f, g \in \mathbf{L}^2(E, d\mu).$$

**Lemma 4.1** *There exists a constant $c_{4.1}$ such that*

$$\mathcal{E}^{(n)}(f,f) \leq c_{4.1} \mathcal{E}^{(n+m)}(f,f), \;\; \forall n, m, \;\; f \in \mathbf{L}^2(E, d\mu).$$

*Proof.* By definition of $\mathcal{E}^{(n)}$ and the fact that $\tilde{P}$ is a projection,

$$\mathcal{E}^{(n)}(f,f) = R_n(\xi) \mathcal{E}_n^{(\xi)}(\tilde{P}_n \tilde{P}_{n+m} f, \tilde{P}_n \tilde{P}_{n+m} f).$$

We have the following result, which is obtained as in [22] Lemma 2.12, combined with the inequality $\sigma_n \leq c_1 T_n$.

$$\sum_{x,y \in I_n} (< u >_{x.I_m} - < u >_{y.I_m})^2 q_{xy}^{(n)} \leq c_2 R_m \mathcal{E}_{n+m}(u,u), \tag{4.1}$$

for all $u \in C(I_{n+m}), n, m \geq 0$. Now if $u \in C(I_{n+m})$, then, by (4.1)

$$\begin{aligned} \mathcal{E}_n^{(\xi)}(\tilde{P}_n u, \tilde{P}_n u) &= \sum_{x,y \in I_n} (< \tilde{P}_n u > (x) - < \tilde{P}_n u > (y))^2 q_{xy}^{(n)} \\ &\leq c_2 R_m(\theta^n \xi) \mathcal{E}_{n+m}^{(\xi)}(u,u) \end{aligned}$$

as required. ∎

Let $\mathcal{D}ch$ be the set of Dirichlet forms associated with the cluster points of $\{Q^{(n)}\}$ and let $\mathcal{D}_0^{(\xi)} = \{f : \sup_n \mathcal{E}^{(n)}(f,f) < \infty\}$.

**Lemma 4.2** *For any $f \in \mathcal{D}_0^{(\xi)}$ and $i \in I_n$, $f \circ \psi_i \in \mathcal{D}^{(\theta^n \xi)}$ holds.*

*Proof.* This follows easily from

$$\mathcal{E}_{m+n}(f,f) \geq \sum_{i \in I_n} \mathcal{E}_m(f \circ \psi_i, f \circ \psi_i). \quad ∎$$

**Lemma 4.3** *(1) For any $\mathcal{E} \in \mathcal{D}ch$, we have $\mathcal{D}^{(\xi)}(\mathcal{E}) = \mathcal{D}_0^{(\xi)}$.*
*(2) There exist $c_{4.2}, c_{4.3} > 0$ such that*

$$c_{4.2} \sup_n \mathcal{E}^{(n)}(f,f) \leq \mathcal{E}^{(\xi)}(f,f) \leq c_{4.3} \liminf_{n\to\infty} \mathcal{E}^{(n)}(f,f), \tag{4.2}$$

*for any $\mathcal{E} \in \mathcal{D}ch$ and $f \in \mathcal{D}_0$.*

*Proof.* The first result follows from the second. For the second we use Lemma 4.1 in the same way as [22] Theorem 5.4. ∎

We can now write down a decomposition of the limiting Dirichlet form which holds for all the cluster points.

**Lemma 4.4** *There exists a constant $c_{4.4}$ such that*

$$\mathcal{E}^{(\xi)}(f,f) \geq c_{4.4} \sum_{i\in I_n} \mathcal{E}^{(\theta^n\xi)}(f\circ\psi_i, f\circ\psi_i) R_n, \quad \forall f \in \mathcal{D}_0.$$

*Proof.* By construction we have

$$\mathcal{E}_{n+k}(f,f) \geq \sum_{i\in I_k} \mathcal{E}_n^{(\theta^k\xi)}(f\circ\psi_i, f\circ\psi_i)$$

so that

$$\begin{aligned}
\mathcal{E}^{(n+k)}(f,f) &\geq R_{n+k} \sum_{i\in I_k} \mathcal{E}_n^{(\theta^k\xi)}(f\circ\psi_i, f\circ\psi_i) \\
&\geq c_{3.1} R_k \sum_{i\in I_k} \mathcal{E}^{(\theta^k\xi)(n)}(f\circ\psi_i, f\circ\psi_i).
\end{aligned}$$

Taking limits as $n \to \infty$ gives

$$\liminf_{n\to\infty} \mathcal{E}^{(n+k)}(f,f) \geq R_k \sum_{i\in I_k} \liminf_{n\to\infty} \mathcal{E}^{(\theta^k\xi)(n)}(f\circ\psi_i, f\circ\psi_i).$$

Then for any cluster point $\mathcal{E}^{(\xi)}$ we have by Lemma 4.3, that

$$\begin{aligned}
\mathcal{E}^{(\xi)}(f,f) &\geq c_{4.2} \sup_n \mathcal{E}^{(n)}(f,f) \geq c_{4.2} \liminf_{n\to\infty} \mathcal{E}^{(n)}(f,f) \\
&\geq c_1 R_k \sum_{i\in I_k} \mathcal{E}^{(\theta^k\xi)}(f\circ\Psi_i, f\circ\Psi_i),
\end{aligned}$$

for any cluster point $\mathcal{E}^{(\theta^k\xi)}$. ∎

In the following we take $\mathcal{E} \in \mathcal{D}ch$ and fix it. For $\{Q_t^{(n)}\}$ and $\{Q_t\}$ as defined in Section 3 and $\lambda > 0$, set $U_n^\lambda = \int_0^\infty e^{\lambda t} Q_t^{(n)} dt$, $U^\lambda = \int_0^\infty e^{\lambda t} Q_t dt$.

**Proposition 4.5** *For each $f \in \mathbf{L}^2(E, d\mu) \cap \mathbf{L}^\infty(E, d\mu)$, $U^\lambda f$ is a continuous function on $E$.*

The proof of this proposition is essentially the same as that in [3] Section 6. Here we only sketch the outline of the proof and refer the reader to the paper for details. First, by the uniform Harnack inequality (Theorem 3.4), one can deduce the following in the same way as in [3] Section 3:
There exist constants $\beta, C > 0$ such that, if $u_n$ is non-negative bounded and harmonic w.r.t. $L^{(n)}$, then

$$|u_n(x) - u_n(y)| \le C\{ \operatorname{dist}(\Psi_x(E), \Psi_y(E)) \vee (\min_{v \in V} l_v)^{-n}\}^\beta \|u_n\|_\infty$$

for all $x, y \in I^n$.

Using this, it is not hard to show that for $f \in \mathbf{L}^2 \cap \mathbf{L}^\infty$, (a suitable continuous modification of) $\{U_n^\lambda f\}_{n=1}^\infty$ is equicontinuous and uniformly bounded. Therefore by the Ascoli-Arzelà theorem, there exists subsequence of which converges uniformly. By Theorem 3.13, the limit should be $U^\lambda f$ so that the continuity of the function is deduced.

Using this proposition, we have the following.

**Theorem 4.6** *$(\mathcal{E}, \mathcal{D}_0)$ is a local regular Dirichlet form on $\mathbf{L}^2(E, d\mu)$.*

*Proof.* The local property follows easily using the right inequality in (4.2). Thus we will only prove the regularity of the form.

We take $G_0, G_1$ as in Theorem 3.3 and fix them. Let $u_{l+k}(x) = P^x(\tau_{G_0 \cdot I_k} < \tau_{G_1 \cdot I_k}) \in C(I_{l+k}; \mathbf{R})$. Then, we see that $u_{l+k}|_{G_0 \cdot I_k} = 1, u_{l+k}|_{G_1 \cdot I_k} = 0$ and $\mathcal{E}_{l+k}(u_{l+k}, u_{l+k}) = R_{l+k}(G_0 \cdot I_k, G_1 \cdot I_k)^{-1}$. Using (3.8), we have

$$\sup_{n \ge l} R_n \mathcal{E}_n(u_n, u_n) = \sup_{n \ge l} \mathcal{E}^{(n)}(\iota_n u_n, \iota_n u_n) < \infty. \tag{4.3}$$

On the other hand, as $\|\iota_n u_n\|_2 \le 1$, by the Banach-Alaoglu theorem, we have a subsequence (which we also denote $\iota_n u_n$) so that $\iota_n u_n$ converges weakly to some $v \in \mathbf{L}^2$. Then, clearly $\tilde{P}_n \iota_k u_k \to \tilde{P}_n v$ pointwise as $k \to \infty$ ($n$ is fixed). This, with (4.3) and Lemma 4.1, gives

$$\mathcal{E}^{(n)}(v, v) = \lim_{k \to \infty} \mathcal{E}^{(n)}(\iota_k u_k, \iota_k u_k) \le \sup_k \mathcal{E}^{(k)}(\iota_k u_k, \iota_k u_k) < \infty \quad \forall n \le 1,$$

so that $v \in \mathcal{D}_0$. It is easy to see that $v \in C(E; \mathbf{R}), v|_{\psi_{G_0}(E)} = 1, v|_{\psi_{G_1}(E)} = 0$ and, by the Stone-Weierstrass theorem, we have proved that $\mathcal{D}_0 \cap C$ is dense in $C$. To show that $\mathcal{D}_0 \cap C$ is dense in $\mathcal{D}_0$ in $\mathcal{E}_1$-norm, it is enough to approximate $f \in \mathcal{D}_0 \cap \mathbf{L}^\infty$ by elements of $\mathcal{D}_0 \cap C$ due to Theorem 1.4.2 *iii*) of [17]. But this is now clear as $U^\lambda f \in \mathcal{D}_0 \cap C$ for $f \in \mathcal{D}_0 \cap \mathbf{L}^\infty$ from Proposition 4.5 and it is a general fact that $U^\lambda f \to f$ in $\mathcal{E}_1$-norm. ∎

As we have a local regular Dirichlet form, there is a one to one correspondence between it and a diffusion process $\{X_t : t \geq 0\}$ ([17]). However this diffusion process is only defined for quasi-every starting point, as the capacity of points could well be zero. As we will see later in this section, we can extend this quasi-everywhere result to everywhere.

We now derive a Poincaré inequality.

**Proposition 4.7** *There exists a constant* $c_{4.5}$*, such that for all* $f \in \mathcal{D}_0$*,*

$$\mathcal{E}^{(\xi)}(f,f) \geq c_{4.5} \|f - \int_E f d\mu\|_2^2. \tag{4.4}$$

*Proof.* This result will come from the construction of the Dirichlet form. We use the fact that the Poincaré constant $\lambda_n$ scales as the time constant $T_n$. Note that it is enough to prove the result for $f \in \mathcal{D}_0 \cap C$ by Theorem 4.6.

Recall that by the definition of the Poincaré constant,

$$\mathcal{E}_n(\tilde{P}_n u, \tilde{P}_n u)\lambda_n \geq \sum_{x \in I_n} (u(x) - < u >_{I_n})^2, \ \ \forall u \in C(I_n; \mathbf{R}).$$

We know that

$$\mathcal{E}^{(\xi)}(u,u) \geq \sup_n R_n \mathcal{E}_n(\tilde{P}_n u, \tilde{P}_n u) \ \ \forall u \in \mathcal{D}_0$$

and hence

$$\mathcal{E}^{(n)}(u,u) \geq \frac{T_n}{\lambda_n} \sum_{x \in I_n} (\tilde{P}_n u(x) - < \tilde{P}_n u >_{I_n})^2 M_n^{-1}.$$

For $u \in \mathcal{D}_0 \cap C$, taking $n \to \infty$, we see that, as $\frac{T_n}{\lambda_n} \geq c_{3.11}$, the Poincaré inequality will follow. ∎

Let $P_t$ be the semigroup of positive operators associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}_0)$ on $\mathbf{L}^2(E,\mu)$. We can prove the Nash inequality using Propositions 4.4 and 4.7. We omit the proof as it is the same as that of [11] Lemma 4.1.

**Lemma 4.8** *There is a constant* $c_{4.6}$ *such that if* $T_n^{-1} \le t \le T_{n-1}^{-1}$*, then*

$$||P_t||_{1\to\infty} \le c_{4.6} M_n. \tag{4.5}$$

We now consider the density of $P_t$ w.r.t. $\mu$. Using the method indicated in the lead up to [2] Proposition 4.14, we can prove the existence of a transition density $p_t(x, y)$ which is jointly measurable and satisfies the Chapman-Kolmogorov equations. In order to prove the joint continuity of the heat kernel we will follow the argument of [16], Lemma 4.6.

**Lemma 4.9** *The transition semigroup* $P_t$ *on* $\mathbf{L}^2(E)$ *has a kernel* $p_t(x, y)$ *which is jointly continuous for* $(t, x, y) \in (0, \infty) \times E \times E$.

*Proof.* We will first show that $P_t$ has the strong Feller property:

$$P_t : \mathbf{L}^1 \cap \mathbf{L}^\infty \to C(E).$$

Note that as the semigroup is the $\mathbf{L}^2$ semigroup associated with a Dirichlet form, it is holomorphic (see [14]). Thus $P_t f \in \mathcal{D}(\mathcal{L})$ for all $f \in \mathbf{L}^2$. Now, as $U^\lambda f$ is continuous for all $f \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ (due to Proposition 4.5), according to Proposition 2.3 and Lemma 2.4 of [26], it is enough to check that

$$\int_0^\infty t^{r/2-1} e^{-t} \|P_t\|_{p\to\infty} dt < \infty \tag{4.6}$$

holds for some $r > 0, 1 < p < \infty$. But we already have a good bound of $||P_t||_{1\to\infty} (= ||P_t||^2_{2\to\infty})$ for small $t$ in Lemma 4.8 so that (4.6) holds for $p = 2$ and large $r$.

Thus given $f \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ we have that $P_t f \in C$. Observe that the transition density $p_t(\cdot, y) \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ as

$$\int_F p_t(x, y)\mu(dx) = 1, \quad \text{and} \quad \sup_x p_t(x, y) \le c(t).$$

Now we can write $p_t(x, y) = P_{t/2}(p_{t/2}(., y))(x)$ by the Chapman-Kolmogorov equations, and hence, by the above, we see that $p_t(x, y)$ is continuous in $x$. Equipped with this result we can follow through the argument of [16] Lemma 4.6 to obtain the joint continuity of the transition density. ∎

This result shows that there is no uncertainty in the starting point for the one to one correspondence between the Dirichlet form and the diffusion process, which was mentioned after the proof of Theorem 4.6.

Finally in this section, we state the coupling result and Harnack inequality for the limiting operator for later use. Given two processes $Y^1, Y^2$, defined on the same state space, we set

$$T_C(X, Y) = \inf\{t \geq 0 : Y_t^1 = Y_t^2\}.$$

Also, let $S_B^z$ denote the exit time from the set $B$, when the process is started from the point $z$.

**Theorem 4.10** *(Coupling) For $x, y \in E$, there exist diffusion processes $W_t^x, W_t^y$ with $W_0^x = x, W_0^y = y$ on $E$ whose laws are equal to $\{X_t\}$ that satisfy the following:*
*For $n \in \mathbf{N}$ and $\epsilon > 0$, there exists a $k_0$ such that*

$$P(T_C(W^x, W^y) < \min\{S^x_{D_n^0(x)}, S^y_{D_n^0(x)}\}) > 1 - \epsilon,$$

*for all $k > k_0$, $y \in D_{n+k}^0(x)$.*

**Theorem 4.11** *Let $\mathcal{L}$ be the generator associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}_0)$. Then for any connected open sets $G_1, G_2$ in $E$ with* dist $(G_1^c, G_2) > 0$, *there exists $\delta > 0$ such that*

$$\delta \max_{x \in G_2} f(x) \leq \min_{x \in G_2} f(x) \tag{4.7}$$

*for any $f \in \mathcal{D}_0$ with $f|_{G_1} \geq 0$ and $\mathcal{L}f|_{G_1} = 0$.*

The proof follows in the same way as Barlow-Bass [8], as $E$ and $W$ have enough symmetries for their arguments to work. As before, we do not write down the proof as it is lengthy and there are no non-trivial modifications from the original one.

# 5 Shortest path metric

In this section we construct the shortest path metric on our random fractals. This is an intrinsic metric which we will use to study the diffusion process on the fractal. This kind of metric is constructed for affine nested fractals in [16] and for homogeneous random Sierpinski gaskets in [11]. Let $B_n$ be the smallest number of steps in the path across the $n$-th approximation to the carpet, as defined in (3.2). Let $x(n), y(n)$ (sometimes denoted by $x(\xi_1 \cdots \xi_n), y(\xi_1 \cdots \xi_n)$) be the extreme points of the shortest path.

**Proposition 5.1** *There exists $c_{5.1} > 0$ such that, for each $n, m \in \mathbf{N}$,*

$$c_{5.1} B_n B_{\theta^n \xi|m} \leq B_{n+m} \leq B_n B_{\theta^n \xi|m}.$$

*Proof.* We first prove the second inequality. Take the $n$-path which attains $B_n$. We will construct the $n+m$-path by putting the contraction of the $m$-path, which attains $B_{\theta^n \xi|m}$ (or less), on each $n$-complex in the path on $B_n$. For this to succeed in giving a path we should connect each contracted $m$-path and construct a connected $n+m$-path. This can be done using homogeneity and symmetry of the fractal. Indeed, if the shortest $m$-path goes from $x(m)$ to $y(m)$, by reflection arguments we can construct an $m$-path from $x(m)$ to the point which is a translation of $x(m)$ and which is in the opposite face of $[0,1]^d$. As this path is constructed using the original shortest $m$-path and the reflection of it, the length of the path is less than or equal to the original $m$-path. On the other hand, by diagonal reflections, we can construct an $m$-path from $x(m)$ to each face of $[0,1]^d$ whose end point is a rotation of $x(m)$ with length less than or equal to the shortest $m$-path. Using the contractions of these $m$-paths, it is easy to construct the desired $n+m$-path.

In order to prove the first inequality, we define for $x \in I_{n+m}$, the domains $D^0_m(x), D^1_m(x)$, in the same way as (3.6), (3.7). Take a $n+m$-path which attains $B_{n+m}$ and separate it into each $n$-complex. Set $x_0 = x(n+m)$ and let $x_1 \in I_{n+m}$ be the first element in the shortest $n+m$-path which is outside $D^1_m(x_0)$. Define inductively $x_{i+1} \in I_{n+m}$ to be the first element in the shortest $n+m$-path which is outside $D^1_m(x_i)$ until it reaches $y(n+m)$ (we denote by $s$ the last such $i$). Clearly the number of the $n+m$-complexes between $x_i$ and $x_{i+1}$ is greater than or equal to $B_{\theta^n \xi|m}$. On the other hand, $\frac{s}{c_1} \geq B_n$ where $c_1 \equiv (\max_{v \in V} m_v)^{-1}$. We thus obtain $c_1 B_n B_{\theta^n \xi|m} \leq B_{n+m}$. ∎

For $x, y \in I_n$, define $d_n(x,y) = \min\{\pi : \pi \text{ is a } n\text{-path between } x \text{ and } y\}/B_n$. Now we assume the following.

**Assumption 5.2** *There exists $c_{5.2} > 0$ such that for all $n, m \in \mathbf{N}$ and for all $x, y \in I_{n+m}$ which are in the same $n$-complex (i.e. $\Psi_x(E)$ and $\Psi_y(E)$ are in the same $n$-complex),*

$$\min\{\pi : \pi \textit{ is a } n+m\textit{-path between } x \textit{ and } y\} \leq c_{5.2} B_{\theta^n \xi|m}.$$

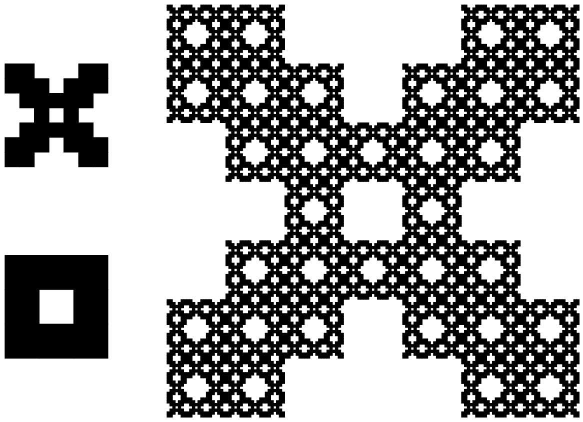

Figure 3: A random Sierpinski carpet with borders not included and its two generators

We believe that this assumption holds for all our fractals but so far we can only prove it in the following situations. An example is shown in Figure 3.

**Proposition 5.3** *Under the following condition (a) or (b), Assumption 5.2 holds.*
*(a) $d = 2$ (i.e. The fractal is in $\mathbf{R}^2$)*
*(b) (Borders included) $\partial F_0$ is contained in $F_1^a$.*

*Proof.* In case (a), if $x, y$ are at the opposite ends of a path which is from one boundary to the other, then any path from $x$ to $y$ intersects the original path. Using this fact, it is easy to deduce that

$$\min\{\pi : \pi \text{ is a } n+m\text{-path from } x \text{ to } \pi'\} \le c_1 \sum_{l=1}^{m} B_{\xi_{n+l}\cdots\xi_{n+m}},$$

where $\pi'$ is an $n+m$-path which attains $B_{\theta^n\xi|m}$ and $c_1 = \max_{v\in V} m_v$. Using Proposition 5.1, the right hand side can be estimated from above by

$$c_1 \sum B_{\xi_{n+1}\cdots\xi_{n+m}} / c_{5.1} B_{\xi_{n+1}\cdots\xi_{n+l-1}}.$$

As $B_{\xi_{n+1}\cdots\xi_{n+l-1}} \le (c_2)^{l-1}$, we have the desired fact.

That case (b) is sufficient for the Assumption can be proved similarly as lines are the shortest paths in this case. ∎

We now construct a metric on $E$.

**Theorem 5.4** *For $x, y \in E$, take (arbitrary) $x_n, y_n \in I_n$ so that $x \in \psi_{x_n}(E)$, $y \in \psi_{y_n}(E)$. Then the following $d(x,y)$ can be defined independently of the choice of $x_n, y_n$ and $d$ is a metric on $E$:*

$$d(x,y) \equiv \limsup_{n\to\infty} d_n(x_n, y_n).$$

*Proof.* Firstly, we remark that from Proposition 5.1 and Assumption 5.2, if $x, y \in I_{n+m}$ are in the same $n$-complex, then

$$d_{n+m}(x,y) \le C' B^{-1}_{\xi_1\cdots\xi_n}, \tag{5.1}$$

for some $C' > 0$. Using this and the fact that $B_{\xi_1\cdots\xi_n} \ge 2^n$, it is easy to show that $d(x,y)$ is independent of the choice of $\{x_n\}, \{y_n\}$.

For the proof that $d$ is a metric, the only non-trivial part is to show that, if $d(x,y) = 0$, then $x = y$. To prove this, suppose $x \ne y$. Then there exists an $m \in \mathbf{N}$ such that $y_{n+m} \notin D^1_m(x_{n+m})$ for all $n \ge 1$. We then see that $d_{n+m}(x_{n+m}, y_{n+m}) \ge B_{\theta^m \xi|n}/B_{n+m} \ge 1/B_m > 0$ and hence $d(x,y) > 0$. ∎

We call this metric the shortest path metric. The following proposition suggests that this metric behaves like a geodesic metric.

**Proposition 5.5** *There exists $c_{5.3} > 0$ such that the following holds. For all $x \ne y \in E$ and all $m \in \mathbf{N}$, there is a sequence $\{x_i\}_{i=1}^m \subset E$, $x_1 = x, x_m = y$ such that*

$$d(x,y) \ge c_{5.3} \sum_{i=1}^{m-1} d(x_i, x_{i+1}), \qquad 1/2 \le \frac{d(x_i, x_{i+1})}{d(x_j, x_{j+1})} \le 2 \quad (1 \le i, j \le m-1).$$

*Proof.* We first prove that there exists $c_1 > 0$ such that for $x, y \in E$,

$$d(x,y) \le c_1 \liminf_{n\to\infty} d_n(x_n, y_n), \tag{5.2}$$

where $x_n, y_n$ are chosen as in Theorem 5.4. To prove this, we will show that for $w, w' \in I_{n+m}$

$$d_n(w|n, w'|n) \le c_1 d_{n+m}(w, w'), \tag{5.3}$$

where $w|n \in I_n$ is the first $n$ letters in the word $w$. Indeed,

$$\begin{aligned} d_{n+m}(w,w') &\geq c_2 \min\{\pi : \pi \text{ is a } n\text{-path from } w|n \text{ to } w'|n\} \frac{B_{\theta^n \xi|m}}{B_{n+m}} \\ &\geq c_3 \min\{\pi : \pi \text{ is a } n\text{-path from } w|n \text{ to } w'|n\} \frac{B_{\theta^n \xi|m}}{B_n B_{\theta^n \xi|m}} \\ &= c_3 d_n(w|n, w'|n), \end{aligned}$$

for some $c_2, c_3 > 0$. The first inequality can be proved in a similar way to the proof of Proposition 5.1 and the second inequality is from Proposition 5.1. Using (5.3),

$$\begin{aligned} d(x,y) &= \limsup_{n\to\infty} d_n(x_n, y_n) \\ &\leq c_1 \lim_{n\to\infty} \inf_{m\geq 1} d_{n+m}(x_{n+m}, y_{n+m}) = c_1 \liminf_{n\to\infty} d_n(x,y), \end{aligned}$$

so that (5.2) is proved. Now, for each $x, y \in E$, corresponding $x_n, y_n \in I_n$ ($n$ large) and for each $m \in \mathbf{N}$, we can choose $\{x_i^n : n = 1, \cdots, m\} \subset I_n$ which satisfies $d_n(x_n, y_n) = \sum_{i=1}^{m-1} d_n(x_i^n, x_{i+1}^n)$, $x_0^n = x, x_m^n = y$ and the ratio of each distance is within $[1/2, 2]$. Using (5.2),

$$d(x,y) \geq \sum \liminf_n d_n(x_i^n, x_{i+1}^n) \geq c_1^{-1} \sum d(x_i, x_{i+1}),$$

where $\{x_i\} \subset E$ is taken as a limit of some subsequence of $\{x_i^n\}$. The proof is completed. ∎

We remark that this proposition will be used for the chaining argument required in the proof of the lower bound for the heat kernel (Theorem 6.8). It follows from (5.1) that there exists $c_{5.4} > 0$ such that

(5.4) $\qquad d(x,y) \leq c_{5.4} B_k^{-1}$ if $x, y$ belong to the same $k$-complex.

Note also that if $d(x,y) \leq B_k^{-1}$ then $x, y$ are either in the same $k$-complex or in "adjacent" $k$-complexes, (which means that $y \in D_k^1(x)$). If $B(x,r) = \{y \in F : d(x,y) \leq r\}$, then as the $\mu$-measure of each $k$-complex is $M_k^{-1}$, we have $cM_k^{-1} \leq \mu(B(x, B_k^{-1})) \leq c' M_k^{-1}$. Set

(5.5) $$d_f(n) = \frac{\log M_n}{\log B_n};$$

it follows that if $B_n^{-1} \le r \le B_{n-1}^{-1}$,

$$c_{5.5} r^{d_f(n)} \le \mu(B(x,r)) \le c_{5.6} r^{d_f(n)}, \quad x \in E. \tag{5.6}$$

The Hausdorff and packing dimension with respect to the metric $d$ are written $\dim_{H,d}(.)$ and $\dim_{P,d}(.)$. The following result follows easily from (5.6) and the density theorems for Hausdorff and packing measure – see [15].

**Lemma 5.6** *(a)* $\dim_{H,d}(E) = \liminf_{n\to\infty} d_f(n)$,
*(b)* $\dim_{P,d}(E) = \limsup_{n\to\infty} d_f(n)$.

In the case that the environment is generated by a stationary and ergodic sequence of random variables, we will have more detailed information of Hausdorff and packing dimensions in Section 7.

# 6 Transition density estimates

From this section, we let $P_t$ be the semigroup of positive operators associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}_0)$ on $\mathbf{L}^2(E,\mu)$, and let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be the infinitesimal generator of $(P_t)$. We will call this operator a Laplacian on $E$. As $(\mathcal{E}, \mathcal{D}_0)$ is regular and local, there exists a diffusion $(X_t, t \ge 0, P^x, x \in E)$ with semigroup $P_t$, which we will call *Brownian motion on* $E$. There could be different processes associated with the different limits of the sequence of Dirichlet forms. We will show that all processes have the same bounds on their transition density.

We first note that from Lemma 4.8 we have the pointwise bound, that if $T_n^{-1} < t \le T_{n-1}^{-1}$, then

$$p_t(x,y) \le c_{4.6} M_n, \quad x, y \in E. \tag{6.1}$$

We can now extend our hitting time estimates for the Markov chains, obtained in Lemma 3.10 to the diffusion itself. We firstly construct the neighbourhood of a point $x \in E$. For a point $x \in [0,1]^d$ we define the cube with center near $x$ by letting $\phi(x_i) = j$ if $(j-(1/2))/B_n \le x_i < (j+(1/2))/B_n$ $(i = 1, \cdots, d)$ and setting

$$\bar{D}_n(x) = [(\phi(x_1)-1)/B_n, (\phi(x_1)+1)/B_n] \times \ldots \times [(\phi(x_d)-1)/B_n, (\phi(x_d)+1)/B_n].$$

Let $S_{\bar{D}_n(x)} = \inf\{t : W_t \in \bar{D}_n^c(x)\}$.

**Lemma 6.1** *There exist constants* $c_{6.1}, c_{6.2}$ *such that, if* $k = k(r, n)$, *then for all* $x \in E$,

$$P^x(S_{\bar{D}_r(x)} \le T_n^{-1}) \le c_{6.1} \exp(-c_{6.2} B_{r+k}/B_r). \tag{6.2}$$

*Proof.* This follows exactly the same approach as for the proof of Lemma 3.10. We first establish the weak bound, that $P^x(S_{\bar{D}_r} \le t) \le c_0 + c_1 t$, and then use Lemma 1.1 of [3]. ∎

The next lemma can be proved in the same way as [11] Lemma 4.4.

**Lemma 6.2** *There exist constants* $c_{6.3}, c_{6.4}$ *such that if* $0 < t < 1$, $0 < r < 1$, *and* $n, m$ *satisfy*

$$T_n^{-1} \le t < T_{n-1}^{-1}, \quad B_m^{-1} \le r < B_{m-1}^{-1},$$

*and* $k = k(m, n)$ *then for* $x \in E$

$$P^x(\sup_{0 \le s \le t} d(X_s, x) \ge r) \le c_{6.3} \exp\Big(-c_{6.4}(\frac{r^{d_w(m+k)}}{t})^{1/(d_w(m+k)-1)}\Big). \tag{6.3}$$

**Theorem 6.3** *There exist constants* $c_{6.5}, c_{6.6}$ *such that if* $0 < t < 1$, $x, y \in E$, *and* $n, m$ *satisfy*

$$T_n^{-1} \le t < T_{n-1}^{-1}, \quad B_m^{-1} \le d(x, y) < B_{m-1}^{-1}, \tag{6.4}$$

*and* $k = k(m, n)$ *then*

$$p_t(x, y) \le c_{6.5} t^{-d_s(n)/2} \exp\Big(-c_{6.6}(\frac{d(x, y)^{d_w(m+k)}}{t})^{1/(d_w(m+k)-1)}\Big). \tag{6.5}$$

*Proof.* Noting that $M_n \le ct^{-d_s(n)/2}$, this is proved from (6.1) and Lemma 6.2 by exactly the same argument as in Theorem 6.2 of [6]. ∎

**Remark.** Note that the bound (6.5) may also be written in the form

$$p_t(x, y) \le c M_n \exp(-c' B_{m+k}/B_m), \tag{6.6}$$

where $m, n$ satisfy (6.4), and $k = k(m, n)$.

We obtain lower bounds on $p_t(x, y)$ using the same approach as [11], though the techniques must be modified to cater for the case when $d_s > 2$. These bounds are identical, apart from the constants, to the upper bound (6.5).

**Lemma 6.4** *There exists a constant* $c_{6.7}$ *such that if* $T_n^{-1} \geq t$ *then*

$$p_t(x,x) \geq c_{6.7} M_n \quad \text{for all } x \in E. \tag{6.7}$$

*Proof.* As in [11] Lemma 5.1. We note that the direction of the inequality of time was mistyped in [11]. ∎

We need to extend this on-diagonal lower bound to a neighbourhood of the diagonal. In the case of finitely ramified fractals with $d_s < 2$ this has been done via an estimate on the Hölder continuity of the heat kernel, derived directly from the control on functions in the domain provided by the effective resistance. As we wish to consider the case in which $d_s \geq 2$ as well, we use the Harnack inequality following [8]. Let $T_n^{-1} < t < T_{n-1}^{-1}$ and set

$$A_x^n = \{y : p_t(x,y) \geq c_{6.8} M_n\}.$$

We can write $c_1(n) = P^x(X_t \in A_x^n)$ and begin by showing that $c_1(n) \geq c_1 > 0$ for some $c_1 > 0$. Using the Chapman-Kolmogorov equations and our on diagonal estimates, we have

$$\begin{aligned} p_{2t}(x,x) &= \int p_t(x,y)p_t(y,x)\mu(dy) \\ &= \int_{A_x^n} p_t(x,y)p_t(y,x)\mu(dy) + \int_{(A_x^n)^c} p_t(x,y)p_t(y,x)\mu(dy), \\ c_{6.7}M_{n-1} &\leq c_{4.6}M_n P^x(X_t \in A_x^n) + c_{6.8}M_n P^x(X_t \in (A_x^n)^c). \end{aligned}$$

Removing $M_n$ and writing $c_1' = c_{6.7}/\max m_v$, we have

$$c_1' \leq c_{4.6}c_1(n) + c_{6.8}(1 - c_1(n)),$$

and thus $c_1(n) \geq c_1 \equiv \frac{c_1' - c_{6.8}}{c_{4.6} - c_{6.8}} > 0$, by choice of $c_{6.8} = c_1'/2 \wedge c_{4.6}/2$.

Now

$$p_t(x,y) \geq \int_{A_x^n} p_{t/2}(x,z)p_{t/2}(z,y)\mu(dz) \geq c_{6.8}M_n P^y(X_{t/2} \in A_x^n).$$

Thus we will have the near diagonal bound if $P^y(X_{t/2} \in A_x^n) \geq c_2$ for $d(x,y) \leq c_3 B_n^{-1}, T_n^{-1} < t$ where $c_2, c_3$ are positive constants.

We prove this in two lemmas.

**Lemma 6.5** *There exists a constant* $c_{6.9} > 0$ *such that*

$$P^y(S_{\bar{D}_{n+l}(x)} > t) \leq c_{6.9}\frac{T_n}{T_{n+l}} \quad \text{if } y \in \bar{D}_{n+l}(x), T_n^{-1} < t \leq T_{n-1}^{-1}.$$

*Proof.* This is a simple application of Markov's inequality, $P^y(S_{\bar{D}_{n+l}(x)} > t) \leq E^y S_{\bar{D}_{n+l}(x)}/t$. For $y \in \bar{D}_{n+l}(x)$ we have $E^y S_{\bar{D}_{n+l}(x)} \leq cT_{n+l}^{-1}$ and hence we have the result. ∎

**Lemma 6.6** *There exist constants $c_{6.10}, k_1$ such that for all $n > 0$,*

$$P^y(X_t \in A_x^n) \geq c_{6.10}, \quad \textit{if } y \in \bar{D}_{n+k_1}(x), T_n^{-1} < t \leq T_{n-1}^{-1}.$$

*Proof.* Let $\epsilon = c_1/4 > 0$. Using the coupling result in Theorem 4.10, there exists a $k_0^l$ such that if $y \in \bar{D}_{n+k}(x)$, then

$$P(T_C < \min\{S^x_{\bar{D}_{n+l}(x)}, S^y_{\bar{D}_{n+l}(x)}\}) > 1 - \epsilon,$$

for $k \geq k_0^l$. Rewriting this we have

$$1 - \epsilon < P(T_C < t/2) + P^x(S_{\bar{D}_{n+l}(x} > t/2) + P^y(S_{\bar{D}_{n+l}(x)} > t/2).$$

Thus for our value of $t$, choosing $l$ such that $c_{6.9}T_n/T_{n+l} \leq c_{6.9}(t^*)^{-l}\epsilon$, and using Lemma 6.5, we have

$$P(T_C < t/2) > 1 - 3\epsilon,$$

for $y \in \bar{D}_{n+k}(x)$. Thus, using the argument from [8] Theorem 6.9,

$$\begin{aligned} P^y(X_t \in A_x^n) &\geq P^y(X_t \in A_x^n; T_C < t/2) \\ &= P^x(X_t \in A_x^n; T_C < t/2) \\ &\geq P^x(X_t \in A_x^n) - P^x(T_C > t/2) \\ &\geq c_1 - 3\epsilon \geq c_{6.10} = c_1/4, \end{aligned}$$

for $y \in \bar{D}_{n+k_1}(x)$, where $k_1 = k_0^l$. ∎

Thus we have the following near diagonal bound.

**Lemma 6.7** *There exist $c_{6.11}, c_{6.12}$ such that if $T_n^{-1} < t \leq T_{n-1}^{-1}$, then*

$$p_t(x,y) \geq c_{6.11}M_n \quad \textit{whenever } d(x,y) \leq c_{6.12}B_n^{-1}. \tag{6.8}$$

We can now use a standard chaining argument to obtain general lower bounds on $p_t$ from Lemma 6.7.

**Theorem 6.8** *There exist constants* $c_{6.13}, c_{6.14}$ *such that if* $x, y$ *in* $E$, $t \in (0,1)$ *and*

$$T_n^{-1} \le t < T_{n-1}^{-1}, \quad B_m^{-1} \le d(x,y) < B_{m-1}^{-1},$$

*then*

$$(6.9) \quad p_t(x,y) \ge c_{6.13} t^{-d_s(n)/2} \exp\Big(-c_{6.14}\big(\frac{d(x,y)^{d_w(m+k)}}{t}\big)^{1/(d_w(m+k)-1)}\Big).$$

*Proof.* Using (6.8) we see that the bound is satisfied if $m \ge n$. Now let $m < n$, write $k = k(m,n)$, and choose $j, l$ with $0 \le j < l < c$ such that

$$2^{l-j} \ge 3b^*/c_{6.12}, \quad (b^*)^l < (2b^*)^j;$$

note that such a choice is possible, with a constant $c$ depending only on $c_{6.12}$ and $b^*$. We then have

$$(6.10) \qquad \frac{B_{m+k+l}}{B_{m+k}} \le \frac{B_{m+k+j}}{B_{m+k}}(b^*)^{l-j} \le \frac{T_{m+k+j}}{T_{m+k}} 2^{-j}(b^*)^{l-j} < \frac{T_{m+k+j}}{T_{m+k}},$$

and

$$(6.11) \qquad \frac{3b^*}{B_{m+k+l}} \le \frac{3b^* 2^{j-l}}{B_{m+k+j}} \le \frac{c_{6.12}}{B_{m+k+j}}.$$

Let $N = B_{m+k+l}/B_m$. Since $d(x,y) \le b^* B_m^{-1}$ there exists a chain $x = z_0, z_1, \dots, z_N = y$ with $d(z_{i-1}, z_i) \le c_{5.3}^{-1} b^* B_{m+k+l}^{-1}$ (here we use Proposition 5.5). Let $G_i = B(z_i, b^* B_{m+k+l}^{-1})$; then, if $x_i \in G_i$, we have

$$(6.12) \qquad d(x_{i-1}, x_i) \le 3c_{5.3}^{-1} b^* B_{m+k+l}^{-1} \le c_{6.12} B_{m+k+j}^{-1}.$$

Let $s = t/N$, then

$$(6.13) \qquad s \ge \frac{B_m}{T_n B_{m+k+l}} \ge \frac{B_{m+k}}{T_{m+k} B_{m+k+l}} > \frac{1}{T_{m+k+j}}.$$

From (6.8), (6.12) and (6.13) we have $p_s(x_{i+1}, x_i) \ge c_{6.11} M_{m+k+j} \ge c_{6.11} M_{m+k}$. Therefore since $\mu(G_i) \ge c_1 M_{m+k}^{-1}$, and $m + k \ge n$,

$$\begin{aligned} p_t(x,y) &\ge \int_{G_1} \cdots \int_{G_{N-1}} p_s(x,x_1) \dots p_s(x_{N-1},y) \mu(dx_1) \dots \mu(dx_{N-1}), \\ &\ge \Big(\prod_{i=1}^{N-1} \mu(G_i)\Big)(c_8 M_{m+k})^N, \\ &\ge c_2 M_{m+k} \exp(-c_3 N) \ge c_2 M_n \exp(-c_4 B_{m+k}/B_m). \end{aligned}$$

Using Lemma 3.9 completes the proof. ∎

# 7 Stationary and ergodic environment

In this section we assume that the environment is generated by a stationary and ergodic sequence of random variables and see how oscillations in the environment sequence $\xi_i$ relate to oscillations in the transition density. In [11] it was possible to explicitly determine the spectral dimension in the case where there was ergodic behavior in the environment. For the homogeneous random carpets we cannot express the spectral or walk dimensions in terms of the time scaling factors for the individual carpet types but we can show the existence of the spectral dimension. We can then use this to find bounds of Aronson type for the transition density.

Let $(\Xi, \mathcal{F}, \mathbf{P})$ be a Borel probability space, on which cylinder sets are measurable. We begin by showing that there is a resistance scale factor.

**Proposition 7.1** *There is a constant $\rho \in (0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{\log R_n}{n} = \log \rho, \quad \mathbf{P}\text{-}a.s..$$

*Proof.* Note that from (3.3) and the fact that $|V| < \infty$, there exist constants $c_1, c_2 > 0$ such that

$$c_1^n \le R_n \le c_2^n \qquad \text{for all } n \in \mathbf{N}. \tag{7.1}$$

Now, from (3.3),

$$\log(c_{3.2} R_{nk}) \le \sum_{i=1}^{n} \log(c_{3.2} R_{\theta^{(i-1)k}\xi|k}).$$

Thus, for any $k \ge 1$, we have

$$\limsup_{n\to\infty} \frac{\log(c_{3.2}R_{nk})}{nk} \le \frac{1}{k} E_{\mathbf{P}} \log[c_{3.2} R_k], \quad \mathbf{P}\text{-a.s.}$$

Moreover, from (3.3), (7.1) we know that if $i \in [(n-1)k+1, nk]$, then

$$c_{3.1} c_1^k R_{(n-1)k} \le R_i \le c_{3.2} c_2^k R_{nk}.$$

From these facts, we see that for any $k \geq 1$,

$$\limsup_{n\to\infty} \frac{\log[c_{3.2}R_n]}{n} \leq \frac{1}{k} E_{\mathbf{P}} \log[c_{3.2}R_k], \quad \mathbf{P}\text{-a.s.}$$

In the same way, we have for any $k \geq 1$,

$$\liminf_{n\to\infty} \frac{\log[c_{3.1}R_n]}{n} \geq \frac{1}{k} E_{\mathbf{P}} \log[c_{3.1}R_k], \quad \mathbf{P}\text{-a.s.}$$

Let $\tilde{R}_n = E_{\mathbf{P}} \log[c_{3.2}R_n]$, then $\tilde{R}_{n+m} \leq \tilde{R}_n + \tilde{R}_m$, $\forall n, m \geq 1$. We thus see that

$$\limsup_{n\to\infty} \frac{\tilde{R}_n}{n} \leq \liminf_{m\to\infty} \frac{\tilde{R}_m}{m} = \liminf_{m\to\infty} \frac{1}{m} E_{\mathbf{P}} \log[c_{3.1}R_m].$$

Therefore, we have

$$\limsup_{n\to\infty} \frac{1}{n} E_{\mathbf{P}} \log[c_{3.2}R_n] = \liminf_{n\to\infty} \frac{1}{n} E_{\mathbf{P}} \log[c_{3.1}R_n].$$

Combining this with (7.1), we see that there exists a constant $\rho \in [c_1, c_2] \subset (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{\log R_n}{n} = \log \rho, \quad \mathbf{P}\text{-a.s.} \qquad \blacksquare$$

In this setting, we can also determine the Hausdorff and Packing dimensions. As in the previous result we can prove that the limit $\lim_n d_f(n)$ exists and it is a constant $\mathbf{P}$-a.s.. So the Hausdorff and Packing dimensions are the same and are given by $d_f = \log m / \log b$ where $\log m = \lim_{n\to\infty} \log M_n/n$ and $\log b = \lim_{n\to\infty} \log B_n/n$. If we let $p_v$ denote the limiting proportion of type $v \in V$ in the sequence, then

$$\log m = \sum_{v \in V} p_v \log m_v.$$

We can see that the asymptotic behaviour of the spectral dimension depends on the convergence of $\log R_n/n$. In general there will not be a simple expression for this limit in terms of the different types of carpets, unlike the scale irregular gasket case. We now define the dimensional exponents for the random carpet $\mathbf{P}$-a.s. as

$$\begin{aligned} d_s &= \lim_{n\to\infty} d_s(n) = \frac{\log m}{\log m\rho}, \\ d_w &= \lim_{n\to\infty} d_w(n) = \frac{\log m\rho}{\log b}. \end{aligned}$$

As the convergence that occurs in this result comes from the sub-additive ergodic theorem we do not have control on the rate of convergence. Thus, all that we have in general is that there exists a set $\Omega \subset \Xi$, with $\mathbf{P}(\Omega) = 1$, such that for each $\epsilon > 0$ and each $\xi \in \Omega$, there exists an $n_0 = n_0(\xi)$ such that

$$\tfrac{1}{2}|d_s(n) - d_s| \leq \epsilon, \quad |d_w(n) - d_w| \leq \epsilon, \quad \forall n \geq n_0 \tag{7.2}$$

**Theorem 7.2** *For each $\epsilon > 0$, there exist constants $c_{7.i} = c_{7.i}(\xi), i = 1, 2, 3, 4$ such that for $0 < t < 1$, $x$, $y \in E^{(\xi)}$, $\mathbf{P}$-a.s.,*

$$p_t(x, y) \leq c_{7.1} t^{-d_s/2-\epsilon} \exp\Big(-c_{7.2}\big(\frac{d(x, y)^{d_w}}{t^{1-\epsilon}}\big)^{1/(d_w-1)}\Big), \tag{7.3}$$

$$p_t(x, y) \geq c_{7.3} t^{-d_s/2+\epsilon} \exp\Big(-c_{7.4}\big(\frac{d(x, y)^{d_w}}{t^{1+\epsilon}}\big)^{1/(d_w-1)}\Big). \tag{7.4}$$

*Proof.* Take $\xi \in \Omega$ and let $T_n^{-1} \leq t < T_{n-1}^{-1}$, $B_m^{-1} \leq r = d(x, y) \leq B_{m-1}^{-1}$. Note that by modifying constants $c_{7.i} = c_{7.i}(\xi)$ it is enough to prove for the case $n \geq n_0(\xi)$. Since $(t_*)^n \leq T_n \leq (t^*)^n$, and similar bounds hold for $B_m$, we have

$$c_1 n \leq \log(1/t) \leq c_2 n, \quad c_1 m \leq \log(1/r) \leq c_2 m. \tag{7.5}$$

So by (7.2)

$$t^{-d_s(n)/2} \leq t^{-d_s/2-\epsilon}. \tag{7.6}$$

For the off-diagonal term we have, writing $u = r^{d_w}/t$,

$$u \leq c_3 \frac{T_n}{B_m^{d_w}} \leq c_3 \frac{T_{m+k}}{B_{m+k} B_m^{d_w-1}} = c_3 \Big(\frac{B_{m+k}}{B_m}\Big)^{d_w-1} B_{m+k}^{d_w(m+k)-d_w},$$

so that

$$B_{m+k}/B_m \geq c_4 u^{1/(d_w-1)} B_{m+k}^{\epsilon/(d_w-1)}. \tag{7.7}$$

If $m < n$ then using (3.16) we have $c_5 n \leq \log B_{m+k} \leq c_6 n$, and so with (7.5),

$$B_{m+k}^{\epsilon/(d_w-1)} \geq c_7 t^{-\epsilon/(d_w-1)}, \tag{7.8}$$

while if $m \geq n$ then $B_{m+k}/B_m = 1$. From (6.6) we have

$$p_t(x, y) \leq c t^{-d_s(n)/2} \exp(-c' B_{m+k}/B_m),$$

and combining this with (7.6), (7.7) and (7.8) we obtain (7.3).

The lower bound is proved in exactly the same way. ∎

**Remark.** 1. The spectral dimension $d_s$ should be a continuous function of the limiting proportions in the sequence and hence we can obtain fractals which take all values of $d_s$ in some interval. In particular there will be examples of homogeneous random fractals for which $d_s = 2$. In [8] it was noted that this was unlikely to occur for deterministic carpets but results were stated that would hold for any such examples. Thus in our setting there will be such examples where the results stated in [8] hold. A case would be an appropriate combination of the three dimensional Sierpinski carpet and the Menger Sponge as defined in [8] Section 9.
2. In order to obtain sharper estimates of Theorem 7.2 as in the case of the Sierpinski gasket ([11]), one needs to obtain good asymptotics for $(\log R_n)/n$. We do not know how to do this even for the case where the environment is generated by an i.i.d. sequence of random variables, as $R_n$ is not expressible as a simple product of some i.i.d. random variables in that case.

We can also obtain bounds on the eigenvalue counting function using the relationship between it and the transition density. As $p_t$ is uniformly continuous, this implies that $P_t$ is a compact operator on $\mathbf{L}^2(E, \mu)$, so that $P_t$, and hence $-\mathcal{L}$, has a discrete spectrum. Let $0 \leq \lambda_1 \leq \ldots$ be the eigenvalues of $-\mathcal{L}$, (with either Dirichlet or Neumann boundary conditions) and let $N(\lambda) = \#\{\lambda_i : \lambda_i < \lambda\}$ be the eigenvalue counting function.

Since

$$\int_E p_t(x, x)\mu(dx) = \int_0^\infty e^{-st} N(ds), \quad t > 0,$$

using (6.5) and (6.9) we have

$$c_1 M_n \leq \int_0^\infty e^{-s/T_n} N(ds) \leq c_2 M_n, \quad n \geq 0. \tag{7.9}$$

We can then convert this into estimates for $N(\lambda)$, using the same proof as [11].

**Proposition 7.3** *There exist constants $c_{7.5}$, $c_{7.6}$, $c_{7.7}$ such that if $\lambda > c_{7.5}$ and $n$ is such that $T_{n-1} \leq \lambda < T_n$ then*

$$c_{7.6} \lambda^{d_s(n)/2} \leq N(\lambda) \leq c_{7.7} \lambda^{d_s(n)/2}. \tag{7.10}$$

Finally, if the sequence $\xi$ is generated by a stationary and ergodic sequence of random variables, and there is no rapid convergence of the proportions, we see that $N(\lambda)/\lambda^{d_s/2}$ is not bounded from above and below, unlike the regular fractals such as (non-random) nested fractals or Sierpinski carpets.

**Corollary 7.4** *For each $\epsilon > 0$, the following holds* $\mathbf{P}$*-a.s.,*

$$0 < \liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2-\epsilon}}, \quad \limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2+\epsilon}} < \infty. \tag{7.11}$$

*Further, for fixed $\xi \in \Omega$, if there is a function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(x) \to \infty$ as $x \to \infty$ and $\{n_k\}_{k=1}^{\infty}$, such that*

$$\frac{1}{2}(d_s(n_k) - d_s) > \frac{g(n_k)}{n_k} \quad (\textit{resp.}\ \frac{1}{2}(d_s(n_k) - d_s) < \frac{g(n_k)}{n_k}). \tag{7.12}$$

*Then, there is a constant $c_{7.8}$ ( resp. $c_{7.9}$) such that*

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{e^{c_{7.8}g(\log\lambda)}\lambda^{d_s/2}} = \infty \quad (\textit{resp.}\ \liminf_{\lambda\to\infty} \frac{N(\lambda)}{e^{c_{7.9}g(\log\lambda)}\lambda^{d_s/2}} = 0). \tag{7.13}$$

*If the following holds instead,*

$$\frac{1}{2}(d_s - d_s(n_k)) > \frac{g(n_k)}{n_k} \quad (\textit{resp.}\ \frac{1}{2}(d_s - d_s(n_k)) < \frac{g(n_k)}{n_k}), \tag{7.14}$$

*then there is a constant $c_{7.10}$ ( resp. $c_{7.11}$) such that*

$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{e^{-c_{7.10}g(\log\lambda)}\lambda^{d_s/2}} = 0 \quad (\textit{resp.}\ \limsup_{\lambda\to\infty} \frac{N(\lambda)}{e^{-c_{7.11}g(\log\lambda)}\lambda^{d_s/2}} = \infty). \tag{7.15}$$

*Proof.* (7.11) comes from Theorem 7.2. For (7.13), taking $T_{n_k-1} \le \lambda < T_{n_k}$,

$$\frac{N(\lambda)}{e^{c_{7.8}g(\log\lambda)}\lambda^{d_s/2}} \ge c_{7.6}\lambda^{\frac{1}{2}(d_s(n_k)-d_s)}e^{-c_{7.8}g(\log\lambda)} \ge c_{7.6}e^{(c_1-c_{7.8})g(\log\lambda)}$$

where the first inequality is by (7.10) and the second is by (7.12). Thus the result holds by taking $c_{7.8} < c_1$. The rest can be proved in the same way. ∎

**Acknowledgment** The authors are grateful to Professor M. T. Barlow for helpful comments.

# References

[1] M.T. Barlow. Random walks, electrical resistance, and nested fractals. *In: K.D. Elworthy, N. Ikeda (eds.) Asymptotic problems in probability theory: stochastic models and diffusion on fractals,* Montreal: Pitman 131–157, 1993.

[2] M.T. Barlow. Diffusions on fractals Lectures in Probability Theory and Statistics: Ecole d'ete de probabilites de Saint-Flour XXV, Springer, 1998.

[3] M.T. Barlow and R.F. Bass. Construction of Brownian motion on the Sierpinski carpet. *Ann. Inst. H. Poincaré*, **25**, 225–257, 1989.

[4] M.T. Barlow and R.F. Bass. Local times for Brownian motion on the Sierpinski carpet. *Probab. Th. Rel. Fields,* **85**, 91–104, 1990.

[5] M.T. Barlow and R.F. Bass. On the resistance of the Sierpinski carpet. *Proc. R. Soc. London A.*, **431**, 345–360, 1990.

[6] M.T. Barlow and R.F. Bass. Transition densities for Brownian motion on the Sierpinski carpet. *Prob. Theory Rel. Fields*, **91**, 307–330, 1992.

[7] M.T. Barlow and R.F. Bass. Coupling and Harnack inequalities for Sierpinski carpets. *Bull. A.M.S.*, **29**, 208–212, 1993.

[8] M.T. Barlow and R.F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *To appear Canadian Journal of Math.*.

[9] M.T. Barlow and R.F. Bass. Random walks on graphical Sierpinski carpets. Preprint 1997.

[10] M.T. Barlow, R.F. Bass, and J.D. Sherwood. Resistance and spectral dimension of Sierpinski carpets. *J. Phys. A*, **23**, L253–L258, 1990.

[11] M.T. Barlow and B.M. Hambly. Transition density estimates for Brownian motion on scale irregular Sierpinski gasket *Ann. Inst. H. Poincaré*, **33**, 531–557, 1997.

[12] R.F. Bass. Diffusions on the Sierpinski carpet. *Trends in Probability and related Analysis: Proceedings of SAP '96*, World Scientific, Singapore, 1–34, 1997.

[13] A.K. Chandra, P. Raghavan, W.L. Ruzzo, R. Smolensky and P. Tiwari The electrical resistance of a graph captures its commute and cover times. *Comput. Complexity*, **6**, 312–340, 1996/97.

[14] E.B. Davies. *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[15] K.J. Falconer. *Fractal Geometry.* Wiley, Chichester, 1990.

[16] P.J. Fitzsimmons, B.M. Hambly and T. Kumagai. Transition density estimates for Brownian motion on affine nested fractals. *Comm. Math. Phys.*, **165**, 595–620, 1994.

[17] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet forms and symmetric Markov processes.* de Gruyter, Berlin, 1994.

[18] B.M. Hambly. Brownian motion on a homogeneous random fractal. *Probab. Theory Rel. Fields*, **94**, 1–38, 1992.

[19] B.M. Hambly. Brownian motion on a random recursive Sierpinski gasket. *Ann. Probab.*, **25**, 1059–1102, 1997.

[20] B.M. Hambly and T. Kumagai. Transition density estimates for diffusions on p.c.f. fractals. *Proc. London Math. Soc.,* **78**, 431–458, 1999.

[21] T. Kumagai, S. Kusuoka and X.Y. Zhou. Resistance and spectral dimension for some homogeneous random fractals. Unpublished.

[22] S. Kusuoka and X.Y. Zhou. Dirichlet forms on fractals: Poincaré constant and resistance. *Probab. Theory Relat. Fields*, **93**, 169–196, 1992.

[23] T. Lindstrøm. Brownian motion on nested fractals. *Memoirs Amer. Math. Soc. 420*, 83, 1990.

[24] R.D. Mauldin and S.C. Williams. Random recursive constructions: asymptotic geometric and topological properties. *Trans. Am. Math. Soc.* **295**, 325-346, 1990.

[25] I. McGillivray Some applications of Dirichlet forms in probability theory. *Ph.D. dissertation, Cambridge Univ., 1992.*

[26] I. McGillivray A feller property for some degenerate elliptic operators. *Dirichlet forms and stochastic processes (Beijing, 1993)*, 271–282, de Gruyter, Berlin, 1995.

[27] H. Osada. Isoperimetric dimension and estimates of heat kernels of pre–Sierpinski carpets. *Probab. Theory Rel. Fields*, **86**, 469–490, 1990.

[28] P. Tetali. Random walks and effective resistance of networks. *J. Theoret. Probab.*, **4**, 101–109, 1991.

Ben M. Hambly
Department of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW, United Kingdom

Takashi Kumagai
Graduate School of Informatics
Kyoto University
Sakyoku, Kyoto 606-8501, Japan

Shigeo Kusuoka
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Tokyo 153-8914, Japan

Xian Yin Zhou
Department of Mathematics
Beijing Normal University
Beijing 100875, P. R. China