Random walk on the incipient infinite cluster for oriented percolation in high dimensions

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August 31, 2007

Abstract

We consider simple random walk on the incipient infinite cluster for the spread-out model of oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$. In dimensions $d > 6$, we obtain bounds on exit times, transition probabilities, and the range of the random walk, which establish that the spectral dimension of the incipient infinite cluster is $\frac{4}{3}$, and thereby prove a version of the Alexander–Orbach conjecture in this setting. The proof divides into two parts. One part establishes general estimates for simple random walk on an arbitrary infinite random graph, given suitable bounds on volume and effective resistance for the random graph. A second part then provides these bounds on volume and effective resistance for the incipient infinite cluster in dimensions $d > 6$, by extending results about critical oriented percolation obtained previously via the lace expansion.

1 Introduction and main results

1.1 Introduction

The problem of random walk on a percolation cluster — the 'ant in the labyrinth' [17] — has received much attention both in the physics and the mathematics literature. Recently, several papers have considered random walk on a supercritical percolation cluster [5, 9, 34, 35]. Roughly speaking, supercritical percolation clusters on $\mathbb{Z}^d$ are d-dimensional, and these papers prove, in various ways, that a random walk on a supercritical percolation cluster behaves in a diffusive fashion similar to a random walk on the entire lattice $\mathbb{Z}^d$.

Although a mathematically rigorous understanding of *critical* percolation clusters is restricted to examples in dimensions $d = 2$ and $d > 6$, or $d > 4$ in the case of *oriented* percolation, it is

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generally believed that critical percolation clusters in dimension d have dimension less than d, and that random walk on a large critical cluster behaves subdiffusively. Critical percolation clusters are believed to be finite in all dimensions, and are known to be finite in the oriented setting [11]. To avoid finite-size issues associated with random walk on a finite cluster, it is convenient to consider random walk on the incipient infinite cluster (IIC), which can be understood as a critical percolation cluster conditioned to be infinite. The IIC has been constructed so far only when $d = 2$ [29], when $d > 6$ (in the spread-out case) [24], and when $d > 4$ for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ (again in the spread-out case) [21]. See [36] for a summary of the high-dimensional results. Also, it is not difficult to construct the IIC on a tree [7, 30].

Random walk on the IIC has been proved to be subdiffusive on $\mathbb{Z}^2$ [30] and on a tree [7, 30]. See also [13, 14] for related results in the continuum limit. In this paper, we prove several estimates for random walk on the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 6$. These estimates, which show subdiffusive behaviour, establish that the spectral dimension of the IIC is $\frac{4}{3}$, thereby proving the Alexander–Orbach [3] conjecture in this setting. For random walk on ordinary (unoriented) percolation for $d < 6$ the Alexander–Orbach conjecture is generally believed to be false [27, Section 7.4].

The upper critical dimension for oriented percolation is 4. Because of this, we initially expected that the spectral dimension of the IIC would be equal to $\frac{4}{3}$ for oriented percolation in all dimensions $d > 4$, but not for $d < 4$. However, our methods require that we take $d > 6$. The random walk is allowed to travel backwards in 'time' (as measured by the oriented percolation process), and this allows the walk to move between vertices that are not connected to each other in the oriented sense. It may be that this effect raises the upper critical dimension for the random walk in the oriented setting to $d = 6$. Or it may be that our conclusions for the random walk remain true for all dimensions $d > 4$, despite the fact that our methods force us to assume $d > 6$. This leads to the open question: Do our results actually apply in all dimensions $d > 4$, or does different behaviour apply for $4 < d \leq 6$?

1.2 Random walk on graphs and in random environments

Our results on the IIC will be consequences of more general results on random walks on a family of random graphs. We now set up our notation for this. Let $\Gamma = (G, E)$ be an infinite graph, with vertex set G and edge set E. The edges $e \in E$ are *not* oriented. We assume that Γ is connected. We write $x \sim y$ if $\{x, y\} \in E$, and assume that (G, E) is locally finite, i.e., $\mu_y < \infty$ for each $y \in G$, where μ_y is the number of bonds that contain y. We extend μ to a measure on G. Let $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ be the discrete-time simple random walk on Γ, i.e., the Markov chain with transition probabilities

$$P^x(X_1 = y) = \frac{1}{\mu_x}, \quad y \sim x. \tag{1.1}$$

We define the transition density (or discrete-time heat kernel) of X by

$$p_n(x, y) = \frac{P^x(X_n = y)}{\mu_y}; \tag{1.2}$$

we have $p_n(x, y) = p_n(y, x)$.

The natural metric on Γ, obtained by counting the number of steps in the shortest path between points, is written $d(x,y)$ for $x,y \in G$. We write

$$B(x,r) = \{y : d(x,y) < r\}, \qquad V(x,r) = \mu(B(x,r)), \quad r \in (0,\infty). \tag{1.3}$$

Following terminology used for manifolds, we call $V(x,r)$ the *volume* of the ball $B(x,r)$. We will assume G contains a marked vertex, which we denote 0, and we write

$$B(R) = B(0,R), \qquad V(R) = V(0,R). \tag{1.4}$$

For $A \subset G$, we write

$$T_A = \inf\{n \geq 0 : X_n \in A\}, \qquad \tau_A = T_{A^c}, \tag{1.5}$$

and let

$$\tau_R = \tau_{B(0,R)} = \min\{n \geq 0 : X_n \notin B(0,R)\}. \tag{1.6}$$

Let $W_n = \{X_0, X_1, \dots, X_n\}$ be the set of vertices hit by X up to time n, and let

$$S_n = \mu(W_n) = \sum_{x \in W_n} \mu_x. \tag{1.7}$$

We write $R_{\text{eff}}(0, B(R)^c)$ for the effective resistance between 0 and $B(R)^c$ in the electric network obtained by making each edge of Γ a unit resistor – see [15]. A precise mathematical definition of $R_{\text{eff}}(\cdot,\cdot)$ will be given in Section 2.

We now consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a family of random graphs $\Gamma(\omega) = (G(\omega), E(\omega), \omega \in \Omega)$. We assume that, for each $\omega \in \Omega$, the graph $\Gamma(\omega)$ is infinite, locally finite and connected, and contains a marked vertex $0 \in G$. We denote balls in $\Gamma(\omega)$ by $B_\omega(x,r)$, their volume by $V_\omega(x,r)$, and write

$$B(R) = B_\omega(R) = B_\omega(0,R), \qquad V(R) = V_\omega(R) = V_\omega(0,R). \tag{1.8}$$

We write $X = (X_n, n \geq 0, P^x_\omega, x \in G(\omega))$ for the simple random walk on $\Gamma(\omega)$, and denote by $p^\omega_n(x,y)$ its transition density with respect to $\mu(\omega)$. Formally, we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega \times \overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$.

The key ingredients in our analysis of the simple random walk are volume and resistance bounds. The following defines a set $J(\lambda)$ of values of R for which we have 'good' volume and effective resistance estimates. The set $J(\lambda)$ depends on the graph Γ, and thus is a random set under $\mathbb{P}$.

Definition 1.1. *Let $\Gamma = (G,E)$ be as above. For $\lambda > 1$, let $J(\lambda)$ be the set of those $R \in [1,\infty]$ such that the following all hold:*
(1) $V(R) \leq \lambda R^2$,
(2) $V(R) \geq \lambda^{-1} R^2$,
(3) $R_{\text{eff}}(0, B(R)^c) \geq \lambda^{-1} R$.

Note that $R_{\text{eff}}(0, B(R)^c) \leq R$ (see Lemma 2.2(c) in Section 2.1), so there is no need for an upper bound complementary to Definition 1.1(3). We now make the following important assumption concerning the graphs $(\Gamma(\omega))$. This involves upper and lower bounds on the volume, as well as an estimate which says that R is likely to be in $J(\lambda)$ for large enough λ.

Assumption 1.2. *There exists $R^* \geq 1$ such that the following hold:*
(1) There exists $p(\lambda) \geq 0$, with $p(\lambda) \leq c_1\lambda^{-q_0}$ for some $q_0, c_1 > 0$, such that for each $R \geq R^*$,

$$\mathbb{P}(R \in J(\lambda)) \geq 1 - p(\lambda), \tag{1.9}$$

(2) $\mathbb{E}[V(R)] \leq c_2 R^2$, for $R \in [R^*, \infty)$,
(3) $\mathbb{E}[1/V(R)] \leq c_3 R^{-2}$ for $R \in [R^*, \infty)$.

Remark. Assumption 1.2(2,3), together with Markov's inequality, provides upper bounds of the form $c\lambda^{-1}$ for the probability of the complements of the events in Definition 1.1(1,2). This creates some redundancy in our formulation, but we state things this way because some of our conclusions for the random walk rely only on Assumption 1.2(1) and do not require the stronger volume bounds given by Assumption 1.2(2,3).

Note that Assumption 1.2 only involves statements about the volume and resistance from one point 0 in the graph. In general, this kind of information would not be enough to give much control of the random walk. However, the graphs considered here have strong recurrence properties, and are therefore simpler to handle than general graphs. We use techniques developed in [6, 7, 37, 38, 39].

We will prove in Theorem 1.7 that Assumption 1.2 holds for the IIC for sufficiently spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ when $d > 6$. As the reader of Sections 4–5 will see, obtaining volume and (especially) resistance bounds on the IIC from one base point is already difficult; it is fortunate that we do not need to assume more.

We have the following four consequences of Assumption 1.2 for random graphs. They give control, in different ways, of the quantities $E^0_\omega \tau_R$, $p_{2n}(0,0)$, $d(0,X_n)$, and S_n, which measure the rate of dispersion of the random walk X from the base point 0. Some statements in the first proposition involve the averaged law defined by the semi-direct product $P^* = \mathbb{P} \times P^0_\omega$.

Theorem 1.3. *Suppose Assumption 1.2(1) holds. Then, uniformly with respect to $n \geq 1$ and $R \geq 1$,*

$$\mathbb{P}(\theta^{-1} \leq R^{-3} E^0_\omega \tau_R \leq \theta) \to 1 \quad \text{as } \theta \to \infty, \tag{1.10}$$
$$\mathbb{P}(\theta^{-1} \leq n^{2/3} p^\omega_{2n}(0,0) \leq \theta) \to 1 \quad \text{as } \theta \to \infty, \tag{1.11}$$
$$P^*(d(0,X_n) n^{-1/3} < \theta) \to 1 \quad \text{as } \theta \to \infty. \tag{1.12}$$
$$P^*(\theta^{-1} < (1 + d(0,X_n)) n^{-1/3}) \to 1 \quad \text{as } \theta \to \infty. \tag{1.13}$$

Since $P^0_\omega(X_{2n} = 0) \approx n^{-2/3}$, we cannot replace $1 + d(0,X_n)$ by $d(0,X_n)$ in (1.13).

Theorem 1.4. *Suppose Assumption 1.2(1,2,3) hold. Then there exists $n^* \geq 1$ (depending only on R^* and the function $p(\cdot)$ in Assumption 1.2), and constants c_i such that*

$$c_1 R^3 \leq \mathbb{E}(E^0_\omega \tau_R) \leq c_2 R^3 \text{ for all } R \geq 1, \tag{1.14}$$
$$c_3 n^{-2/3} \leq \mathbb{E}(p^\omega_{2n}(0,0)) \leq c_4 n^{-2/3} \text{ for all } n \geq n^*. \tag{1.15}$$
$$c_5 n^{1/3} \leq \mathbb{E}(E^0_\omega d(0,X_n)) \text{ for all } n \geq n^*. \tag{1.16}$$

We do not have an upper bound in (1.16); this is discussed further in Example 2.6 below.

Remark. The above two theorems in fact do not require the polynomial decay of $p(\lambda)$; it is enough to have $p(\lambda) \to 0$ as $\lambda \to \infty$.

Let $d_s(G)$ be the *spectral dimension of* G, defined by

$$d_s(G) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log n}, \tag{1.17}$$

if this limit exists. Here $x \in G$; it is easy to see that the limit is independent of the base point x. Note that $d_s(\mathbb{Z}^d) = d$.

In (c) below, recall that $\overline{\Omega}$ is the second probability space, on which the random walk X is defined.

Theorem 1.5. *Suppose Assumption 1.2(1) holds. Then there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 < \infty$, and a subset Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that the following statements hold.*
(a) For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exists $N_x(\omega) < \infty$ such that

$$(\log n)^{-\alpha_1} n^{-2/3} \le p^{\omega}_{2n}(x,x) \le (\log n)^{\alpha_1} n^{-2/3}, \quad n \ge N_x(\omega). \tag{1.18}$$

In particular, $d_s(G) = \frac{4}{3}$, $\mathbb{P}$-a.s., and the random walk is recurrent.
(b) For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exists $R_x(\omega) < \infty$ such that

$$(\log R)^{-\alpha_2} R^3 \le E^x_\omega \tau_R \le (\log R)^{\alpha_2} R^3, \quad R \ge R_x(\omega). \tag{1.19}$$

Hence

$$\lim_{R\to\infty} \frac{\log E^x_\omega \tau_R}{\log R} = 3.$$

(c) Let $Y_n = \max_{0\le k\le n} d(0, X_k)$. For each $\omega \in \Omega_0$ and $x \in G(\omega)$ there exist $N_x(\omega,\varpi), R_x(\omega,\varpi)$ such that $P^x_\omega(N_x < \infty) = P^x_\omega(R_x < \infty) = 1$, and such that

$$(\log n)^{-\alpha_3} n^{1/3} \le Y_n(\omega,\varpi) \le (\log n)^{\alpha_3} n^{1/3}, \quad n \ge N_x(\omega,\varpi), \tag{1.20}$$

$$(\log R)^{-\alpha_4} R^3 \le \tau_R(\omega,\varpi) \le (\log R)^{\alpha_4} R^3, \quad R \ge R_x(\omega,\varpi). \tag{1.21}$$

Remark. One cannot expect (1.18) or (1.19) to hold with $\alpha_1 = 0$ or $\alpha_2 = 0$, since it is known that $\log\log$ fluctuations occur in the analogous limits for the IIC on regular trees [7]. (This example is discussed further in Example 1.8(i) below).

Let $W_n = \{X_0, X_1, \dots, X_n\}$ as before and let $|W_n|$ denote its cardinality. For a sufficiently regular recurrent graph one expects that $|W_n| \approx n^{d_s/2}$. The original formulation of the Alexander-Orbach conjecture [3] was that, in all dimensions, for the IIC,

$$|W_n| \approx n^{2/3}, \tag{1.22}$$

so that $d_s = \frac{4}{3}$ in all dimensions. As noted already above, the conjecture is now not believed to hold in low dimensions. The following theorem shows that a version of the Alexander–Orbach conjecture does hold for random graphs that satisfy Assumption 1.2(1). As we will see in Theorem 1.7, this is the case for the IIC for sufficiently spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d > 6$.

Theorem 1.6. *(a) Suppose Assumption 1.2(1) holds. Then there exists a subset* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that for each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$,

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{2}{3}, \quad P^x_\omega\text{-a.s.}. \tag{1.23}$$

(b) Suppose in addition there exists a constant c_0 *such that all vertices in* G *have degree less than* c_0. *Then*

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{2}{3}, \quad P^x_\omega\text{-a.s.} \tag{1.24}$$

See Example 1.8 for a graph with unbounded degree which satisfies Assumption 1.2, but for which (1.24) fails.

Remark. See [32] for results which generalise the above theorems to the situation where there exist indices $\alpha < \beta$ such that $V(R)$ is comparable to R^α and $R_{\text{eff}}(0, B(R)^c)$ is comparable to $R^{\beta-\alpha}$. Our case is $\alpha = 2$, $\beta = 3$.

1.3 The IIC

In this section, we define the oriented percolation model and recall the construction of the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 4$ [21]. For simplicity, we will consider only the most basic example of a spread-out model. (In the physics literature, oriented percolation is usually called *directed* percolation; see [28].)

The spread-out oriented percolation model is defined as follows. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and directed bonds $((x, n), (y, n+1))$, for $n \geq 0$ and $x, y \in \mathbb{Z}^d$ with $0 \leq \|x-y\|_\infty \leq L$. Here L is a fixed positive integer and $\|x\|_\infty = \max_{i=1,\dots,d} |x_i|$ for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$. Let $p \in [0, 1]$. We associate to each directed bond $((x, n), (y, n + 1))$ an independent random variable taking the value 1 with probability p and 0 with probability $1 - p$. We say a bond is *occupied* when the corresponding random variable is 1, and *vacant* when the random variable is 0. Given a configuration of occupied bonds, we say that (x, n) is *connected to* (y, m), and write $(x, n) \longrightarrow (y, m)$, if there is an oriented path from (x, n) to (y, m) consisting of occupied bonds, or if $(x, n) = (y, m)$. Let $C(x, n)$ denote the forward cluster of (x, n), i.e., $C(x, n) = \{(y, m) : (x, n) \longrightarrow (y, m)\}$, and let $|C(x, n)|$ denote its cardinality.

The joint probability distribution of the bond variables will be denoted $\mathbb{P}$, with corresponding expectation denoted $\mathbb{E}$; these depend on p and are defined on a probability space $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$. Let $\theta(p) = \mathbb{P}(|C(0, 0)| = \infty)$. For all dimensions $d \geq 1$ and for all $L \geq 1$, there is a critical value $p_c = p_c(d, L) \in (0, 1)$ such that $\theta(p) = 0$ for $p \leq p_c$ and $\theta(p) > 0$ for $p > p_c$. In particular, there is no infinite cluster when $p = p_c$ [11, 19]. For the remainder of this paper, we fix $p = p_c$, so that $\mathbb{P} = \mathbb{P}_{p_c}$.

To define the IIC, some terminology is required. A *cylinder event* is an event that is determined by the occupation status of a finite set of bonds. We denote the algebra of cylinder events by $\mathcal{F}_0$, and define $\mathcal{F}$ to be the σ-algebra generated by $\mathcal{F}_0$. The most natural definition of the IIC is as follows. Let $\{(x, m) \longrightarrow n\}$ denote the event that there exists (y, n) such that $(x, m) \longrightarrow (y, n)$. Let

$$\mathbb{Q}_n(E) = \mathbb{P}(E|(0, 0) \longrightarrow n) \quad (E \in \mathcal{F}_0) \tag{1.25}$$

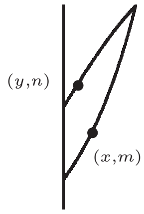

Figure 1: Although the vertex (x,m) is not connected to (y,n), or vice versa, in the sense of oriented percolation (oriented upwards), it is nevertheless possible for a random walk to move from one of these vertices to the other.

and define the IIC by

$$\mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad (E \in \mathcal{F}_0), \tag{1.26}$$

assuming the limit exists. A possible alternate definition of the IIC is to define

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_{x\in\mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0) \longrightarrow (x,n)\}) \quad (E \in \mathcal{F}_0) \tag{1.27}$$

with $\tau_n = \sum_{x\in\mathbb{Z}^d} \mathbb{P}((0,0) \longrightarrow (x,n))$, and to let

$$\mathbb{P}_\infty(E) = \lim_{n\to\infty} \mathbb{P}_n(E) \quad (E \in \mathcal{F}_0), \tag{1.28}$$

assuming the limit exists.

Let $d+1 > 4+1$ and $p = p_c$. It was proved in [21] that there is an $L_0 = L_0(d)$ such that for $L \ge L_0$ the limit (1.28) exists for every cylinder event $E \in \mathcal{F}_0$. Moreover, $\mathbb{P}_\infty$ extends to a probability measure on the σ-algebra $\mathcal{F}$, and, writing $\mathcal{C} = C(0,0)$, $\mathcal{C}$ is $\mathbb{P}_\infty$-a.s. an infinite cluster. It was also proved in [21] that if the critical survival probability $\mathbb{P}((0,0) \longrightarrow n)$ is asymptotic to a multiple of n^{-1} as $n \to \infty$, then for $L_0 = L_0(d)$ the limit (1.26) exists and defines a probability measure on $\mathcal{F}$, and moreover $\mathbb{Q}_\infty = \mathbb{P}_\infty$ so both constructions yield the same measure. Subsequently, it was shown in [22, 23] that the survival probability is indeed asymptotic to a multiple of n^{-1} when $d+1 > 4+1$ and $L \ge L_0(d)$. We will find both of the equivalent definitions (1.26) and (1.28) to be useful.

We call $(\mathcal{C}, \mathbb{Q}_\infty)$= $(\mathcal{C}, \mathbb{P}_\infty)$ the IIC, and this provides the random environment for our random walk. We write $\mathbb{E}_\infty$ for expectation with respect to $\mathbb{Q}_\infty$. It will be convenient to remove a $\mathbb{Q}_\infty$-null set $\mathcal{N}$ from the configuration space Ω, so that for all $\omega \in \Omega_0 = \Omega - \mathcal{N}$ the cluster $\mathcal{C}(\omega)$ is infinite (and connected). The IIC $\mathcal{C}(\omega)$, $\omega \in \Omega$ under the law $\mathbb{Q}_\infty$ gives a family of random graphs, with marked vertex $\mathbf{0} = (0,0)$, so as in Section 1.2 we can define a random walk $X = (X_j, j \in \mathbb{Z}_+, P_\omega^{(x,n)}, (x,n) \in \mathcal{C}(\omega))$. Note that although the orientation is used to construct the cluster $\mathcal{C}$, once $\mathcal{C}$ has been determined the random walk on $\mathcal{C}$ can move in any direction — see Figure 1.

Theorem 1.7. *For $d > 6$, there is an $L_1 = L_1(d) \ge L_0(d)$ such that for all $L \ge L_1$, Assumption 1.2(1)–(3) hold with $q_0 = 1$ and constants c_1, c_2, c_3 independent of d and L. Consequently, the* *conclusions of Theorems 1.3, 1.4, 1.5 and 1.6 all hold for the random walk on the* IIC. *In particular, the Alexander–Orbach conjecture holds in the form of* (1.24).

As we will see later, the restriction to $d > 6$ is required only for our estimate of the effective resistance.

Remark. Since the constants in Assumption 1.2 are independent of d, L for the IIC (provided $d > 6$ and $L \geq L_1(d)$), the constants $\alpha_1, \cdots, \alpha_4$ in Theorem 1.5 are also independent of d and L when applied to the IIC.

The proof of our main results are performed in two principal steps, corresponding to the results in Section 1.2 and Theorem 1.7 respectively.

The results in Section 1.2 are proved in Section 2. The first step is to obtain estimates for a fixed (non-random) graph Γ. In Section 2.1, using arguments based on those in [6] and [7], we show that volume and resistance bounds on Γ lead to bounds on transition probabilities and hitting times. Then, in Section 2.2 we translate these results into the random graph context, and prove Theorems 1.3–1.6.

The second step is the proof of Theorem 1.7. Section 3 states three properties of the IIC for critical spread-out oriented percolation in dimensions $d > 6$, and show that these imply Theorem 1.7. These properties are proved in Sections 4–5, using an extension of results of [21, 22, 26] that were obtained using the lace expansion.

1.4 Further Examples

We have some other examples of random graphs which satisfy Assumption 1.2.

Example 1.8. (i) Assumption 1.2 holds for random walk on the IIC for the binomial tree; see [7, Corollary 2.12]. Therefore the conclusions of Theorems 1.3–1.6 hold for random walk on this IIC. The results of [7] go beyond Theorem 1.5(a) and (b) in this context, but Theorem 1.5(c) and Theorem 1.6 here are new.

(ii) It is shown in [4] that the invasion percolation cluster on a regular tree is stochastically dominated by the IIC for the binomial tree. Consequently, upper bounds on the volume and lower bounds on the effective resistance of the invasion percolation cluster follow from the corresponding bounds for the IIC (using Lemma 2.2(e) in Section 2.1). Assumption 1.2(1,2) for the invasion percolation cluster therefore follows from its counterpart for the IIC for the binomial tree. In addition, the lower bound on the volume in Assumption 1.2(3) is proved for the invasion percolation cluster in [4]. Therefore Assumption 1.2 holds for the invasion percolation cluster on a regular tree, and hence simple random walk on the invasion percolation cluster also obeys the conclusions of Theorems 1.3–1.6. See [4] for further details about this example.

(iii) Consider the incipient infinite branching random walk (IIBRW), obtained as the limit as $n \to \infty$ of critical branching random walk (say with binomial offspring distribution) conditioned to survive to at least n generations [20, Section 2]. We interpret the IIBRW as a random infinite subgraph of $\mathbb{Z}^d \times \mathbb{Z}_+$. There is the option of considering either one edge per particle jump, leading to the occurrence of multiple edges between vertices, or identifying any such multiple edges as a single edge; we believe both options will behave similarly in dimensions $d > 4$. Consider simple random walk on the IIBRW. Our volume estimates for the IIC for oriented percolation for $d > 4$ will adapt to give similar estimates for the IIBRW for $d > 4$. The effective resistance $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW is lower than it is for the IIC on a tree, due to cycles in the IIBRW. It is an interesting open problem to obtain a lower bound on $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW, to establish Assumption 1.2 and hence its consequences Theorems 1.3–1.6 for random walk on the IIBRW. Our main interest is the question: Does random walk on the IIBRW have the same behaviour in all dimensions $d > 4$, or is there different behaviour for $4 < d \le 6$ and $d > 6$? An answer would shed light on the question raised at the end of Section 1.1. It would also be of interest to consider this question in the continuum limit: Brownian motion on the canonical measure of super-Brownian motion conditioned to survive for all time (see [20]).

(iv) A non-random graph Γ satisfies Assumption 1.2 if and only if there exists λ such that $J(\lambda) = [1, \infty)$. If Γ_i, $1 \le i \le n$ are graphs satisfying Assumption 1.2 then the graph Γ obtained by joining the Γ_i at their marked vertices also satisfies Assumption 1.2.

(v) Consider the non-random graph consisting of $\mathbb{Z}_+$ with for each n a finite subgraph G_n connected by one point in G_n to the vertex n. If $\mu(G_n) \asymp n$ and the diameter of G_n is $o(n)$ then Assumption 1.2 holds. In particular, if we take G_n to be the complete graph with $r_n = \lfloor n^{1/2} \rfloor$ vertices, then while $V(R) \asymp R^2$, we have $|B(R)| \asymp R^{3/2}$. In this case (1.23) holds, whereas

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{1}{2}, \quad P_\omega^x\text{-a.s.} \tag{1.29}$$

The rough idea behind (1.29) is as follows. By (1.20), the distance travelled up to time n is approximately $n^{1/3}$. The proof of Theorem 1.6 shows that the random walk will visit a positive fraction of the vertices within this distance, and there are of order $(n^{1/3})^{3/2} = n^{1/2}$ such vertices, leading to (1.29). This shows that some bound on vertex degree is necessary before one can pass from (1.23) to (1.24).

Throughout the paper, we use c, c' to denote strictly positive finite constants whose values are not significant and may change from line to line. We write c_i for positive constants whose values are fixed within theorems and lemmas.

2 Random walk on a random graph

In this section we prove Theorems 1.3–1.6. First, in Section 2.1, we study the random walk on a fixed graph; then, in Section 2.2 we apply these results to a family of random graphs satisfying Assumption 1.2.

2.1 Random walk on a fixed graph

In this section, we fix an infinite locally-finite connected graph $\Gamma = (G, E)$, and will show that bounds on the quantities $V(R)$ and $R_{\text{eff}}(0, B(R)^c)$ lead to control of $E^0\tau_R$, $p_n(0,0)$ and $E^0 d(0, X_n)$. The results in [6] (see [6, Theorem 1.3, Lemma 2.2]) cover the case where, for all $x \in G$ and $R \ge 1$,

$$c_1 R^2 \le V(x, R) \le c_2 R^2, \quad c_3 R \le R_{\text{eff}}(x, B(x, R)^c) \le c_4 R. \tag{2.1}$$

Here, we treat the case where we only have information available on the volume and effective resistance from one fixed point 0 in the graph, and only for certain values of R. Our methods are very close to those of [6], but the need to keep track of the values of R for which we make use of the bounds makes the details of the proofs more complicated.

The following Proposition gives the majority of the bounds on τ_R, $p_n(0,0)$ and $d(0,X_n)$ that will be used in Section 2.2.

Recall the definition of $J(\lambda)$ from Definition 1.1. In the following proposition, we will take $\lambda \geq 1$ and assume that R, and certain multiples of R, are in $J(\lambda)$. We then obtain (for example) bounds on $E^0\tau_R$; these bounds will involve constants depending on λ. For the limit Theorems 1.5 and 1.6 we need to know that the dependence of these constants on λ is polynomial in λ. To indicate this, we write $C_i(\lambda)$ to denote positive constants of the form $C_i(\lambda) = C_i\lambda^{\pm q_i}$, which will be fixed throughout this section. The sign accompanying $q_i > 0$ is such that statements become weaker as λ increases.

Proposition 2.1. *Let $\lambda \geq 1$. There exist $C_1(\lambda), \cdots, C_9(\lambda)$ such that the following hold.*
(a) Suppose that $R \in J(\lambda)$. Then

$$E^x\tau_R \leq 2\lambda R^3 \quad \textit{for } x \in B(R). \tag{2.2}$$

Suppose that $R, R/(4\lambda) \in J(\lambda)$. Then

$$E^x\tau_R \geq C_1(\lambda)R^3, \ \textit{for } x \in B(0, R/(4\lambda)). \tag{2.3}$$

Let $\varepsilon < 1/(4\lambda)$ and $R, \varepsilon R, \varepsilon R/(4\lambda) \in J(\lambda)$. Then

$$P^y\Big(\tau_R \leq C_2(\lambda)(\varepsilon R)^3\Big) \leq C_3(\lambda)\varepsilon, \quad \textit{for } y \in B(\varepsilon R). \tag{2.4}$$

(b) Suppose that $R \in J(\lambda)$. Then

$$p_n(0,y) + p_{n+1}(0,y) \leq C_4(\lambda)n^{-2/3} \quad \textit{for } y \in B(R) \textit{ if } n = 2\lfloor R\rfloor^3. \tag{2.5}$$

Suppose that $R, R/(4\lambda) \in J(\lambda)$. Then

$$p_{2n}(x,x) \geq C_5(\lambda)n^{-2/3} \qquad \textit{for } \tfrac{1}{4}C_1(\lambda)R^3 \leq n \leq \tfrac{1}{2}C_1(\lambda)R^3, \quad x \in B(0, R/(4\lambda)). \tag{2.6}$$

(c) Let $n \geq 1$, $M \geq 1$, and set $R = Mn^{1/3}$. If $R, C_6(\lambda)R/M, C_6(\lambda)R/(4\lambda M) \in J(\lambda)$, then

$$P^0\Big(n^{-1/3}d(0,X_n) > M\Big) \leq \frac{C_7(\lambda)}{M}. \tag{2.7}$$

We have $C_7(\lambda) \leq c\lambda^{22/3}$.
(d) Let $R = (n/2)^{1/3}$ and $M \geq 1$. If $R, R/M \in J(\lambda)$ then

$$P^0\Big(d(0,X_n) < R/M\Big) \leq \frac{\lambda C_4(\lambda)}{M^2}. \tag{2.8}$$

Also, if $R, C_8(\lambda)R \in J(\lambda)$ then

$$E^0 d(0,X_n) \geq C_9(\lambda)n^{1/3}. \tag{2.9}$$

The overall strategy for the proof of these various inequalities is as follows. We begin with obtaining bounds on the mean exit time $E^0\tau_R$. Using the Green function (see (2.17) below for the definition) we can write

$$E^z\tau_B = \sum_{y\in B} g_B(z,y)\mu_y. \tag{2.10}$$

Since $g_B(x,x) = R_{\text{eff}}(x,B^c)$ (see (2.20)), this leads to the upper and lower bounds on $E^x\tau_R$ for x sufficiently close to 0 given in (2.2) and (2.3). The final inequality concerning τ_R is (2.4), which bounds from above the lower tail of τ_R. (This is equivalent to bounding from above the speed at which X can move from its starting point 0.) The proof for this takes the bounds in (2.2) and (2.3) as its starting point, but also uses a simple inequality relating effective resistance and hitting probabilities – see Lemma 2.3 below.

The next set of inequalities we prove are those for the heat kernel $p_n(x,y)$. In the continuous time setting these are proved using differential inequalities which relate the derivative of the heat kernel to its energy. Unfortunately in discrete time the differential inequalities are replaced by rather less intuitive difference equations, which in addition take a slightly more complicated form. The estimate (2.5) is proved from an inequality which bounds the heat kernel just in terms of the volume of balls – see (2.31). Adding information on τ_R then enables one to obtain the lower bound (2.6).

The final bounds on $d(0,X_n)$ then follow easily from the bounds on τ_R and $p_n(0,x)$.

2.1.1 Bounds on τ_R

We begin by giving a precise definition of effective resistance. Let $\mathcal{E}$ be the quadratic form given by

$$\mathcal{E}(f,g) = \tfrac{1}{2}\sum_{\substack{x,y\in G\\ x\sim y}} (f(x)-f(y))(g(x)-g(y)), \tag{2.11}$$

where $x\sim y$ means $\{x,y\}\in E$. If we regard Γ as an electrical network with a unit resistor on each edge in E, then $\mathcal{E}(f,f)$ is the energy dissipation when the vertices of G are at a potential f. Set $H^2 = \{f\in\mathbb{R}^G : \mathcal{E}(f,f)<\infty\}$. Let A,B be disjoint subsets of G. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : f\in H^2, f|_A = 1, f|_B = 0\}. \tag{2.12}$$

Let $R_{\text{eff}}(x,y) = R_{\text{eff}}(\{x\},\{y\})$, and $R_{\text{eff}}(x,x)=0$. For general facts on effective resistance and its connection with random walks see [2, 15, 33]. We recall some basic properties of $R_{\text{eff}}(\cdot,\cdot)$.

Lemma 2.2. *Let $\Gamma=(G,E)$ be an infinite connected graph.*
(a) R_{eff} is a metric on G.
(b) If $A'\subset A$, $B'\subset B$ then $R_{\text{eff}}(A',B')\ge R_{\text{eff}}(A,B)$.
(c) $R_{\text{eff}}(x,y)\le d(x,y)$.
(d) If $x,y\in G\setminus A$ then $R_{\text{eff}}(x,A)\le R_{\text{eff}}(x,y)+R_{\text{eff}}(y,A)$.
(e) If $\Gamma'=(G',E')$ is a subgraph of Γ, with effective resistance R'_{eff}, and if $A'=A\cap G'$ and $B'=B\cap G'$, then $R'_{\text{eff}}(A',B')\ge R_{\text{eff}}(A,B)$.
(f) For all $f\in\mathbb{R}^G$ and $x,y\in G$,

$$|f(x)-f(y)|^2 \le R_{\text{eff}}(x,y)\mathcal{E}(f,f). \tag{2.13}$$

Proof. For (a) see [31, Section 2.3]. The monotonicity in (b) and (e) is immediate from the variational definition of R_{eff}. (c) is easy, and there is a proof in [6, Lemma 2.1]. (d) follows from (a) by considering the graph Γ' in which all vertices in A are connected by short circuits, which reduces A to a single vertex a.
(f) If $f(x) = f(y)$ then (2.13) is immediate. If not, then set $u(z) = (f(z) - f(y))/(f(x) - f(y))$, so that $u(x) = 1$ and $u(y) = 0$. Then by (2.12)

$$R_{\text{eff}}(x, y)^{-1} \le \mathcal{E}(u, u) = \mathcal{E}(f, f)|f(x) - f(y)|^{-2},$$

which gives (2.13). $\square$

The inequality (2.13) will play an important role in obtaining pointwise information on functions from resistance or energy estimates.

Recall that T_A was defined in (1.5) to be the hitting time of $A \subset G$. If A and B are disjoint subsets of G and $x \notin A \cup B$, then (see [10, Fact 2, p. 226])

$$P^x(T_A < T_B) \le \frac{R_{\text{eff}}(x, B)}{R_{\text{eff}}(x, A)}. \tag{2.14}$$

Lemma 2.3. *Let $\lambda \ge 1$ and suppose $R \in J(\lambda)$. Let $0 < \varepsilon \le 1/(2\lambda)$, and $y \in B(\varepsilon R)$. Then*

$$P^y(T_0 < \tau_R) \ge 1 - \frac{\lambda\varepsilon}{1 - \varepsilon\lambda} \ge 1 - 2\varepsilon\lambda, \tag{2.15}$$

$$P^0(T_y < \tau_R) \ge 1 - \varepsilon\lambda. \tag{2.16}$$

Proof. By Lemma 2.2(c) $R_{\text{eff}}(y, 0) \le d(y, 0)$, while by Lemma 2.2(d) and the definition of $J(\lambda)$,

$$R_{\text{eff}}(y, B(R)^c) \ge R_{\text{eff}}(0, B(R)^c) - R_{\text{eff}}(0, y) \ge \frac{R}{\lambda} - \varepsilon R.$$

So by (2.14)

$$P^y(\tau_R < T_0) \le \frac{R_{\text{eff}}(y, 0)}{R_{\text{eff}}(y, B(R)^c)} \le \frac{\varepsilon\lambda}{1 - \varepsilon\lambda}.$$

Similarly, $P^0(\tau_R < T_y) \le R_{\text{eff}}(0, y)/R_{\text{eff}}(0, B(R)^c) \le \varepsilon\lambda$. $\square$

The initial steps in bounding τ_R use the Green kernel for the random walk X, so we now recall its definition. (These facts about Green functions will only be used in this subsubsection.) Let $B \subset G$,

$$L(y, n) = \sum_{k=0}^{n-1} 1_{(X_k = y)},$$

and set

$$g_B(x, y) = \mu_y^{-1} E^x L(y, \tau_B) = \mu_y^{-1} \sum_{k=0}^{\infty} P^x(X_k = y, k < \tau_B). \tag{2.17}$$

Then $g_B(x, y) = g_B(y, x)$ and $g_B(x, \cdot)$ is harmonic on $B \setminus \{x\}$, and zero outside B. Using the Markov property at T_y gives

$$g_B(x, y) = P^x(T_y < \tau_B) g_B(y, y). \tag{2.18}$$

Summing (2.17) over $y \in B$ gives

$$E^z \tau_B = \sum_{y \in B} g_B(z,y)\mu_y. \tag{2.19}$$

The final property of $g_B(\cdot,\cdot)$ we will need is that

$$R_{\text{eff}}(x, B^c) = g_B(x,x). \tag{2.20}$$

One way to see this is to note that $g_B(x,\cdot)$ is the potential due to a unit current flow from x to B^c, so that $g_B(x,x)$ is the effective resistance from x to B^c. Alternatively, writing $p_B^x(y) = g_B(x,y)/g_B(x,x)$, one can verify that p_B^x attains the minimum in (2.12), and that $\mathcal{E}(p_B^x, p_B^x) = g_B(x,x)^{-1}$.

Proof of Proposition 2.1(a), (2.2). It is easy to use (2.19) to obtain an upper bound for the exit time from a ball. By Lemma 2.2(d) we have $R_{\text{eff}}(z, B^c) \leq 2R$ for any $z \in B = B(R)$. So,

$$E^z \tau_B = \sum_{y \in B} g_B(z,y)\mu_y \leq \sum_{y \in B} g_B(z,z)\mu_y = R_{\text{eff}}(z, B^c)V(R) \leq 2\lambda R^3, \tag{2.21}$$

which gives (2.2). □

Proof of Proposition 2.1(a), (2.3). Write $B = B(R)$. To obtain a lower bound for $E^0 \tau_B$ we restrict the sum in (2.19) to a smaller ball $B' = B(R/(4\lambda))$, and use Lemma 2.3 to bound $g_B(0,y)$ from below on B'. If $y \in B'$ then Lemma 2.3 gives $P^y(T_0 < \tau_B) \geq \frac{1}{2}$, so by (2.18) and (2.20)

$$g_B(0,y) = g_B(0,0)P^y(T_0 < \tau_B) \geq \tfrac{1}{2} g_B(0,0) = \tfrac{1}{2} R_{\text{eff}}(0, B^c) \geq \tfrac{1}{2} R/\lambda.$$

As $R/(4\lambda) \in J(\lambda)$ we have $\mu(B') \geq \lambda^{-1}(R/(4\lambda))^2$, and therefore we obtain,

$$E^0 \tau_B \geq \sum_{y \in B'} g_B(0,y)\mu_y \geq \tfrac{1}{2} g_B(0,0)\mu(B') \geq c\lambda^{-4} R^3. \tag{2.22}$$

Then for $x \in B'$ we have $E^x \tau_B \geq P^x(T_0 < \tau_B) E^0 \tau_B$, which gives (2.3). □

The upper and lower bounds on $E^x \tau_R$ lead to a preliminary inequality on the distribution of τ_R.

Lemma 2.4. *Suppose that $R, R/(4\lambda) \in J(\lambda)$. Let $x \in B(0, R/4\lambda)$ and $n \geq 1$. Then*

$$P^x(\tau_R > n) \geq \frac{C_1(\lambda)R^3 - n}{2\lambda R^3} \quad \textit{for } n \geq 0. \tag{2.23}$$

Proof. By the Markov property, (2.2) and (2.3),

$$C_1(\lambda)R^3 \leq E^x \tau_R \leq n + E^x[1_{\{\tau_R > n\}} E^{X_n}(\tau_R)] \leq n + 2\lambda R^3 P^x(\tau_R > n).$$

Rearranging this gives (2.23). □

Setting $n = \delta R^3$ in (2.23) gives

$$P^x(\tau_R \leq \delta R^3) \leq 1 - \frac{C_1(\lambda) - \delta}{2\lambda}. \tag{2.24}$$

This inequality has the defect that the right hand side of (2.24) does not converge to 0 as $\delta \to 0$. We will need a better bound in order to control $d(0, X_n)$, and this is given in (2.4).

Proof of Proposition 2.1(a), (2.4). This proof takes a little more work; we obtain it by a kind of bootstrap from (2.23) and Lemma 2.3. The basic point is that, starting at $y \in B(\varepsilon R)$, X is very likely to visit 0 before escaping from $B(R)$. So X will with high probability have made many excursions from 0 to $\partial B(\varepsilon R)$ before time τ_B. Thus τ_B is stochastically larger than a sum of independent random variables, each of which, by (2.23), has a probability at least $p > 0$ of being greater than cR^3. Rather than following this intuition directly and using stochastic inequalities, it is simpler to obtain a pair of inequalities (2.25) and (2.26) which contain the same information.

Let $t_0 > 0$, and set

$$q(y) = P^y(\tau_R \le T_0), \qquad a(y) = P^y(\tau_R \le t_0).$$

Then

$$\begin{aligned} a(y) = P^y(\tau_R \le t_0) &= P^y(\tau_R \le t_0, \tau_R \le T_0) + P^y(\tau_R \le t_0, \tau_R > T_0) \\ &\le P^y(\tau_R \le T_0) + P^y(T_0 < \tau_R, \tau_R - T_0 \le t_0) \\ &\le q(y) + (1 - q(y))a(0) \le q(y) + a(0), \end{aligned} \tag{2.25}$$

using the strong Markov property for the second inequality. Starting X at 0 we have

$$a(0) = P^0(\tau_R \le t_0) \le E^0[1_{\{\tau_{\varepsilon R} \le t_0\}} P^{X_{\tau_{\varepsilon R}}}(\tau_R \le t_0)] \le P^0(\tau_{\varepsilon R} \le t_0) \max_{y \in \partial B(\varepsilon R)} a(y). \tag{2.26}$$

Combining (2.25) and (2.26) gives

$$a(0) \le \frac{\max_{y \in \partial B(\varepsilon R)} q(y)}{P^0(\tau_{\varepsilon R} > t_0)}. \tag{2.27}$$

Note that as $J(\lambda)$ is defined to be a subset of $[1, \infty)$, the condition that $\varepsilon R/(4\lambda) \in J(\lambda)$ implies that $R \ge 4\lambda/\varepsilon$. Since $\varepsilon < 1/(4\lambda)$, $\varepsilon R + 1 \le 2\varepsilon R < R/(2\lambda)$, and Lemma 2.3 (used with 2ε) gives

$$q(y) \le \frac{2\varepsilon\lambda}{1 - 2\varepsilon\lambda} \le 4\varepsilon\lambda. \tag{2.28}$$

Let $t_0 = \frac{1}{2} C_1(\lambda)(\varepsilon R)^3$; then using (2.23) for the ball $B(\varepsilon R)$ we obtain

$$P^0(\tau_{\varepsilon R} > t_0) \ge \frac{C_1(\lambda)}{4\lambda};$$

combining this with (2.28), (2.27) and (2.25) completes the proof of (2.4). $\square$

2.1.2 Heat kernel bounds

We now turn to the heat kernel bounds in Proposition 2.1(b). Our first result Proposition 2.5 follows from [6, Lemmas 1.1, 1.2 and 3.10], but as the proof is short we give it here. To deal with issues related to the possible bipartite structure of the graph it proves helpful to consider $p_n(x, y) + p_{n+1}(x, y)$. The main result of the proposition below is the inequality (2.31), which gives an upper bound for $p_n(x, x)$ just in terms of the volume. The proof of the analogous inequality in continuous time is a bit easier – see [7, Theorem 4.1].

Proposition 2.5. *Let $x_0 \in G$ and $f_n(y) = p_n(x_0, y) + p_{n+1}(x_0, y)$.*
(a) We have

$$\mathcal{E}(f_n, f_n) \le \frac{2}{n} f_{2\lfloor n/2 \rfloor}(x_0). \tag{2.29}$$

(b) We have

$$|f_n(y) - f_n(x_0)|^2 \le \frac{2}{n} d(x_0, y) f_{2\lfloor n/2 \rfloor}(x_0). \tag{2.30}$$

(c) Let $r \in [1, \infty)$ and $n = 2\lfloor r \rfloor^3$. Then

$$f_n(x_0) \le c_1 n^{-2/3} (1 \vee (r^2/V(x_0, r))). \tag{2.31}$$

Proof. (a) It is easy to check that

$$\mathcal{E}(f_n, f_n) = f_{2n}(x_0) - f_{2n+2}(x_0).$$

The spectral decomposition (see for example, Chapter 3 (32) of [2]) gives that $k \to f_{2k}(x_0) - f_{2k+2}(x_0)$ is non-increasing. Thus

$$\begin{aligned} n\left(f_{2n}(x_0) - f_{2n+2}(x_0)\right) &\le (2\lfloor n/2 \rfloor + 1) \left(f_{4\lfloor n/2 \rfloor}(x_0) - f_{4\lfloor n/2 \rfloor + 2}(x_0)\right) \\ &\le 2 \sum_{i=\lfloor n/2 \rfloor}^{2\lfloor n/2 \rfloor} \left(f_{2i}(x_0) - f_{2i+2}(x_0)\right) \le 2 f_{2\lfloor n/2 \rfloor}(x_0), \end{aligned}$$

and (2.29) is obtained.
(b) Using Lemma 2.2(c),(f),

$$|f_n(y) - f_n(x)|^2 \le R_{\mathrm{eff}}(x, y) \mathcal{E}(f_n, f_n) \le d(x, y) \mathcal{E}(f_n, f_n).$$

We then use (2.29) to bound $\mathcal{E}(f_n, f_n)$.
(c) Choose $x_* \in B(x_0, r)$ such that $f_n(x_*) = \min_{x \in B(x_0, r)} f_n(x)$. Then

$$f_n(x_*) V(x_0, r) \le \sum_{x \in B(x_0, r)} f_n(x) \mu_x \le \sum_{x \in G} p_n(x_0, x) \mu_x + \sum_{x \in G} p_{n+1}(x_0, x) \mu_x \le 2,$$

so that $f_n(x_*) \le 2/V(x_0, r)$. Since n is even, by (2.30) we have

$$f_n(x_0)^2 \le 2\left(f_n(x_*)^2 + |f_n(x_0) - f_n(x_*)|^2\right) \le \frac{8}{V(x_0, r)^2} + \frac{c r f_n(x_0)}{n}.$$

Using $a + b \le 2(a \vee b)$, we see that $f_n(x_0) \le (c'/V(x_0, r)) \vee (c' r/n)$. $\square$

Remark. In fact, (2.29) can be sharpened to give $\mathcal{E}(f_n, f_n) \le c_1 n^{-1} p_{2\lfloor n/2 \rfloor}(x_0, x_0)$, – see [6, Lemma 3.10], but we do not need this.

Proof of Proposition 2.1(b). Let $f_n(y) = p_n(0, y) + p_{n+1}(0, y)$. As $R \in J(\lambda)$, $R^2/V(R) \le \lambda$, so by Proposition 2.5(c),

$$f_n(0) \le c_1 \lambda n^{-2/3}. \tag{2.32}$$

By Proposition 2.5(b), if n is even

$$f_n(y) \le f_n(0) + |f_n(y) - f_n(0)| \le f_n(0) + (2d(0, y)n^{-1} f_n(0))^{1/2} \le c\lambda n^{-2/3}, \tag{2.33}$$

which proves (2.5).

To prove the lower bound (2.6) we use Lemma 2.4. For sufficiently small n this bounds from above the probability that X has left B by time n, and so bounds from below $P^0(X_n \in B)$. This leads easily to a lower bound on $p_{2n}(x, x)$. Here are the details. Let $n \le \frac{1}{2} C_1(\lambda) R^3$. Then using (2.23)

$$P^x(X_n \in B) \ge P^x(\tau_B > n) \ge \tfrac{1}{4} \lambda^{-1} C_1(\lambda). \tag{2.34}$$

By Chapman–Kolmogorov and Cauchy–Schwarz

$$P^x(X_n \in B)^2 = (\sum_{y \in B} p_n(x, y)\mu_y)^2 \le \mu(B) \sum_{y \in B} p_n(x, y)^2 \mu_y \le p_{2n}(x, x)\lambda R^2,$$

and using (2.34) gives (2.6). □

2.1.3 Bounds on $d(0, X_n)$

The main work for these bounds has already been done in the proofs of Proposition 2.1(a) and (b), and in particular the proof of (2.4).

Proof of Proposition 2.1. (c) The proof of (2.7) follows from (2.4) after suitable checking, since

$$P^0(d(0, X_n)n^{-1/3} > M) = P^0(d(0, X_n) > R) \le P^0(\tau_R \le n). \tag{2.35}$$

We now fill in the details. Define ε by the relation $n = C_2(\lambda)(\varepsilon R)^3$; so that $\varepsilon = C_6(\lambda)/M$. Using (2.35) and (2.4),

$$P^0(d(0, X_n)n^{-1/3} > M) \le P^0(\tau_R \le C_2(\lambda)(\varepsilon R)^3) \le C_3(\lambda)\varepsilon \le \frac{C_7(\lambda)}{M}, \tag{2.36}$$

which proves (2.7). Tracking the powers of λ gives that $C_7(\lambda) \le \lambda^{22/3}$.

(d) We can bound the probability that X is in a ball B' by the volume of the ball and the maximum of the heat kernel on the ball. By (2.5), writing $B' = B(0, R/M) \subset B(0, R)$ and $f_n(0, y) = p_n(0, y) + p_{n+1}(0, y)$,

$$P^0(X_n \in B') = \sum_{y \in B'} p_n(0, y)\mu_y \le \sum_{y \in B'} f_n(0, y)\mu_y \le V(R/M) C_4(\lambda) R^{-2} \le \lambda C_4(\lambda)/M^2, \tag{2.37}$$

proving (2.8). The final inequality in (d) now follows easily, since all we need is that $d(0, X_n)$ is greater than $cn^{1/3}$ with positive probability. Let $M = C_8(\lambda)$ satisfy $M^2 = 2\lambda C_4(\lambda)$. Then using (c), $P^0(d(0, X_n) < R/M) \le \frac{1}{2}$, so $E^0 d(0, X_n) \ge \frac{1}{2} R/M$. □

We do not have an upper bound on $E^0 d(0, X_n)$ to complement the lower bound of Proposition 2.1(d), which uses volume and resistance bounds from a single base point, i.e., bounds on $V(0, R)$ and $R_{\text{eff}}(0, B(R)^c)$. Suppose that $J(\lambda) = [1, \infty)$ for some $\lambda \geq 1$, and let $Z_n = n^{-1/3} d(0, X_n)$. Then we are able to bound $E^0 Z_n^p$ for $p < 1$, since (2.7) gives

$$\begin{aligned} E^0[Z_n^p] &\leq \sum_{m=1}^{\infty} (2^{m+1})^p P^0\Big(2^m \leq n^{-1/3} d(0, X_n) < 2^{m+1}\Big) \\ &\leq \sum_{m=1}^{\infty} (2^{m+1})^p P^0\Big(n^{-1/3} d(0, X_n) \geq 2^m\Big) \leq c_1 \sum_{m=1}^{\infty} 2^{m(p-1)} = c_2 < \infty. \end{aligned}$$

On the other hand the following example indicates that, under our hypotheses, we cannot expect to have a uniform bound on $E^0(Z_n^p)$ when $p > 1$. We sketch this argument below.

Example 2.6. Let Γ be the subgraph of $\mathbb{Z}^2$ with vertex set $G = G_0 \cup G_1$, where $G_0 = \{(n, 0), n \in \mathbb{Z}\}$, and $G_1 = \{(n, m) : 0 \leq m \leq n\}$. Let the edges be $\{(n, 0), (n + 1, 0)\}$, for $n \in \mathbb{Z}$, and $\{(n, m), (n, m + 1)\}$ if $n \geq 1$ and $0 \leq m \leq n - 1$. Thus Γ consists of $\mathbb{Z}_-$ and a comb-type graph of vertical branches with base $\mathbb{Z}_+$. Write 0 for $(0, 0)$. It is easily checked that $V(0, R) \asymp R^2$, and $R_{\text{eff}}(0, B(0, R)^c) \geq R/4$. Thus there exists $\lambda_0 < \infty$ such that $J(\lambda_0) = [1, \infty)$. Let

$$H(a, b) = \{(n, m) \in G : a \leq n \leq b\}.$$

Let X_n be the simple random walk on Γ. If we time-change out the excursions of X away from $\mathbb{Z}$ then we obtain a simple random walk Y_n on $\mathbb{Z}$. Now let $R \geq 1$, and $r = R^{2/3} \in \mathbb{Z}$. Let $A = H(-r, r)$. Since $B(0, r/2) \subset A \subset B(0, 2r)$, Proposition 2.1(a) implies that $E^0 \tau_A \approx r^3 \approx R^2$. Since X only moves horizontally when it is on the x-axis, $P^0(X_{\tau_A} = (-r, 0)) = 1/2$. If $X_{\tau_A} = (-r, 0)$ then the probability that X reaches $H(-\infty, -R)$ before returning to 0 is $r/R \approx R^{-1/3}$; also, if X does this then the time taken to do so will be of order R^2.

These arguments lead us to expect that if $n = R^2$ then

$$P^0\Big(X_n \in H(-\infty, -R/2)\Big) \geq c R^{-1/3}. \tag{2.38}$$

Given (2.38), it follows from Markov's inequality that

$$E^0 Z_n^p \geq n^{-p/3} (R/2)^p P^0\Big(X_n \in H(-\infty, -R/2)\Big) \geq c n^{(p-1)/6},$$

and the lower bound diverges if $p > 1$. This concludes Example 2.6.

2.2 Results for random graphs

We now consider a family of random graphs, as described in Section 1.2, and prove Theorems 1.3–1.6. Most of the hard work has been done in the previous section, where we obtained bounds for a fixed graph Γ.

We begin by obtaining tightness of the quantities $R^{-3} E^0 \tau_R$, $n^{2/3} p_{2n}(0, 0)$, and $n^{-1/3} d(0, X_n)$. We recall the definition of the function $p(\lambda)$ in Assumption 1.2(1), and that $p(\lambda) \leq c_0 \lambda^{-q_0}$.

Proof of Theorem 1.3. The basic idea here is straightforward. For each of the quantities we are interested in, the estimates in Proposition 2.1 tell us that provided the environment is 'good' at the scale R (that is, more precisely, that $c_i R \in J(\lambda)$ for suitable c_i) then the quantity takes the value we want. The bounds we get will only hold if R or n is large enough, but it is easy to handle the small values of R or n.

We begin with (1.10). Let $\varepsilon > 0$. Choose $\lambda \geq 1$ such that $2p(\lambda) < \varepsilon$. Let $R/(4\lambda) \geq R^*$, and set $F_1 = \{R, R/(4\lambda) \in J(\lambda)\}$. Then, by Assumption 1.2(1), $\mathbb{P}(F_1) \geq 1 - 2p(\lambda)$. For $\omega \in F_1$, by Proposition 2.1(a), there exists $c_1 < \infty$, $q_1 \geq 0$ such that

$$(c_1 \lambda^{q_1})^{-1} \leq R^{-3} E_\omega^x \tau_R \leq c_1 \lambda^{q_1} \text{ for } x \in B(R/(4\lambda)). \tag{2.39}$$

So, if $\theta \geq c_1 \lambda^{q_1}$ then for $R \in [4\lambda R^*, \infty)$,

$$\mathbb{P}\Big(\theta^{-1} \leq R^{-3} E_\omega^0 \tau_R \leq \theta\Big) \geq \mathbb{P}(F_1) \geq 1 - 2p(\lambda) \geq 1 - \varepsilon. \tag{2.40}$$

Let $R_0 \geq 1$. Since $0 < \sup_{1 \leq r \leq R_0} r^{-3} E_\omega^0 \tau_r < \infty$, we have

$$\lim_{\theta \to \infty} \mathbb{P}(\theta^{-1} \leq r^{-3} E_\omega^0 \tau_r \leq \theta) = 1 \quad \text{uniformly for } r \in [1, R_0].$$

Combining this with (2.40) gives (1.10).

A similar argument enables us to handle the cases of small n in (1.11)–(1.13), and we do not provide further details on this point below.

For (1.11) let $n \geq 1$, $\lambda \geq 1$, and let R_0, R_1 be defined by $n = \frac{1}{2} C_1(\lambda) R_1^3 = 2R_0^3$. Let $F_2 = \{R_0, R_1, R_1/(4\lambda) \in J(\lambda)\}$. Suppose that R_0 and $R_1/(4\lambda)$ are both greater than R^*; then $\mathbb{P}(F_2) \geq 1 - 3p(\lambda)$. If $\omega \in F_2$ then by Proposition 2.1(b)

$$(c_2 \lambda^{q_2})^{-1} \leq n^{2/3} p_{2n}^\omega(0,0) \leq c_2 \lambda^{q_2}.$$

So,

$$\mathbb{P}\Big((c_2 \lambda^{q_2})^{-1} \leq n^{2/3} p_{2n}^\omega(0,0) \leq c_2 \lambda^{q_2}\Big) \geq \mathbb{P}(F_2) \geq 1 - 3p(\lambda), \tag{2.41}$$

proving (1.11).

We now prove (1.12). Let $n \geq 1$ and $\lambda \geq 1$. Let $M = \lambda^8$ and set

$$R_0 = M n^{1/3}, \quad R_1 = C_6(\lambda) n^{1/3}, \quad R_2 = C_6(\lambda) n^{1/3}/(4\lambda),$$

$F_3 = \{R_0, R_1, R_2 \in J(\lambda)\}$. If n is large enough so that $R_i \geq R^*$ for $0 \leq i \leq 2$ then by (2.7), if $\omega \in F_3$ then

$$P_\omega^0\Big(n^{-1/3} d(0, X_n) > \lambda^8\Big) \leq \frac{C_7(\lambda)}{\lambda^8} \leq \frac{c\lambda^{22/3}}{\lambda^8} = \frac{c}{\lambda^{2/3}}.$$

Taking $\theta = \lambda^8$, we have

$$P^*\Big(n^{-1/3} d(0, X_n) > \theta\Big) \leq \mathbb{P}(F_3^c) + \mathbb{E}\Big(P_\omega^0(n^{-1/3} d(0, X_n) > \lambda^8) 1_{F_3}\Big) \leq 3p(\theta^{1/8}) + c_3 \theta^{-1/12}, \tag{2.42}$$

and (1.12) follows.

Finally, we prove (1.13). Let $R = (n/2)^{1/3}$, $M \geq 1$. If $R, R/M \in J(\lambda)$ then by (2.8)

$$P_\omega^0\Big(n^{-1/3} d(0, X_n) < 2^{-1/3} M^{-1}\Big) \leq \frac{\lambda C_4(\lambda)}{M^2}. \tag{2.43}$$

Given $\varepsilon > 0$ choose λ so that $p(\lambda) < \varepsilon$ and M so that $\lambda C_4(\lambda)/M^2 < \varepsilon$. Let $F_4 = \{R, R/M \in J(\lambda)\}$. Then (2.43) holds for $\omega \in F_4$, so taking expectations with respect to $\mathbb{P}$

$$\begin{aligned} P^*\Big(n^{-1/3}(1+d(0,X_n)) < 2^{-1/3}M^{-1}\Big) &\le P^*\Big(n^{-1/3}d(0,X_n) < 2^{-1/3}M^{-1}\Big) \\ &= \mathbb{E}P^0_\omega\Big(n^{-1/3}d(0,X_n) < 2^{-1/3}M^{-1}\Big) \\ &\le \mathbb{P}(F_4^c) + \varepsilon < 3\varepsilon. \end{aligned}$$

This deals with the case of large n; for small n we just use $1 + d(0, X_n) \ge 1$. □

Proof of Theorem 1.4. We begin with the upper bounds in (1.14)–(1.15). Here all we need do is to use the bounds on $\mathbb{E}V(R)$ and $\mathbb{E}(1/V(R))$ given by Assumption 1.2(2), together with the bounds on $E^0\tau_R$ and $p_{2n}(0,0)$ obtained above.

By (2.21) and Assumption 1.2(2),

$$\mathbb{E}(E^0_\omega \tau_R) \le \mathbb{E}(2RV(R)) \le cR^3,$$

provided $R \ge R^*$. If $R \le R^*$ then since $\tau_R \le \tau^*_R$ we obtain the upper bound in (1.14) by adjusting the constant c_2. Also, by Proposition 2.5(c), if $r = (n/2)^{1/3}$ then using Assumption 1.2(3)

$$\mathbb{E}p^\omega_{2n}(0,0) \le cn^{-2/3}\mathbb{E}(1 + r^2/V(r)) \le c' n^{-2/3},$$

again provided $r \ge R^*$.

For each of the lower bounds, it is sufficient to find a set $F \subset \Omega$ of 'good' graphs with $\mathbb{P}(F) \ge c > 0$ such that, for all $\omega \in F$ we have suitable lower bounds on $E^0_\omega\tau_R$, $p^\omega_{2n}(0,0)$ or $E^0_\omega d(0, X_n)$. We assume that $R \ge 1$ is large enough so that $R/(4\lambda_0) \ge R^*$, where λ_0 is chosen large enough that $p(\lambda_0) < 1/8$. Again, we obtain the lower bound in (1.14) for small R using the fact that $\mathbb{E}(E^0_\omega\tau_R) \ge 1$ and adjusting the constant c_1.

Let $F = \{R, R/(4\lambda_0) \in J(\lambda_0)\}$. Then $\mathbb{P}(F) \ge \frac{3}{4}$, and for $\omega \in F$, by (2.3), $E^0_\omega\tau_R \ge c_1(\lambda_0)R^3$. So,

$$\mathbb{E}(E^0_\omega\tau_R) \ge \mathbb{E}(E^0_\omega\tau_R 1_F) \ge c_1(\lambda_0)R^3\mathbb{P}(F) \ge c_2(\lambda_0)R^3.$$

Given $n \in \mathbb{N}$, choose R so that $n = \frac{1}{2}C_1(\lambda_0)R^3$. Then there exists n^* (depending on λ_0 and R^*) such that $n \ge n^*$ implies that $R/(4\lambda_0) \ge R^*$. Let F be as above. Then using (2.6) to bound $p_{2n}(0,0)$ from below,

$$\mathbb{E}p^\omega_{2n}(0,0) \ge \mathbb{P}(F)c_3(\lambda_0)n^{-2/3} \ge c_4(\lambda_0)n^{-2/3},$$

giving the lower bound in (1.15).

A similar argument uses (2.9) to conclude (1.16). □

Proof of Theorem 1.5. These results will follow from the bounds already obtained in Proposition 2.1 and in the proof of Theorem 1.3 by a straightforward Borel–Cantelli argument.

We will take $\Omega_0 = \Omega_a \cap \Omega_b \cap \Omega_c$ where the sets Ω_* are defined in the proofs of (a), (b) and (c). Recall that by Assumption 1.2(1), $p(\lambda) = \mathbb{P}(R \notin J(\lambda)) \le c_0\lambda^{-q_0}$.

(a) We begin with the case $x = 0$, and write $w(n) = p^\omega_{2n}(0,0)$. By (2.41) we have

$$\mathbb{P}((c_1\lambda^{q_1})^{-1} < n^{2/3}w_n \le c_1\lambda^{-q_1}) \ge 1 - 3p(\lambda).$$

Let $n_k = \lfloor e^k \rfloor$ and $\lambda_k = k^{2/q_0}$. Then, since $\sum p(\lambda_k) < \infty$, by Borel–Cantelli there exists $K_0(\omega)$ with $\mathbb{P}(K_0 < \infty) = 1$ such that $c_1^{-1} k^{-2q_1/q_0} \le n_k^{2/3} w(n_k) \le c_1 k^{2q_1/q_0}$ for all $k \ge K_0(\omega)$. Let $\Omega_a = \{K_0 < \infty\}$. For $k \ge K_0$ we therefore have

$$c_2^{-1} (\log n_k)^{-2q_1/q_0} n_k^{-2/3} \le w(n_k) \le c_2 (\log n_k)^{2q_1/q_0} n_k^{-2/3},$$

so that (1.18) holds for the subsequence n_k. The spectral decomposition (see for example [2]) gives that $p^\omega_{2n}(0,0)$ is monotone decreasing in n. So, if $n > N_0 = e^{K_0} + 1$, let $k \ge K_0$ be such that $n_k \le n < n_{k+1}$. Then

$$w(n) \le w(n_k) \le c_2 (\log n_k)^{2q_1/q_0} n_k^{-2/3} \le 2e^{2/3} c_2 (\log n)^{2q_1/q_0} n^{-2/3}.$$

Similarly $w(n) \ge w(n_{k+1}) \ge c_3 n^{-2/3} (\log n)^{-2q_1/q_0}$. Taking $q_2 > 2q_1/q_0$, so that the constants c_2, c_3 can be absorbed into the $\log n$ term, we obtain

$$(\log n)^{-q_2} n^{-2/3} \le p^\omega_{2n}(0,0) \le (\log n)^{q_2} n^{-2/3} \quad \text{for all } n \ge N_0(\omega). \tag{2.44}$$

That $\lim_n \log p^\omega_{2n}(0,0)/\log n = -2/3$, $\mathbb{P}$-a.s. is then immediate. Since $\sum_n p^\omega_{2n}(0,0) = \infty$, X is recurrent.

If $x, y \in \mathcal{C}(\omega)$ and $k = d_\omega(x,y)$, then the Chapman–Kolmogorov equations give that

$$p^\omega_{2n}(x,x) (p^\omega_k(x,y) \mu_x(\omega))^2 \le p^\omega_{2n+2k}(y,y),$$

and using this it is easy to obtain (1.18) from (2.44).

(b) Let $R_n = e^n$ and $\lambda_n = n^{2/q_0}$. Let $F_n = \{R_n, R_n/(4\lambda_n) \in J(\lambda_n)\}$. Then (provided $R_n/(4\lambda_n) \ge 1$) we have $\mathbb{P}(F_n^c) \le 2p(\lambda_n) \le 2n^{-2}$. So, by Borel–Cantelli, if $\Omega_b = \liminf F_n$, then $\mathbb{P}(\Omega_b) = 1$. Hence there exists M_0 with $M_0(\omega) < \infty$ on Ω_b, and such that $\omega \in F_n$ for all $n \ge M_0(\omega)$.

Now fix $\omega \in \Omega_b$, and let $x \in \mathcal{C}(\omega)$. Write $F(R) = E^x_\omega \tau_R$. By (2.39) there exist constants c_4, q_4 such that

$$(c_4 \lambda_n^{q_4})^{-1} \le R_n^{-3} F(R_n) \le c_4 \lambda_n^{q_4}. \tag{2.45}$$

provided $n \ge M_0(\omega)$ and n is also large enough so that $x \in B(R_n/(4\lambda_n))$. Writing $M_x(\omega)$ for the smallest such n,

$$c_4^{-1} (\log R_n)^{-2q_4/q_0} R_n^3 \le F(R_n) \le c_4 (\log R_n)^{2q_4/q_0} R_n^3, \quad \text{for all } n \ge M_x(\omega).$$

As $F(R)$ is monotonic, the same argument as in (a) enables us to replace $F(R_n)$ by $F(R)$, for all $R \ge R_x = 1 + e^{M_x}$. Taking $\alpha_2 > 2q_4/q_0$ we obtain (1.19).

(c) Recall that $Y_n = \max_{0 \le k \le n} d(0, X_k)$. We begin by noting that

$$\{Y_n \ge R\} = \{\tau_R \le n\}. \tag{2.46}$$

Using this, (1.20) follows easily from (1.21).

It remains to prove (1.21). Since τ_R is monotone in R, as in (b) it is enough to prove the result for the subsequence $R_n = e^n$.

The estimates in (b) give the upper bound. In fact, if $\omega \in \Omega_b$, and $n \ge M_x(\omega)$, then by (2.45)

$$P^x_\omega(\tau_{R_n} \ge n^2 c_4 \lambda_n^{q_4} R_n^3) \le \frac{F(R_n)}{n^2 c_4 \lambda_n^{q_4} R_n^3} \le n^{-2}.$$

So, by Borel–Cantelli (with respect to the law P_ω^x), there exists $N'_x(\omega,\varpi)$ with

$$P_\omega^x(N'_x<\infty)=P_\omega^x(\{\varpi:N'_x(\omega,\varpi)<\infty\})=1$$

such that

$$\tau_{R_n}\le c_5(\log R_n)^{q_5}R_n^3,\qquad \text{for all } n\ge N'_x.$$

For the lower bound, write $C_2(\lambda)=c_6\lambda^{-q_6}$, $C_3(\lambda)=c_7\lambda^{q_7}$. Let $\lambda_n=n^{2/q_0}$, and $\varepsilon_n=n^{-2}\lambda_n^{-q_6-q_7}$. Set $G_n=\{R_n,\varepsilon_nR_n,\varepsilon_nR_n/(4\lambda_n)\in J(\lambda_n)\}$. Then, for n sufficiently large so that $\varepsilon_nR_n/(4\lambda_n)\ge 1$, we have $\mathbb{P}(G_n^c)\le 3p(\lambda_n)\le 3c_0n^{-2}$. Let $\Omega_c=\Omega_b\cap(\liminf G_n)$; then by Borel–Cantelli $\mathbb{P}(\Omega_c)=1$ and there exists M_1 with $M_1(\omega)<\infty$ for $\omega\in\Omega_c$ such that $\omega\in G_n$ whenever $n\ge M_1(\omega)$. By (2.4), if $n\ge M_1$ and $x\in B(\varepsilon_nR_n)$ then

$$P_\omega^x(\tau_{R_n}\le c_6\lambda_n^{-q_6}\varepsilon_n^3R_n^3)\le c_7\lambda_n^{q_7}\varepsilon_n\le c_7n^{-2}. \tag{2.47}$$

So, using Borel–Cantelli, we deduce that (for some q_8)

$$\tau_{R_n}\ge c_6\lambda^{-q_6}\varepsilon_n^3R_n^3\ge n^{-q_8}R_n^3=(\log R_n)^{-q_8}R_n^3,$$

for all $n\ge N''_x(\omega,\varpi)$. This completes the proof of (1.21). □

Proof of Theorem 1.6. (a) We first consider the case $x=0$. The upper bound on $\log S_n/\log n$ follows easily from the bounds on τ_R and $V(R)$, as follows. A Borel–Cantelli argument similar to those above implies that

$$V(R)\le R^2(\log R)^c \tag{2.48}$$

for all sufficiently large R. Recall that $Y_n=\max_{0\le k\le n}d(0,X_n)$. We have $W_n\subset B(Y_n)$, so $S_n\le V(Y_n)$. So, for sufficiently large n, using (1.20),

$$S_n\le V((\log n)^{\alpha_3}n^{1/3})\le n^{2/3}(\log n)^{c'}, \tag{2.49}$$

proving the upper bound in (1.23).

For the lower bound, we need to show that a positive proportion of the points in $B(Y_n)$ have been hit by time n, and for this we use Lemma 2.3.

Choose $q_1\ge 1$, $q_2\ge 1$ so that we can write $C_2(\lambda)=c_1\lambda^{-q_1}$ and $C_3(\lambda)=c_2\lambda^{q_2}$. Let $R_k=e^k$, and $\lambda_k=k^{q_3}$ where $q_3\ge 2$ is chosen large enough so that $\sum p(\lambda_k)<\infty$. Let $\varepsilon_k=c_2^{-1}\lambda_k^{-q_2}k^{-q_3}$. Set

$$F_k=\{R_k,\varepsilon_kR_k,\varepsilon_kR_k/4\lambda_k\in J(\lambda_k)\}.$$

Write $\xi(x,R)=1_{\{T_x>\tau_R\}}$. If $R\in J(\lambda)$ and $\varepsilon<1/2\lambda$ then by Lemma 2.3,

$$P_\omega^0(\xi(x,R)=1)\le\varepsilon\lambda,\qquad \text{for } x\in B(\varepsilon R).$$

Set

$$Z_k=V(\varepsilon_kR_k)^{-1}\sum_{x\in B(\varepsilon_kR_k)}\xi(x,R_k)\mu_x;$$

this is the proportion of points in $B(\varepsilon_kR_k)$ which are not hit by time τ_{R_k}. Then if $\omega\in F_k$,

$$P_\omega^0(Z_k\ge\tfrac12)\le 2E_\omega^0Z_k\le 2\varepsilon_k\lambda_k\le k^{-q_3}.$$

Let $m(k) = k^{q_3}\lambda_k R_k^3$. Then if $\omega \in F_k$, by (2.2),

$$P_\omega^0(\tau_{R_k} \geq m(k)) \leq 2\lambda_k R_k^3 m(k)^{-1} = 2k^{-q_3}.$$

Thus

$$P^*(F_k^c \cup \{Z_k \geq \tfrac{1}{2}\} \cup \{\tau_{R_k} \geq m(k)\}) \leq 3p(\lambda_k) + 3k^{-q_3},$$

so by Borel–Cantelli, P^*-a.s. there exists a $k_0(\omega,\overline{\omega}) < \infty$ such that, for all $k \geq k_0$, F_k holds, $\tau_{R_k} \leq m(k)$, and $Z_k \leq 1/2$. So, for $k \geq k_0$,

$$S_{m(k)} \geq S_{\tau_{R_k}} = \sum_{x \in B(\varepsilon_k R_k)} (1 - \xi(x, R_k))\mu_x = V(\varepsilon_k R_k)(1 - Z_k) \geq \tfrac{1}{2}\lambda_k^{-1}(\varepsilon_k R_k)^2.$$

Let n be large enough so that $m(k) \leq n < m(k+1)$ for some $k \geq k_0$. Then

$$\frac{\log S_n}{\log n} \geq \frac{\log S_{m(k)}}{\log m(k+1)} \geq \frac{2k - c\log k}{3(k+1) + c'\log(k+1)},$$

and the lower bound in (1.23) follows. This proves (1.23) when $x = 0$.

Now let

$$\Omega_0 = \{\omega : G(\omega) \text{ is recurrent and } P_\omega^0(\lim_n(\log S_n/\log n) = \tfrac{2}{3}) = 1\}.$$

We have $\mathbb{P}(\Omega_0) = 1$. If $\omega \in \Omega_0$, and $x \in G(\omega)$ then X hits 0 with P_ω^x–probability 1. Since the limit does not depend on the initial segment $X_0, \ldots, X_{T_0}$, we obtain (1.23).

(b) We have $|W_n| \leq S_n \leq c_0|W_n|$, so (1.24) is immediate from (1.23). □

Remark. Note that the constants c_i in Theorem 1.4 and α_i in Theorem 1.5 depend only on the constants c_1, c_2, c_3, q_0 in Assumption 1.2.

3 Verification of Assumption 1.2 for the IIC

In Section 3.1, we state three propositions which give estimates for the volume and effective resistance for the IIC. Propositions 3.1–3.2, which pertain to the volume growth of $\mathcal{C}$, are proved in Section 4. Proposition 3.3, which will be used to estimate the effective resistance, is proved in Section 5. In Section 3.2, we use the three propositions to verify Assumption 1.2 for the IIC, and complete the proof of our main result Theorem 1.7.

3.1 Three propositions

We will use the following notation for the IIC. Let $U(R) = \{(x,n) : n \geq R\}$, $B(R) = \{(x,n) \in \mathcal{C} : 0 \leq n < R\}$, and $\partial B(R) = \{(x,R) : (x,R) \in \mathcal{C}\}$. We note that, using the graph distance d on $\mathcal{C}$, $B(R)$ is just the ball $B(\mathbf{0}, R)$, and $\partial B(R)$ is its exterior boundary. Let

$$Z_R = b_0 R^{-2} V(R), \tag{3.1}$$

where b_0 is a constant that will be specified below (4.25). The constant b_0 has limit $\frac{1}{2}$ as $L \to \infty$.

Proposition 3.1. *Let $d > 4$ and $L \geq L_0$. Under the* IIC *measure, the random variables Z_R converge in distribution to a strictly positive limit Z, whose distribution is independent of d and L. Also, all moments converge, i.e., $\mathbb{E}_\infty Z_R^l \to \mathbb{E}Z^l$ for each $l \in \mathbb{N}$. In particular,*

$$c_1(d)R^2 \leq \mathbb{E}_\infty V(R) \leq c_2(d)R^2, \quad R \geq 1.$$

Moreover, c_1 and c_2 do not depend on d, if we further require that $L \geq L_1$, for some $L_1 = L_1(d)$.

Remark. We do not need the full strength of Proposition 3.1 to establish Assumption 1.2 for the IIC. However, since the scaling limit of $V(R)$ is also of independent interest, we will prove the stronger result, and, moreover, identify the limiting random variable Z in terms of super-Brownian motion.

Proposition 3.2. *Let $d > 4$ and $L \geq L_0$.*

$$\mathbb{Q}_\infty(V(R)R^{-2} < \lambda) \leq c_1(d) \exp\{-c_2(d)\lambda^{-1/2}\}, \quad R \geq 1. \tag{3.2}$$

Moreover, c_1 and c_2 do not depend on d, if we further require that $L \geq L_1$, for some $L_1 = L_1(d)$.

The third proposition gives an estimate on the expected number of edges at level $n-1$ that need to be cut in order to disconnect 0 from level R. We say that $(x, n), (x', n') \in \mathcal{C}$ are *RW-connected*, if there is a path, not necessarily oriented, in $\mathcal{C}$ from (x, n) to (x', n'). We reserve the term *connected* to mean oriented connection, that is $(x, n) \longrightarrow (x', n')$. Let

$$D(n) = \left\{ e = ((w, n-1), (x, n)) \subset \mathcal{C} : \begin{array}{l} (x,n) \text{ is RW-connected to} \\ \text{level } R \text{ by a path in } \mathcal{C} \cap U(n) \end{array} \right\}, \quad 0 < n \leq R. \tag{3.3}$$

It follows from the definition that all edges in $D(n)$ need to be cut in order to RW-disconnect 0 from level R. Also, cutting all the edges in $D(n)$ RW-disconnects 0 from $B(R)^c$, since for any RW-path from 0 to $B(R)^c$ the last crossing of level n occurs at an edge in $D(n)$.

Proposition 3.3. *Let $d > 6$. There exists $L_1 = L_1(d) \geq L_0(d)$ such that for $L \geq L_1$, $R \geq 1$ and $0 < a < 1$,*

$$\mathbb{E}_\infty(|D(n)|) \leq c_1(a), \qquad 0 < n \leq \lfloor aR \rfloor. \tag{3.4}$$

The constant $c_1(a)$ is independent of the dimension d and also of L.

Remark. Proposition 3.3 is the only place where we need $d > 6$ rather than $d > 4$.

3.2 Verification of Assumption 1.2 for the IIC

We begin with a lemma that relates $|D(n)|$ and the effective resistance.

Lemma 3.4. *For oriented percolation in any dimension $d \geq 1$,*

$$R_{\text{eff}}(0, \partial B(R)) \geq \sum_{n=1}^{R} \frac{1}{|D(n)|}. \tag{3.5}$$

Proof. We have that $R_{\text{eff}}(0, \partial B(R))$ is the minimum energy dissipation of a unit current from $\mathbf{0}$ to $\partial B(R)$ – see [15, p. 63]. Let I be such a unit current. Fix $1 \le n \le R$, let $k = |D(n)|$, and let $J_1, \dots J_k$ be the currents in the bonds in $D(n)$. Then since all the current must flow through the edges in $D(n)$, we have $\sum_{i=1}^k |J_i| \ge 1$. Hence the energy dissipation for I in the bonds in $D(n)$, which is $\sum_{i=1}^k |J_i|^2$, is greater than $1/k = |D(n)|^{-1}$. Summing then gives (3.5). ☐

Now we combine Proposition 3.3 and Lemma 3.4 to show that it is unlikely that the effective resistance $R_{\text{eff}}(0, \partial B(R))$ is less than a small multiple of R.

Proposition 3.5. *There is a constant c such that for $d > 6$, $L \ge L_1$, $R \ge 2$ and $\epsilon > 0$,*

$$\mathbb{Q}_\infty(R_{\text{eff}}(0, \partial B(R)) \le \varepsilon R) \le c\varepsilon. \tag{3.6}$$

Proof. Let $R \ge 2$. Fix $\frac{1}{2} < a < 1$ and let $r = \lfloor aR \rfloor$; note that $r \ge 1$. By Lemma 3.4 and the Cauchy–Schwarz inequality,

$$R_{\text{eff}}(0, \partial B(R))^{-1} \le \Big(\sum_{n=1}^r |D(n)|^{-1} \Big)^{-1} \le r^{-2} \sum_{n=1}^r |D(n)|. \tag{3.7}$$

Therefore, by Proposition 3.3, Markov's inequality and (3.7)

$$\begin{aligned}
\mathbb{Q}_\infty(R_{\text{eff}}(0, \partial B(R)) \le \varepsilon R) &= \mathbb{Q}_\infty(R_{\text{eff}}(0, \partial B(R))^{-1} \ge \varepsilon^{-1} R^{-1}) \\
&\le \varepsilon R \mathbb{E}_\infty \Big(R_{\text{eff}}(0, \partial B(R))^{-1} \Big) \\
&\le \varepsilon R r^{-2} \mathbb{E}_\infty (\sum_{n=1}^r |D(n)|) \le \varepsilon R r^{-1} c_1(a) \le 2a^{-1} c_1(a) \varepsilon.
\end{aligned}$$

☐

Proof of Theorem 1.7. Let $W_R = V(R)/R^2$. By Proposition 3.1 we have (2) and

$$\mathbb{Q}_\infty(W_R \ge \lambda) \le \lambda^{-1} \mathbb{E}_\infty W_R \le c\lambda^{-1}. \tag{3.8}$$

Also, Proposition 3.2 gives

$$\mathbb{Q}_\infty(W_R < \lambda^{-1}) \le c \exp(-c' \lambda^{1/2}), \tag{3.9}$$

and (3) is then immediate after integration. The combination of (3.8)–(3.9) and (3.6) (with $\varepsilon = \lambda^{-1}$), together with the fact that each of the bounds is less than $c\lambda^{-1}$ for large λ, gives (1) with $q_0 = 1$ and $R^* = 2$. The fact that all constants here are independent of d, L implies that the constants in Assumption 1.2 share this independence. ☐

4 IIC volume estimates: Proof of Propositions 3.1–3.2

In Section 4.2 we prove Proposition 3.1, and in Section 4.3 we prove Proposition 3.2. The proofs make use of results from several previous papers involving the lace expansion; these results are gathered together and slightly extended in Section 4.1.

We assume throughout that $d > 4$ and that L is large; these assumptions will often not be mentioned explicitly in the following. Throughout:

$\beta = L^{-d}$, *K denotes a constant that only depends on d, and $\bar{K}$ denotes an absolute constant.*

The values of the constants K and $\bar{K}$ may change from one occurrence to the next.

4.1 Preliminaries

In this section, we recall and slightly extend various results from [20, 21, 25, 26]. These results isolate the necessary ingredients from other papers that will be used in the proof of Propositions 3.1–3.2.

4.1.1 Critical oriented percolation r-point functions

The critical oriented percolation two-point function $\tau_n(x)$ is defined by

$$\tau_n(x) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x,n)). \tag{4.1}$$

Let $\tau_n = \sum_{x\in\mathbb{Z}^d} \tau_n(x)$. By [26, Theorem 1.1],

$$\sup_{x\in\mathbb{Z}^d} \tau_n(x) \le K\beta(n+1)^{-d/2}, \quad n \ge 1, \tag{4.2}$$

$$\tau_n = A(1+\mathcal{O}(n^{(4-d)/2})), \quad \text{as } n \to \infty, \tag{4.3}$$

where $|A-1| \le K\beta$. The estimate [25, (4.2)] shows that the error term in (4.3) is bounded by $K\beta n^{(4-d)/2}$ (note that $f_n(0,z_c)$ of [25] corresponds to our τ_n). Hence for $L \ge L_1 = L_1(d)$, we have

$$\bar{K}^{-1} \le A \le \bar{K}, \qquad |\tau_n - A| \le \bar{K} n^{(4-d)/2}, \quad n \ge 1, \qquad \bar{K}^{-1} \le \tau_n \le \bar{K}, \quad n \ge 0. \tag{4.4}$$

Also, noting that τ_1 is called p_c in [26], we see from [26, Eqn. (1.12)] that $|\tau_1 - 1| \le K\beta \le \bar{K}$ for $L \ge L_1(d)$ sufficiently large.

For all $r \ge 2$, the critical oriented percolation r-point function $\tau_n^{(r)}(x)$ is defined by

$$\tau^{(r)}_{n_1,\ldots,n_{r-1}}(x_1,\ldots,x_{r-1}) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x_i,n_i) \text{ for all } i = 1,\ldots,r-1), \tag{4.5}$$

with $x_i \in \mathbb{Z}^d$, $n_i \in \mathbb{Z}_+$. The asymptotic behaviour of the Fourier transforms of the r-point functions is given in [26, Theorem 1.2]. A very special case of [26, Theorem 1.2] is that there is a $\delta > 0$ such that for $t_1, t_2 > 0$,

$$\sum_{x_1,x_2\in\mathbb{Z}^d} \tau^{(3)}_{\lfloor nt_1\rfloor,\lfloor nt_2\rfloor}(x_1,x_2) = nV^*A^3\left[t_1 \wedge t_2 + O(n^{-\delta})\right] \tag{4.6}$$

as $n \to \infty$ (see [26, (1.22)]). The *vertex factor* V^* is written V in [26] but written V^* here to avoid confusion with the volume. The vertex factor is a constant with $|V^*-1| \le K\beta$, and we assume that L_1 has been chosen so that $\bar{K}^{-1} \le V^* \le \bar{K}$.

4.1.2 The IIC r-point functions

Let $\vec{y} = (y_1,\ldots,y_{r-1})$ and $\vec{m} = (m_1,\ldots,m_{r-1})$ with $y_i \in \mathbb{Z}^d$, $m_i \in \mathbb{Z}_+$. For $r \ge 2$, the IIC r-point function is defined by

$$\rho^{(r)}_{\vec{m}}(\vec{y}) = \mathbb{Q}_\infty((0,0) \longrightarrow (y_i,m_i) \text{ for all } i = 1,\ldots,r-1). \tag{4.7}$$

Let

$$\hat{\rho}^{(r)}_{\vec{m}} = \sum_{y_1,\ldots,y_{r-1}\in\mathbb{Z}^d} \rho^{(r)}_{\vec{m}}(\vec{y}). \tag{4.8}$$

Let A be the constant of (4.3), and let V^* be the vertex factor of (4.6). Let $r \geq 2$, $\vec{t} = (t_1, \ldots, t_{r-1}) \in (0,1]^{r-1}$, and for a positive integer m, let $m\vec{t}$ be the vector with components $\lfloor mt_i \rfloor$. It is immediate from [21, (5.15)] (with $\vec{k} = \vec{0}$) that for $r \geq 2$,

$$\lim_{m\to\infty} \frac{1}{(mA^2V^*)^{r-1}} \hat{\rho}^{(r)}_{m\vec{t}} = \hat{M}^{(r)}_{1,\vec{t}}, \tag{4.9}$$

where the limit $\hat{M}^{(r)}_{1,\vec{t}}$ is defined recursively as follows (see [21, Section 4.2]).

For $r = 1$, we have simply

$$\hat{M}^{(1)}_s = 1. \tag{4.10}$$

For $r > 2$ and $\bar{s} = (s_1, \ldots, s_r)$ with each $s_i > 0$, the $\hat{M}^{(r)}_{\bar{s}}$ are given recursively by

$$\hat{M}^{(r)}_{\bar{s}} = \int_0^{\underline{s}} ds\, \hat{M}^{(1)}_s \sum_{I\subset J_1:|I|\geq 1} \hat{M}^{(i)}_{\bar{s}_I - s} \hat{M}^{(r-i)}_{\bar{s}_{J\setminus I} - s}, \tag{4.11}$$

where $i = |I|$, $J = \{1, \ldots, l\}, J_1 = J\backslash\{1\}$, $\underline{s} = \min_i s_i$, $\vec{s}_I$ denotes the vector consisting of the components s_i of $\vec{s}$ with $i \in I$, and $\vec{s}_I - s$ denotes subtraction of s from each component of $\vec{s}_I$. The explicit solution to the recursive formula (4.11) can be found, e.g., in [26, (1.25)]. In particular, $\hat{M}^{(2)}_{s_1,s_2} = s_1 \wedge s_2$. It is shown in [21, Lemma 4.2] that for $r \geq 1$ and $t > 0$,

$$\hat{M}^{(r)}_{t,\ldots,t} = t^{r-1} 2^{-(r-1)} r!. \tag{4.12}$$

To this we add the following elementary fact.

Lemma 4.1. *For $r \geq 1$, $\hat{M}^{(r)}_{s_1,\ldots,s_r}$ is nondecreasing in each s_i.*

Proof. The proof is by induction on r. For $r = 1$, $\hat{M}^{(1)}_{s_1} = 1$ by (4.10), which is nondecreasing. Assume the result holds for all $j \leq r$. Then it holds also for $r + 1$ by (4.11), since increasing an s_i can only increase the integrand (by the induction hypothesis) or the domain of integration in (4.11). □

4.1.3 Super-Brownian motion

As discussed in [21, Section 4], the quantity $\hat{M}^{(r)}_{\bar{s}}$ appearing in (4.9) is the r^{th} moment of the canonical measure $\mathbb{N}$ of super-Brownian motion X_t, namely

$$\hat{M}^{(r)}_{s_1,\ldots,s_r} = \mathbb{N}\Big(X_{s_1}(\mathbb{R}^d) \cdots X_{s_r}(\mathbb{R}^d)\Big). \tag{4.13}$$

For an introduction to the canonical measure, see [36, Chapter 17].

Let Y_t denote the canonical measure of super-Brownian motion conditioned to survive for all time (see [20]). Let

$$Z = \int_0^1 dt\, Y_t(\mathbb{R}^d), \tag{4.14}$$

so that Z is a positive random variable. It is clear that the distribution of Z does not depend on L. It also does not depend on d, since it is equal to the mass up to time 1 of the continuum random tree conditioned to survive forever. The moments of Z are given, for integers $l \geq 1$, by

$$\mathbb{E}Z^l = \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,\vec{t}} \tag{4.15}$$

(see [20, Section 3.4]). We will use the fact that Z has an exponential moment. This follows from

$$\mathbb{E}Z^l \leq \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,1,\dots,1} = 2^{-l}(l+1)!, \tag{4.16}$$

where we have used (4.15), Lemma 4.1 and (4.12).

4.1.4 Rate of convergence to the IIC

For the proof of Proposition 3.2, we will need an estimate for the rate of convergence of $\mathbb{P}_n$ to $\mathbb{P}_\infty$ (recall the definitions from (1.27)–(1.28)). Let $\mathcal{E}_m$ denote the set of cylinder events measurable with respect to the set of edges up to level $m-1$. In [21, Eqn. (2.19)], the following representation was obtained for $\mathbb{P}_n(E)$, $E \in \mathcal{E}_m$:

$$\mathbb{P}_n(E) = \frac{1}{\tau_n}\left[\sum_{l=m}^{n-1} \varphi_l(E)\tau_1\tau_{n-l-1} + \varphi_n(E)\right], \tag{4.17}$$

where $\varphi_l(E)$ is a function arising in the lace expansion. The factor τ_1 was called p_c in [21]. By [21, Lemma 2.2], φ_l satisfies

$$|\varphi_l(E)| \leq K\beta m(l-m+1)^{-d/2}, \qquad l \geq m+1. \tag{4.18}$$

However, a very slight modification of the proof of [21, Lemma 2.2] actually shows that

$$|\varphi_l(E)| \leq K\beta(l-m+1)^{(2-d)/2}, \quad l \geq m \geq 1 \tag{4.19}$$

(replace the upper bound $Km(l-m+1)^{-d/2}$ on $\sum_{a=0}^{m-1}(l-a)^{-d/2}$ used in [21, (2.33),(2.35)] by the more careful upper bound $K(l-m+1)^{(2-d)/2}$), and we will use this variant. The IIC measure is given in [21, Eqn. (2.29)] as

$$\mathbb{P}(E) = \sum_{l=m}^{\infty} \tau_1\varphi_l(E), \qquad E \in \mathcal{E}_m. \tag{4.20}$$

The following lemma bounds the rate at which the measure $\mathbb{P}_{2m}$ converges to $\mathbb{P}_\infty$.

Lemma 4.2. *Let $d > 4$. For $E \in \mathcal{E}_m$,*

$$|\mathbb{P}_{2m}(E) - \mathbb{P}_\infty(E)| = \mathcal{O}((m+1)^{(4-d)/2}) \tag{4.21}$$

where the constant in the error term is uniform in E and $L \geq L_0$. The error term can be guaranteed to be uniform in d as well, by further requiring that $L \geq L_1$ for some $L_1 = L_1(d)$.

Proof. By the triangle inequality,

$$|\mathbb{P}_{2m}(E) - \mathbb{P}_\infty(E)| \leq \left|\mathbb{P}_{2m}(E) - \sum_{l=m}^{2m}\tau_1\varphi_l(E)\right| + \left|\mathbb{P}_\infty(E) - \sum_{l=m}^{2m}\tau_1\varphi_l(E)\right|. \tag{4.22}$$

For the second term on the right-hand side, we use (4.20) and (4.19) to obtain

$$\left|\mathbb{P}_\infty(E) - \sum_{l=m}^{2m}\tau_1\varphi_l(E)\right| \leq \sum_{l=2m+1}^{\infty}\tau_1|\varphi_l(E)| \leq K\beta \sum_{l=2m+1}^{\infty}(l-m+1)^{(2-d)/2} \leq K\beta m^{(4-d)/2}. \tag{4.23}$$

For the first term on the right-hand side of (4.22), we use (4.17) to obtain

$$\left|\mathbb{P}_{2m}(E) - \sum_{l=m}^{2m} \tau_1 \varphi_l(E)\right| \leq \sum_{l=m}^{2m-1} \tau_1 |\varphi_l(E)| \left|\frac{\tau_{2m-l-1}}{\tau_{2m}} - 1\right| + |\varphi_{2m}(E)| \left|\frac{1}{\tau_{2m}} - \tau_1\right|. \tag{4.24}$$

By (4.19), the last term is bounded by $K\beta m^{(2-d)/2}$. To bound the sum, we split it into the cases $m \leq l < 3m/2$ and $3m/2 \leq l \leq 2m-1$. In the first case, we use (4.3) to obtain $|(\tau_{2m-l-1}/\tau_{2m})-1| \leq K\beta m^{(4-d)/2}$. Then inserting the bound (4.19) and summing over l, we obtain a bound $K\beta m^{(4-d)/2}$ for the first case. In the second case, we bound $|\tau_{2m-l-1}/\tau_{2m} - 1| \leq K$. Inserting the bound on φ_l, and summing over l, we obtain a bound $K\beta m^{(4-d)/2}$ for the second case. Thus, in either case, (4.24) is bounded by $K\beta m^{(4-d)/2}$. For $L \geq L_1$ this bound is at most $\bar{K} m^{(4-d)/2}$. With (4.22)–(4.23), this proves (4.21). □

4.2 Volume convergence: Proof of Proposition 3.1

In this section, we prove Proposition 3.1. We now choose $b_0 = (2\tau_1 A^2 V^* R^2)^{-1}$ in (3.1), so that Z_R is defined by

$$Z_R = (2\tau_1 A^2 V^* R^2)^{-1} V(R). \tag{4.25}$$

As pointed out in Section 4.1, the constants τ_1, A, V^* all have limit 1 as $L \to \infty$. Let

$$\tilde{Z}_R = (A^2 V^* R^2)^{-1} |B(R)|. \tag{4.26}$$

Thus $\tilde{Z}_R$ is defined in terms of the vertices in $B(R)$, whereas Z_R is defined in terms of the edges. Recall the random variable Z defined in (4.14). We use (4.9) to prove that $\lim_{R\to\infty} \mathbb{E}\tilde{Z}_R^l = \mathbb{E}Z^l$ for all $l \geq 1$, and then adapt this to Z_R.

Let $l \geq 1$. By definition,

$$\begin{aligned}
\mathbb{E}\tilde{Z}_R^l &= \frac{1}{(A^2 V^* R^2)^l} \sum_{n_1=0}^{R-1} \cdots \sum_{n_l=0}^{R-1} \sum_{x_1 \in \mathbb{Z}^d} \cdots \sum_{x_l \in \mathbb{Z}^d} \rho^{(l+1)}_{n_1,\ldots,n_l}(x_1, \ldots, x_l) \\
&= \frac{1}{R} \sum_{n_1=0}^{R-1} \cdots \frac{1}{R} \sum_{n_l=0}^{R-1} \frac{1}{(A^2 V^* R)^l} \hat{\rho}^{(l+1)}_{\vec{t}R},
\end{aligned} \tag{4.27}$$

where $\vec{t} = (n_1 R^{-1}, \ldots, n_l R^{-1})$. The summand on the right hand side is bounded by a constant, by standard tree-graph inequalities [1] (see [21, Section 5.1] for the details when $l = 1$). Therefore, by (4.9), the dominated convergence theorem, and (4.15),

$$\lim_{R\to\infty} \mathbb{E}\tilde{Z}_R^l = \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,\vec{t}} = \mathbb{E}Z^l. \tag{4.28}$$

The next lemma implies that it is also the case that $\lim_{R\to\infty} \mathbb{E}Z_R^l = \mathbb{E}Z^l$ for all $l \geq 1$.

Lemma 4.3. *For all $l \geq 1$ and $R \geq 3$,*

$$(1 - 2/R)^{2l} \mathbb{E}\tilde{Z}_{R-2}^l \leq \mathbb{E}Z_R^l \leq \mathbb{E}\tilde{Z}_{R-1}^l + c(d, L, l) R^{-1}. \tag{4.29}$$

Since Z was shown in (4.16) to has a moment generating function with radius of convergence at least 2, the convergence of moments established in Lemma 4.3 implies that Z_R converges weakly to Z (see [12, Theorem 30.2]).. Note that for $L \geq L_1$, the constants A, V^* and τ_1 satisfy bounds independent of d, hence c_1 and c_2 in Proposition 3.1 do not depend on d. This completes the proof of Proposition 3.1, subject to Lemma 4.3.

Proof of Lemma 4.3. For $l \geq 1$, we define

$$\sigma_{\vec{m}}^{(l+1)}(\vec{x}, \vec{y}) = \mathbb{Q}_\infty((0,0) \longrightarrow (x_i, m_i) \longrightarrow (y_i, m_i+1) \text{ for all } i = 1, \ldots, l).$$

Note that

$$2|\text{edges in } B(R-1)| \leq \sum_{(x,m)\in B(R)} \mu_{(x,m)} = V(R) \leq 2|\text{edges in } B(R)|, \tag{4.30}$$

since edges on the boundary of $B(R)$ are counted once in $V(R)$, while other edges are counted twice. Therefore

$$\mathbb{E}Z_R^l \geq \frac{1}{(\tau_1 A^2 V^* R^2)^l} \sum_{n_1=0}^{R-2} \cdots \sum_{n_l=0}^{R-2} \sum_{x_1, y_1 \in \mathbb{Z}^d} \cdots \sum_{x_l, y_l \in \mathbb{Z}^d} \sigma_{n_1, \ldots, n_l}^{(l+1)}(x_1, \ldots, x_l, y_1, \ldots, y_l), \tag{4.31}$$

with a corresponding upper bound if the summations over the n_i's extend to $R-1$.

Lower bound. The Harris–FKG inequality [16, 18] implies that for increasing events A and B we have $\mathbb{Q}_n(A \cap B) \geq \mathbb{Q}_n(A)\mathbb{P}(B)$. If A and B are cylinder events, then by passing to the limit, we have $\mathbb{Q}_\infty(A \cap B) \geq \mathbb{Q}_\infty(A)\mathbb{P}(B)$. Hence

$$\sigma_{\vec{n}}^{(l+1)}(\vec{x}, \vec{y}) \geq \rho_{\vec{n}}^{(l+1)}(\vec{x}) \prod_{i=1}^{l} \tau_1(y_i - x_i). \tag{4.32}$$

With (4.27), this gives $\mathbb{E}Z_R^l \geq [(R-2)/R]^{2l} \mathbb{E}\tilde{Z}_{R-2}^l$.

Upper bound. Let

$$A_{\vec{m}}(\vec{x}) = \{(0,0) \longrightarrow \infty,\, (0,0) \longrightarrow (x_i, m_i),\, i = 1, \ldots, l\}.$$

Let $F_{\vec{m}}(\vec{x}, \vec{y})$ denote the event that the following $l+1$ events occur on disjoint sets of edges:

$$A_{\vec{m}}(\vec{x}),\, \{(x_1, m_1) \longrightarrow (y_1, m_1+1)\}, \ldots, \{(x_l, m_l) \longrightarrow (y_l, m_l+1)\}. \tag{4.33}$$

Then

$$\sigma_{\vec{m}}^{(l+1)}(\vec{x}, \vec{y}) \leq \mathbb{Q}_\infty(F_{\vec{m}}(\vec{x}, \vec{y})) + \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}) \cap_{i=1}^l \{(x_i, m_i) \longrightarrow (y_i, m_i+1)\} \setminus F_{\vec{m}}(\vec{x}, \vec{y})). \tag{4.34}$$

The BK inequality implies that for increasing events A and B that depend on only finitely many edges we have $\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B)$, where $A \circ B$ denotes disjoint occurrence [8, 18]. We will bound the first term by passing to the limit in the BK inequality. Let

$$A_{\vec{m},n}(\vec{x}) = \{(0,0) \longrightarrow n,\, (0,0) \longrightarrow (x_i, m_i),\, i = 1, \ldots, l\},$$

and define $F_{\vec{m},n}(\vec{x},\vec{y})$ analogously, by replacing $A_{\vec{m}}(\vec{x})$ in (4.33) by $A_{\vec{m},n}(\vec{x})$. Then each event in the definition of $F_{\vec{m},n}(\vec{x},\vec{y})$ only depends on finitely many edges, hence by BK,

$$\mathbb{P}(F_{\vec{m},n}(\vec{x},\vec{y})) \leq \mathbb{P}(A_{\vec{m},n}(\vec{x})) \prod_{i=1}^{l} \tau_1(y_i - x_i).$$

Dividing both sides by $\mathbb{P}((0,0) \longrightarrow n)$ and letting $n \to \infty$, we get

$$\mathbb{Q}_\infty(F_{\vec{m}}(\vec{x},\vec{y})) \leq \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x})) \prod_{i=1}^{l} \tau_1(y_i - x_i) = \rho_{\vec{m}}^{(l+1)}(\vec{x}) \prod_{i=1}^{l} \tau_1(y_i - x_i). \tag{4.35}$$

The sum of this bound over $\vec{x}$ and $\vec{y}$ is $\hat{\rho}_{\vec{m}}^{(l+1)} \tau_1^l$. With (4.27), this gives a contribution $\mathbb{E}\tilde{Z}_{R-1}^l$ to the upper bound version of (4.31).

We claim that on the event $A_{\vec{m}}(\vec{x}) \cap_{i=1}^l \{(x_i,m_i) \longrightarrow (y_i, m_i+1)\} \setminus F_{\vec{m}}(\vec{x},\vec{y})$, there exists $1 \leq i \leq l$ such that either $(x_i,m_i) \longrightarrow (x_j,m_j)$ for some $j \neq i$, or $(x_i,m_i) \longrightarrow \infty$. To see this, we may assume that all the (x_i,m_i)'s are different, otherwise there is nothing to prove. Under this assumption, the last l events in (4.33) occur disjointly. As in a tree-graph bound [1], choose a set of disjoint paths showing that $A_{\vec{m}}(\vec{x})$ occurs. Then at least one of the paths uses an edge $((x_i,m_i),(y_i,m_i+1)$, otherwise $F_{\vec{m}}(\vec{x},\vec{y})$ would occur. This path includes a connection $(x_i,m_i) \longrightarrow (x_j,m_j)$ or $(x_i,m_i) \longrightarrow \infty$, proving the claim.

By the claim, the second term on the right hand side of (4.34) is at most

$$\sum_{1\leq i\leq l} \left[\sum_{j\neq i} \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}),\, (x_i,m_i) \longrightarrow (x_j,m_j)) + \mathbb{Q}_\infty(A_{\vec{m}}(\vec{x}),\, (x_i,m_i) \longrightarrow \infty) \right]. \tag{4.36}$$

Each term in (4.36) can be bounded using a tree-graph inequality where the number of internal vertices in the tree-graph bound is $l-1$, one less than it would be for $\rho^{(l+1)}$. This implies that the sum of (4.36) over $\vec{x}$ and $\vec{y}$ inside $B(R)$ is bounded by $c(d,L,l)R^{l-1}$. It follows that

$$\mathbb{E}Z_R^l \leq \mathbb{E}\tilde{Z}_{R-1}^l + c(d,L,l)R^{-1},$$

which gives the desired upper bound and completes the proof of (4.29). $\square$

4.3 Volume estimate: Proof of Proposition 3.2

In this section, we prove Proposition 3.2. Recall the definitions of $\mathbb{P}_n$ and $\mathbb{P}_\infty$ from (1.27)–(1.28). It is enough to show that we can find constants $R_0(d), c_1(d), c_2(d), c_3(d)$ such that for $R \geq R_0$ and $\lambda \leq c_3$ we have

$$\mathbb{P}_\infty(V(R)R^{-2} < \lambda) \leq c_1 \exp\{-c_2 \lambda^{-1/2}\}. \tag{4.37}$$

Indeed, the restrictions on λ and R can be removed by adjusting the constant c_1 as follows. First, for $\lambda > c_3$, if $c_1 > \exp\{c_2(c_3)^{-1/2}\}$, the right hand side of (4.37) is larger than 1. As for $R < R_0$, due to the (deterministic) inequality $V(R) \geq R$, we have $V(R)R^{-2} \geq R\,R^{-2} > R_0^{-1}$. Therefore, if $\lambda < R_0^{-1}$, the left hand side of (4.37) is 0. For $\lambda \geq R_0^{-1}$, it is enough to require that $c_1 > \exp\{c_2 R_0^{1/2}\}$. Finally, note that if initially R_0, c_1, c_2, c_3 are independent of d, then so is the adjusted c_1.

We begin with a simple consequence of Proposition 3.1.

Corollary 4.4. *Given $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, d)$, such that*

$$\mathbb{Q}_\infty(V(R)R^{-2} < \lambda_0) < \varepsilon, \qquad R \geq 1. \tag{4.38}$$

For $L \geq L_1$, λ_0 can be chosen independent of d.

Proof. This follows from Proposition 3.1 and the fact that Z is strictly positive. □

Let $c = c(d) = \sup_{m \geq 1} \tau_m$. According to (4.38), there is a constant $c_3 = c_3(d)$ such that

$$\mathbb{P}_\infty(V(R) < 4c_3(R+1)^2) < \frac{1}{3c}, \qquad R \geq 1. \tag{4.39}$$

We fix $m_0 = m_0(d)$ such that for $m \geq m_0$ the error term on the right-hand side of (4.21) is at most $(3c)^{-1}$. Let $R_0 = 16c_3 m_0^2$. Fix $\lambda \leq c_3$ and $R \geq R_0$. We will prove that (4.37) holds for λ and R with the choice of c_3 made and with $c_1 = 1$ and $c_2 = \frac{1}{2}\log(3/2)c_3^{1/2}$.

There is nothing to prove if $\lambda < R_0/R^2$, since, in this case

$$\mathbb{P}_\infty(V(R)R^{-2} < \lambda) \leq \mathbb{P}_\infty(V(R) < R_0) \leq \mathbb{P}_\infty(V(R) < R) = 0 \tag{4.40}$$

and (4.37) holds trivially. Hence, without loss of generality, we assume that

$$\frac{16c_3 m_0^2}{R^2} = \frac{R_0}{R^2} \leq \lambda \leq c_3. \tag{4.41}$$

To estimate $\mathbb{P}_\infty(V(R) < \lambda R^2)$, we subdivide the time interval $[0, R]$ into blocks that provide roughly independent contributions to the volume, and apply (4.39) in each block. The number of blocks is $S = \lfloor (c_3/\lambda)^{1/2} \rfloor$, which is at least 1 by (4.41). The length of a block is $2m$, with $m = \lfloor R/2S \rfloor$. Note that $m \geq m_0$, since

$$\frac{R}{2S} \geq \frac{R}{2(c_3/\lambda)^{1/2}} \geq \frac{R_0^{1/2}}{2c_3^{1/2}} = 2m_0 > 1, \tag{4.42}$$

and hence

$$m = \left\lfloor \frac{R}{2S} \right\rfloor \geq \frac{R}{4S} \geq \frac{R}{4(c_3/\lambda)^{1/2}} \geq m_0. \tag{4.43}$$

Set $n_i = i(2m)$, $i = 0, \ldots, S$, so that the i-th block starts at level n_{i-1} and ends at level n_i.

By (1.28),

$$\mathbb{P}_\infty(V(R) < \lambda R^2) = \lim_{N \to \infty} \frac{1}{\tau_N} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}(V(R) < \lambda R^2, (0,0) \longrightarrow (x, N)). \tag{4.44}$$

The path $(0,0) \longrightarrow (x, N)$ on the right-hand side passes through the levels $n_1, \ldots, n_S$, and hence there exist $0 = x_0, x_1, \ldots, x_S \in \mathbb{Z}^d$ such that

$$(0,0) \longrightarrow (x_1, n_1) \longrightarrow \cdots \longrightarrow (x_S, n_S) \longrightarrow (x, N).$$

We write $\boldsymbol{x}_i = (x_i, n_i)$ for $i = 0, \ldots, S$, and write $\boldsymbol{x} = (x, N)$. It follows that

$$\begin{aligned} &\mathbb{P}_{p_c}(V(R) < \lambda R^2,\, (0,0) \longrightarrow (x, N)) \\ &\quad = \mathbb{P}_{p_c}\Bigg(\bigcup_{x_1,\ldots,x_S \in \mathbb{Z}^d} \{V(R) < \lambda R^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i,\, i = 1, \ldots, S\} \cap \{\boldsymbol{x}_S \longrightarrow \boldsymbol{x}\} \Bigg) \\ &\quad \le \sum_{x_1,\ldots,x_S \in \mathbb{Z}^d} \mathbb{P}_{p_c}(V(R) < \lambda R^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i,\, i = 1, \ldots, S,\, \boldsymbol{x}_S \longrightarrow \boldsymbol{x}). \end{aligned} \tag{4.45}$$

Let

$$\mathcal{C}(\boldsymbol{y}; n) = C(\boldsymbol{y}) \cap (\mathbb{Z}^d \times \{0, 1, \ldots, n\}). \tag{4.46}$$

On the event on the right-hand side of (4.45), $\boldsymbol{x}_{i-1}$ is contained in $B(R)$, and hence $\mathcal{C}(\boldsymbol{x}_{i-1}; n_{i-1} + m) \subset B(R)$. Denote $V_i = \mu(\mathcal{C}(\boldsymbol{x}_{i-1}; n_{i-1} + m))$. Then on the event in the right-hand side of (4.45), since $\lambda \le c_3/S^2$ by the choice of S, we have

$$V_i \le V(R) < \lambda R^2 \le \frac{c_3}{S^2} R^2 = 4c_3 \left(\frac{R}{2S}\right)^2 \le 4c_3 (m+1)^2. \tag{4.47}$$

Hence, the right-hand side of (4.45) is at most

$$\sum_{x_1,\ldots,x_S \in \mathbb{Z}^d} \mathbb{P}_{p_c}\left(\bigcap_{i=1}^{S} \{V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i\} \cap \{\boldsymbol{x}_S \longrightarrow \boldsymbol{x}\} \right). \tag{4.48}$$

The $S+1$ events in (4.48) depend on disjoint sets of bonds, so the probability factors as

$$\sum_{x_1,\ldots,x_S \in \mathbb{Z}^d} \mathbb{P}_{p_c}(\boldsymbol{x}_S \longrightarrow \boldsymbol{x}) \prod_{i=1}^{S} \mathbb{P}_{p_c}(V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i). \tag{4.49}$$

We insert this into (4.45), and use (4.44), (4.3) and (1.27) to obtain

$$\begin{aligned} \mathbb{P}_\infty(V(R) < \lambda R^2) &\le \prod_{i=1}^{S} \left(\sum_{x_i \in \mathbb{Z}^d} \mathbb{P}_{p_c}(V_i < 4c_3(m+1)^2,\, \boldsymbol{x}_{i-1} \longrightarrow \boldsymbol{x}_i) \right) \limsup_{N\to\infty} \frac{\tau_{N-n_S}}{\tau_N} \\ &= \Big[\tau_{2m} \mathbb{P}_{2m}(V(m) < 4c_3(m+1)^2) \Big]^S. \end{aligned} \tag{4.50}$$

By Lemma 4.2, the right-hand side equals

$$\tau_{2m}^S \Big[\mathbb{P}_\infty(V(m) < 4c_3(m+1)^2) + \mathcal{O}((m+1)^{(4-d)/2}) \Big]^S. \tag{4.51}$$

By the choice of m_0 and (4.39), both terms inside the square brackets are at most $(3c)^{-1}$. Since

$$S = \Big\lfloor (c_3/\lambda)^{1/2} \Big\rfloor \ge \frac{1}{2} (c_3/\lambda)^{1/2},$$

it follows from our choice of c that

$$\mathbb{P}_\infty(V(R) < \lambda R^2) \le \tau_{2m}^S \left(\frac{2}{3c}\right)^S \le \left(\frac{2}{3}\right)^S \le \exp\{-\tfrac{1}{2}\log(3/2) c_3^{1/2} \lambda^{-1/2}\}. \tag{4.52}$$

The choice $c_2 = \frac{1}{2}\log(3/2) c_3^{1/2}$ gives (4.37). Noting that for $L \ge L_1$, c, c_3 and m_0 (and hence all further constants chosen) are independent of d, this completes the proof of Proposition 3.2.

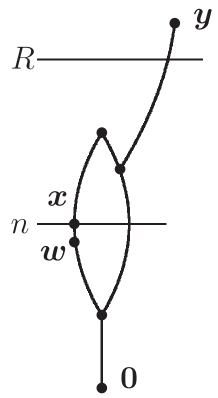

Figure 2: The configuration bounded in (5.4). The vertices $\boldsymbol{w} = (w, n-1)$, $\boldsymbol{x} = (x, n)$, $\boldsymbol{y} = (y, N)$ are summed over $w, x, y \in \mathbb{Z}^d$, and the three unlabelled vertices are summed over space and time.

5 IIC resistance estimates: Proof of Proposition 3.3

In this section we prove Proposition 3.3. Throughout, we use $\boldsymbol{x}, \boldsymbol{y}, \ldots$ to denote space-time vertices in $\mathbb{Z}^d \times \mathbb{Z}_+$, we denote the spatial component of a vertex $\boldsymbol{x}$ by x, and we write $|\boldsymbol{x}| = n$ when $\boldsymbol{x} = (x, n)$. According to (3.3),

$$D(n) = \left\{ e = (\boldsymbol{w}, \boldsymbol{x}) \subset \mathcal{C} : \begin{array}{l} |\boldsymbol{x}| = n,\ \boldsymbol{x} \text{ is RW-connected to} \\ \text{level } R \text{ by a path in } \mathcal{C} \cap U(n) \end{array} \right\}, \quad 0 < n \le R. \tag{5.1}$$

Our goal is to prove that for $d > 6$, L sufficiently large and $0 < a < 1$,

$$\mathbb{E}_\infty(|D(n)|) \le c_1(a), \quad 0 < n \le \lfloor aR \rfloor. \tag{5.2}$$

Writing $\boldsymbol{y} = (y, N)$, by (1.28) and (4.3) we have

$$\begin{aligned} \mathbb{E}_\infty |D(n)| &= \sum_{w,x \in \mathbb{Z}^d} \mathbb{P}_\infty \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n) \right] \\ &= \frac{1}{A} \lim_{N \to \infty} \sum_{w,x,y \in \mathbb{Z}^d} \mathbb{P}_{p_c} \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y} \right]. \end{aligned} \tag{5.3}$$

Hence we will focus on the event $\{(\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\}$, for fixed n, $\boldsymbol{w} = (w, n-1)$, $\boldsymbol{x} = (x, n)$ and $\boldsymbol{y} = (y, N)$.

Remark. For a quick indication of why we need to assume $d > 6$, consider the configuration in Figure 2, which contributes to the right-hand side of (5.3). Using the fact that τ_n is bounded by a constant by (4.4), and using (4.2) (see also (5.32) below), the configuration in Figure 2 can be bounded above using the BK inequality by

$$c \sum_{l=n}^{\infty} \sum_{k=n}^{l} \sum_{j=0}^{n} (l-j+1)^{-d/2} \le c \sum_{l=n}^{\infty} \sum_{k=n}^{l} (l-n+1)^{(2-d)/2} \le c \sum_{l=n}^{\infty} (l-n+1)^{(4-d)/2} = c \sum_{m=1}^{\infty} m^{(4-d)/2}, \tag{5.4}$$

where j, k, l are the time coordinates of the unlabelled vertices, from bottom to top. Here, the connection from the lower unlabelled vertex to the upper unlabelled vertex via $\boldsymbol{w}$ and $\boldsymbol{x}$ contributes

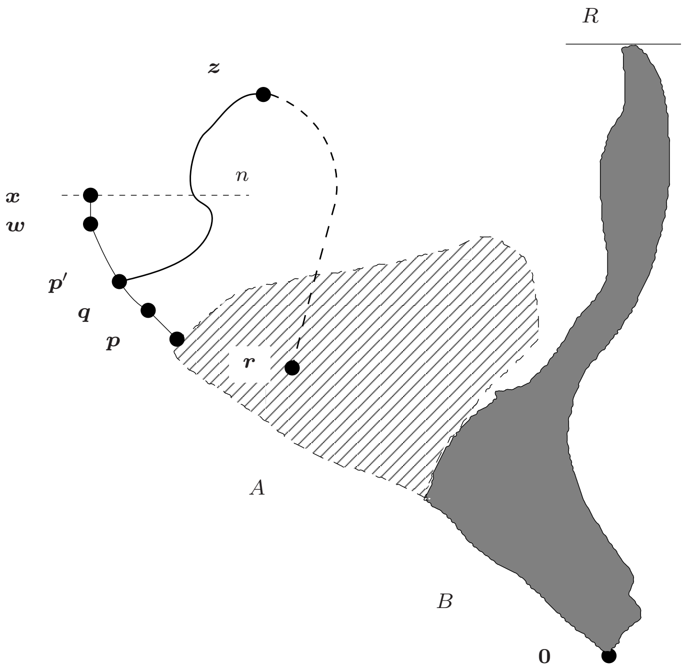

Figure 3: Illustration of the setup in Lemma 5.1.

$K(l-j+1)^{-d/2}$, and the other connections all contribute constants. The right-hand side is bounded only for $d>6$. Our complete proof of (5.2) is more involved since we must estimate the contributions to (5.3) due also to more complex zigzag random walk paths.

In Section 5.1, we prove Lemma 5.1, which explores the geometry of the event $\{(\boldsymbol{w},\boldsymbol{x})\in D(n),\, \mathbf{0}\longrightarrow \boldsymbol{y}\}$. Then, in Section 5.2, we apply Lemma 5.1 to construct events $A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})$, $J\geq 0$, such that

$$\{(\boldsymbol{w},\boldsymbol{x})\in D(n),\, \mathbf{0}\longrightarrow \boldsymbol{y}\}\subset \bigcup_{J=0}^{\infty} A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y}). \tag{5.5}$$

In Section 5.3, the BK inequality [8] is used to obtain a diagrammatic bound for the probability of the event $A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})$. Finally, in Section 5.4, we estimate the diagrams in this diagrammatic bound, to prove (5.2) and hence Proposition 3.3. The need to restrict to $d>6$, rather than $d>4$, occurs only in our last lemma, Lemma 5.6.

5.1 An intersection lemma

We will need the existence of certain intersections within the cluster $\mathcal{C}$ that are implied by the presence of a random walk path from $\boldsymbol{x}$ to R. These intersections are isolated in the following lemma. The following notation will be convenient:

$$\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}=\{\boldsymbol{v}:\mathbf{0}\longrightarrow \boldsymbol{v} \text{ disjointly from the edge } (\boldsymbol{p},\boldsymbol{q})\},\quad (\boldsymbol{p},\boldsymbol{q})\subset\mathcal{C}.$$

Also, we write $\overline{\boldsymbol{y}_1\boldsymbol{y}_2}$ for an occupied oriented path $\boldsymbol{y}_1\longrightarrow \boldsymbol{y}_2$. Such paths are in general not unique, but context will often identify a unique path for consideration.

We first describe informally the statement of the lemma, whose setup is illustrated in Figure 3. Suppose that $(\boldsymbol{w},\boldsymbol{x})\in D(n)$, and $\mathbf{0}\longrightarrow \boldsymbol{y}$. Let $(\boldsymbol{p},\boldsymbol{q})$ be an edge on an occupied path that starts at

$\mathbf{0}$ and ends with the edge $(\boldsymbol{w},\boldsymbol{x})$. Assume that $\boldsymbol{q} \not\longrightarrow R$. Then $\mathcal{C}(\boldsymbol{q})$ must intersect $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$, otherwise a RW-connection from $\boldsymbol{x}$ to R in $\mathcal{C} \cap U(n)$ could not occur. Indeed, $\mathcal{C}(\boldsymbol{x})$ would have to intersect $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$, but the lemma gives a more sophisticated version of the intersection requirements, which allows us to have some control over the way the intersection occurs. This is needed, because we will use the lemma recursively to construct a set of paths realizing the intersections. Assume that we are given a subgraph $A \cup B$ of $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$, that will represent a set of paths already constructed, where A will be a certain 'preferred region.' Assume that $A \cup B$ is disjoint from $\mathcal{C}(\boldsymbol{q})$, and $\mathbf{0} \in A \cup B$. Then there will be upwards occupied paths from some vertex $\boldsymbol{r} \in A \cup B$ and some vertex $\boldsymbol{p}' \in \overline{\boldsymbol{qx}}$ to an intersection point $\boldsymbol{z}$. It will be convenient, if we can also conclude that $\boldsymbol{r}$ is in the preferred region A. For this reason, we will also assume that any occupied path from B to $\mathcal{C}(\boldsymbol{q})$ passes through A. Now we state the lemma precisely.

Lemma 5.1. *Assume the event $\{(\boldsymbol{w},\boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\}$. In addition, assume the following:*

(i) $(\boldsymbol{p},\boldsymbol{q}) \subset \mathcal{C}$ *and either* $\boldsymbol{q} \longrightarrow \boldsymbol{w}$ *or* $(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{w},\boldsymbol{x})$*;*

(ii) $\boldsymbol{q} \not\longrightarrow R$*;*

(iii) *A and B are subgraphs of $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ with $\mathbf{0} \in A \cup B$, and such that $(A \cup B) \cap \mathcal{C}(\boldsymbol{q}) = \varnothing$;*

(iv) *every occupied oriented path from B to $\mathcal{C}(\boldsymbol{q})$ passes through a vertex of A.*

Then there exist $\boldsymbol{p}' \in \overline{\boldsymbol{qx}}$, $\boldsymbol{r} \in A$ and $\boldsymbol{z}$ with $|\boldsymbol{p}| < |\boldsymbol{z}| < R$, such that

$$\boldsymbol{p}' \longrightarrow \boldsymbol{z} \text{ and } \boldsymbol{r} \longrightarrow \boldsymbol{z} \text{ edge-disjointly, and edge-disjointly from } \overline{\boldsymbol{px}} \cup A \cup B.$$

Here $\boldsymbol{z}$ may coincide with $\boldsymbol{p}'$ or $\boldsymbol{r}$.

Proof. We first show that $\mathcal{C}(\boldsymbol{q})$ and $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ must have a common vertex $\boldsymbol{v}$. Fix a random walk path Γ from $\boldsymbol{x}$ to R in $U(n)$, showing that $(\boldsymbol{w},\boldsymbol{x}) \in D(n)$. Note that $\mathcal{C}$ (as a set of vertices) is the union $\tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})} \cup \mathcal{C}(\boldsymbol{q})$. Since Γ starts at $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{q})$, but $\boldsymbol{q} \not\longrightarrow R$, there is an edge $(\boldsymbol{v},\boldsymbol{v}') \subset \Gamma$ such that $\boldsymbol{v} \in \mathcal{C}(\boldsymbol{q})$ but $\boldsymbol{v}' \notin \mathcal{C}(\boldsymbol{q})$, and therefore $\boldsymbol{v}' \in \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$. We need to have $|\boldsymbol{v}'| = |\boldsymbol{v}| - 1$ (otherwise $\boldsymbol{v}' \in \mathcal{C}(\boldsymbol{q})$). We can rule out $(\boldsymbol{v}',\boldsymbol{v}) = (\boldsymbol{p},\boldsymbol{q})$, since Γ stays in $U(n)$, and $|\boldsymbol{p}| \le n-1$. It follows that $\boldsymbol{v} \in \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$, and hence is in the intersection $\mathcal{C}(\boldsymbol{q}) \cap \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$.

Choose $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{q}) \cap \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$ with $|\boldsymbol{z}|$ minimal. Since $\boldsymbol{q} \not\longrightarrow R$, $|\boldsymbol{p}| < |\boldsymbol{q}| \le |\boldsymbol{z}| < R$.

We can find occupied oriented paths $\overline{\boldsymbol{qz}} \subset \mathcal{C}(\boldsymbol{q})$ and $\overline{\mathbf{0}\boldsymbol{z}} \subset \tilde{\mathcal{C}}^{(\boldsymbol{p},\boldsymbol{q})}$. These two paths must be edge-disjoint by minimality of $|\boldsymbol{z}|$. Let $\boldsymbol{p}'$ be the last visit of $\overline{\boldsymbol{qz}}$ to $\overline{\boldsymbol{qx}}$, and let $\boldsymbol{r}$ be the last visit of $\overline{\mathbf{0}\boldsymbol{z}}$ to $A \cup B$. Such a last visit exists, since we assumed $\mathbf{0} \in A \cup B$. Since $\boldsymbol{z} \notin A \cup B$, due to $(A \cup B) \cap \mathcal{C}(\boldsymbol{q}) = \varnothing$, the last visit has to be in A by assumption (iv).

The path $\overline{\boldsymbol{p}'\boldsymbol{z}}$ is edge-disjoint from $\overline{\boldsymbol{px}}$, by the definition of $\boldsymbol{p}'$. It is also edge-disjoint from $A \cup B$, by minimality of $|\boldsymbol{z}|$. Likewise, the path $\overline{\boldsymbol{rz}}$ is edge-disjoint from $A \cup B$ by definition of $\boldsymbol{r}$. It is also edge-disjoint from $\overline{\boldsymbol{p}'\boldsymbol{z}}$, by minimality of $|\boldsymbol{z}|$. $\square$

Remark. Note that in the proof, we have first found a vertex $\boldsymbol{r} \in A \cup B$, and assumption (iv) was only used to show that we must have $\boldsymbol{r} \in A$. In fact, without assumption (iv), we would get the statement of the Lemma with $\boldsymbol{r} \in A \cup B$. The significance of being able to ensure that $\boldsymbol{r}$ is in the smaller set A, as well as the roles played by A and B will become apparent in Section 5.2.

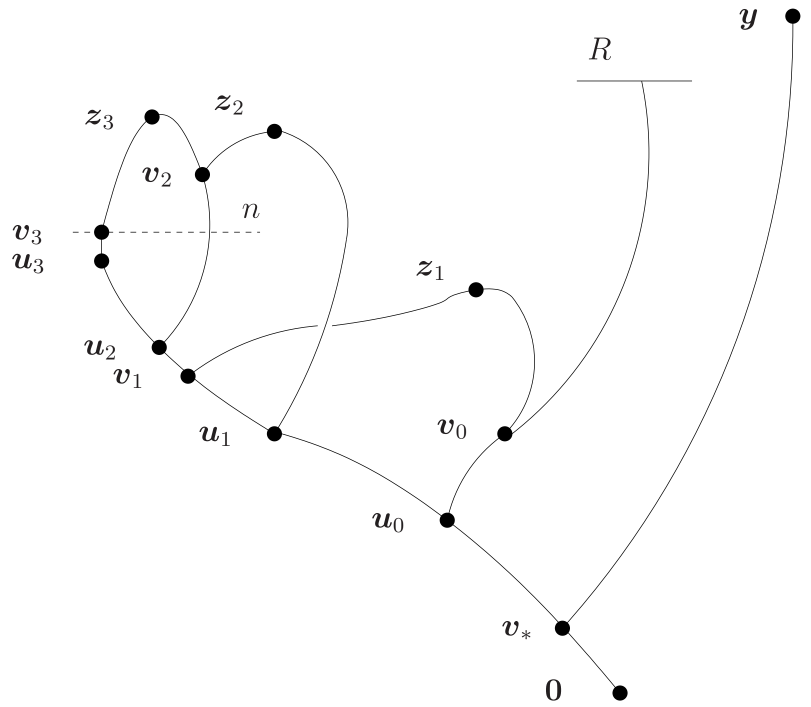

Figure 4: The vertices and disjoint paths of $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ for $J = 3$. Here $\boldsymbol{x} = \boldsymbol{v}_3$ and $\boldsymbol{w} = \boldsymbol{u}_3$.

5.2 The event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$

In this section, we define the event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ and prove (5.5). The following lemma is key.

Lemma 5.2. *Let $e = (\boldsymbol{w}, \boldsymbol{x})$, and assume the event $\{e \in D(n), \boldsymbol{0} \longrightarrow \boldsymbol{y}\}$. Then there exists $J \geq 0$, such that the following vertices and paths (all edge-disjoint) exist:*

(i) vertices $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_J = \boldsymbol{w}$ such that $0 \leq |\boldsymbol{u}_0| \leq |\boldsymbol{u}_1| \leq \cdots \leq |\boldsymbol{u}_J| = n - 1$;

(ii) vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_J = \boldsymbol{x}$, and, if $J \geq 1$, vertices $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_J$ such that

$$|\boldsymbol{u}_{i-1}| \leq |\boldsymbol{v}_{i-1}| \leq |\boldsymbol{z}_i|, \quad 1 \leq i \leq J; \tag{5.6}$$

$$|\boldsymbol{u}_{i-1}| < |\boldsymbol{z}_i| < R, \quad 1 \leq i \leq J; \tag{5.7}$$

(iii) $\boldsymbol{0} \longrightarrow \boldsymbol{u}_0$ and $\boldsymbol{u}_{i-1} \longrightarrow \boldsymbol{u}_i$, $1 \leq i \leq J$;

(iv) $\boldsymbol{u}_{i-1} \longrightarrow \boldsymbol{z}_i$, $1 \leq i \leq J$;

(v) $\boldsymbol{v}_{i-1}$ lies either on $\overline{\boldsymbol{u}_{i-1}\boldsymbol{u}_i}$ or $\overline{\boldsymbol{u}_{i-1}\boldsymbol{z}_i}$, and $\boldsymbol{v}_i \longrightarrow \boldsymbol{z}_i$, $1 \leq i \leq J$.

In addition, at least one of the following holds: Case (a) $\boldsymbol{v}_0 \longrightarrow \boldsymbol{y}$; Case (b) $\boldsymbol{v}_0 \longrightarrow R$ and there exists $\boldsymbol{v}_*$ on $\overline{\boldsymbol{0}\boldsymbol{u}_0}$ such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$.

Definition 5.3. We denote by $A_J = A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ the event that the vertices and disjoint paths listed in Lemma 5.2 exist, and $(\boldsymbol{w}, \boldsymbol{x})$ is occupied. See Figure 4.

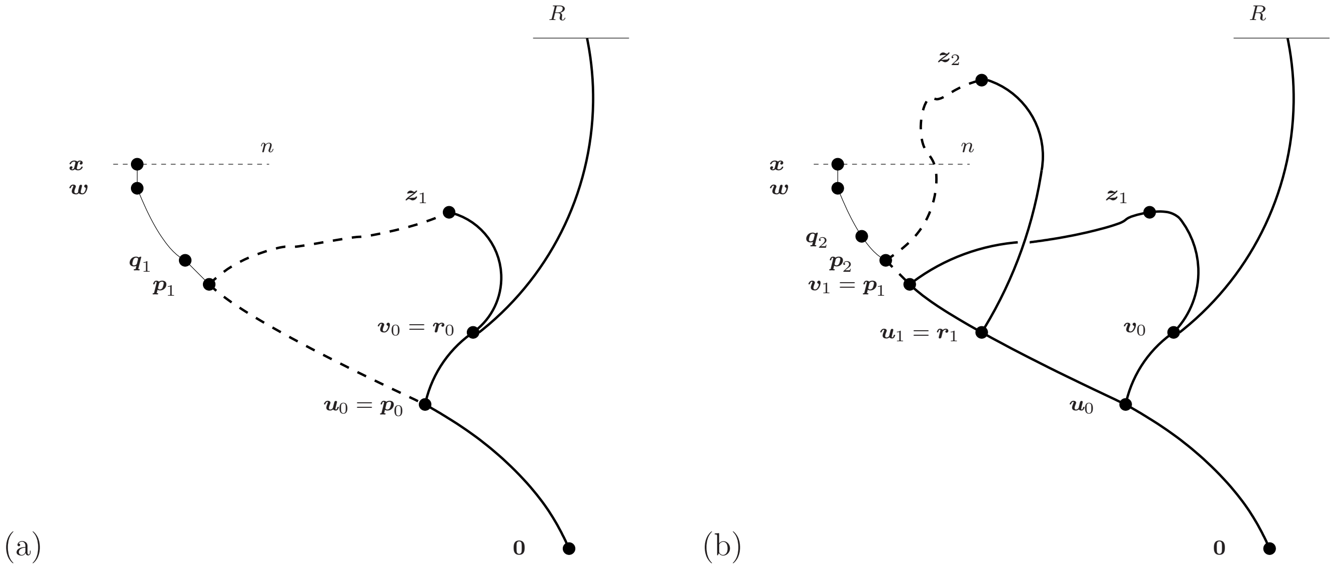

Figure 5: Assumptions of the recursion hypothesis for (a) $I = 1$; (b) $I = 2$. The thick solid lines indicate the sets (a) B_1 and (b) B_2, and the thick dashed lines the sets (a) A_1 and (b) A_2. The intersection lemma is used to produce paths that join the thick dashed lines to the thin solid lines.

The inclusion (5.5) then follows immediately from Lemma 5.2.

Proof of Lemma 5.2. Throughout the proof, we assume the event $\{e = (\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \longrightarrow \boldsymbol{y}\}$.

We first show that if $\boldsymbol{x} \longrightarrow R$ then the lemma holds with $J = 0$. Indeed, take $\boldsymbol{u}_0 = \boldsymbol{w}$ and $\boldsymbol{v}_0 = \boldsymbol{x}$. Then $\mathbf{0} \longrightarrow \boldsymbol{u}_0$, since $\boldsymbol{u}_0 \in \mathcal{C}$. Hence it is left to show that at least one of Cases (a) and (b) holds. If $\boldsymbol{v}_0 = \boldsymbol{x} \longrightarrow \boldsymbol{y}$, then Case (a) holds. If not, then since $\mathbf{0} \longrightarrow \boldsymbol{y}$ we can find $\boldsymbol{v}_* \in \overline{\mathbf{0}\boldsymbol{u}_0}$ such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$ edge-disjointly from $\overline{\mathbf{0}\boldsymbol{u}_0}$. The connection $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ has to be edge-disjoint from $\overline{\boldsymbol{w}\boldsymbol{x}R}$, otherwise we are in Case (a). Hence Case (b) holds.

For the rest of the proof, we assume $\boldsymbol{x} \not\longrightarrow R$.

We construct the paths claimed in the lemma recursively. Hence our proof will be based on a recursion hypothesis whose statement involves an integer $I \geq 0$, and which says that a subset of the paths claimed in the lemma (depending on I) have already been constructed. In order to advance the recursion, the hypothesis also specifies graphs A_I and B_I such that Lemma 5.1 can be applied with $A = A_I$ and $B = B_I$.

The outline of the proof is the following. Since the statement of the hypothesis for $I = 0$ is slightly different than for $I \geq 1$, we state and verify the hypothesis for $I = 0$ separately. This will show that the recursion can be started. Since the general step of the recursion is complex, we explain the first two steps of the recursion ($I = 1$ and $I = 2$) in some detail, before formulating the recursion hypothesis precisely in the general case $I \geq 1$. The recursion will lead to the proof of the lemma by the following steps. We prove that if the hypothesis holds for some value of $I \geq 0$, then either the conclusion of Lemma 5.2 follows with $J = I + 1$, or else the hypothesis also holds for $I + 1$. If, for some $i > 0$, the hypothesis holds for $I = 0, 1, \ldots, i$, then its statement will guarantee the existence of vertices $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_i$ with

$$|\boldsymbol{p}_0| < |\boldsymbol{p}_1| < \cdots < |\boldsymbol{p}_i| < n. \tag{5.8}$$

Consequently the hypothesis cannot hold for all $I = 0, 1, \ldots, n$, and the implications just mentioned provide a proof of Lemma 5.2. We now carry out the details.

(R) Recursion hypothesis for $I = 0$**.** *There exists* $\boldsymbol{p}_0, \boldsymbol{q}_0$ *such that*

$$\mathbf{0} \longrightarrow \boldsymbol{p}_0, \qquad \boldsymbol{p}_0 \longrightarrow R, \tag{5.9}$$

$$\boldsymbol{p}_0 \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_0 \not\longrightarrow R, \tag{5.10}$$

where $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ *is the first edge in the path* $\overline{\boldsymbol{p}_0\boldsymbol{x}}$*. All paths stated are edge-disjoint. Letting*

$$A_0 = \{\overline{\mathbf{0}\boldsymbol{p}_0}, \overline{\boldsymbol{p}_0 R}\} = \{\textit{paths in } (5.9)\},$$
$$B_0 = \varnothing,$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_0$*,* $\boldsymbol{q} = \boldsymbol{q}_0$*,* $A = A_0$ *and* $B = B_0$*.*

Verification of (R) for $I = 0$**.** Since $\mathbf{0} \longrightarrow \boldsymbol{w}$ and $\mathbf{0} \longrightarrow R$, there exists $\boldsymbol{p}_0$ such that

$$\mathbf{0} \longrightarrow \boldsymbol{p}_0, \qquad \boldsymbol{p}_0 \longrightarrow \boldsymbol{w} \quad \text{and} \quad \boldsymbol{p}_0 \longrightarrow R \quad \text{disjointly.}$$

Fix the paths $\overline{\mathbf{0}\boldsymbol{p}_0}$, $\overline{\boldsymbol{p}_0\boldsymbol{w}}$ and $\overline{\boldsymbol{p}_0 R}$, and let $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ be the first step of the path $\overline{\boldsymbol{p}_0\boldsymbol{x}}$. If we select $\boldsymbol{p}_0$ so that $|\boldsymbol{p}_0|$ is maximal, then we have $\boldsymbol{q}_0 \not\longrightarrow R$. We verify the hypotheses of Lemma 5.1 with these choices. First, (i), (ii) and $\mathbf{0} \in A_0 \cup B_0$ are immediate. Also, $\mathcal{C}(\boldsymbol{q}_0) \cap (A_0 \cup B_0) = \mathcal{C}(\boldsymbol{q}_0) \cap A_0 = \varnothing$, since otherwise $\boldsymbol{q}_0 \longrightarrow R$. Finally, (iv) is vacuous, since B_0 is empty.

Next, to illustrate the main idea of the proof, we explain the first two steps of the recursion.

Since we have verified **(R)** in the case $I = 0$, we can apply Lemma 5.1 with $\boldsymbol{p} = \boldsymbol{p}_0$, $\boldsymbol{q} = \boldsymbol{q}_0$, $A = A_0$ and $B = B_0$. Lemma 5.1 shows that there exist $\boldsymbol{p}' \in \overline{\boldsymbol{q}_0\boldsymbol{x}}$ and $\boldsymbol{r} \in A_0 = \overline{\mathbf{0}\boldsymbol{p}_0} \cup \overline{\boldsymbol{p}_0 R}$ and a vertex $\boldsymbol{z}$ such that $\boldsymbol{p}' \longrightarrow \boldsymbol{z}$ and $\boldsymbol{r} \longrightarrow \boldsymbol{z}$. For reasons that will be explained in the third paragraph below, we select $\boldsymbol{p}'$ with $|\boldsymbol{p}'|$ maximal such that the conclusions of Lemma 5.1 hold. With this choice of $\boldsymbol{p}'$, we set $\boldsymbol{p}_1 = \boldsymbol{p}'$, $\boldsymbol{z}_1 = \boldsymbol{z}$ and $\boldsymbol{r}_0 = \boldsymbol{r}$. Note that $|\boldsymbol{p}_1| > |\boldsymbol{p}_0|$. We define the vertices $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ as follows. Note that $\boldsymbol{r}_0 \in A_0$, which is the union of the paths $\overline{\mathbf{0}\boldsymbol{p}_0}$ and $\overline{\boldsymbol{p}_0 R}$. If $\boldsymbol{r}_0 \in \overline{\boldsymbol{p}_0 R}$ then we set $\boldsymbol{v}_0 = \boldsymbol{r}_0$ and $\boldsymbol{u}_0 = \boldsymbol{p}_0$, and if $\boldsymbol{r}_0 \in \overline{\mathbf{0}\boldsymbol{p}_0}$ then we set $\boldsymbol{v}_0 = \boldsymbol{p}_0$, $\boldsymbol{u}_0 = \boldsymbol{r}_0$. In either case, we have $|\boldsymbol{u}_0| \leq |\boldsymbol{p}_0| < |\boldsymbol{z}_1| < R$, and hence (5.7) holds for $i = 1$.

The paths constructed so far are depicted in Figure 5 (a). For the moment, the reader should disregard $\boldsymbol{q}_1$, and the distinction between thin, thick and dashed paths in the figure. We either have $|\boldsymbol{p}_1| < |\boldsymbol{x}| = n$, as depicted in Figure 5(a), or $\boldsymbol{p}_1 = \boldsymbol{x}$.

We first argue that in the case $\boldsymbol{p}_1 = \boldsymbol{x}$, Lemma 5.2 holds with $J = 1$. Indeed, if $\boldsymbol{p}_1 = \boldsymbol{x}$, we set $\boldsymbol{u}_1 = \boldsymbol{w}$ and $\boldsymbol{v}_1 = \boldsymbol{x}$. Then apart from the claim regarding Cases (a) and (b), the vertices and paths required by Lemma 5.2 for $J = 1$ have been constructed. (Note that the conclusion of Lemma 5.1 guarantees that the newly constructed paths are edge-disjoint from the old ones.) It is not difficult to also show that either Case (a) or (b) holds, and we leave the details of this to when we deal with the general recursion step.

Next we explain how to continue the construction if $|\boldsymbol{p}_1| < |\boldsymbol{x}| = n$. Let $\boldsymbol{q}_1$ denote the first vertex on the path $\overline{\boldsymbol{p}_1\boldsymbol{x}}$ following $\boldsymbol{p}_1$. Let B_1 denote the union of the thick solid lines in Figure 5(a), that is, $B_1 = \overline{\mathbf{0}\boldsymbol{p}_0} \cup \overline{\boldsymbol{p}_0 R} \cup \overline{\boldsymbol{r}_0\boldsymbol{z}_1} = A_0 \cup \overline{\boldsymbol{r}_0\boldsymbol{z}_1}$. Let A_1 denote the union of the dashed lines in Figure 5(a), that is, $A_1 = \overline{\boldsymbol{p}_0\boldsymbol{p}_1} \cup \overline{\boldsymbol{p}_1\boldsymbol{z}_1}$. We want to apply Lemma 5.1 with $A = A_1$, $B = B_1$, etc. It is easy to verify conditions (i)–(iii) of the lemma. The crucial condition here is (iv), which allows us to conclude that $\boldsymbol{r} \in A_1$, and hence the two new paths produced by Lemma 5.1 will connect the dashed lines to the thin solid lines in Figure 5(a). The reason condition (iv) is satisfied is that we chose $|\boldsymbol{p}_1|$ to be maximal. Indeed, a glance at Figure 5(a) suggests that if we had paths from $\overline{\boldsymbol{q}_1\boldsymbol{x}}$ and $B_1 \setminus A_1$ to a vertex $\boldsymbol{z}$ that are edge-disjoint from $A_1 \cup B_1$, then that would contradict the maximality of $|\boldsymbol{p}_1|$. (Recall the earlier application of Lemma 5.1 with $A = A_0$, $B = B_0$, etc., and the choice of $\boldsymbol{p}_1$.) We will verify the details of this when we deal with the general case $I \geq 1$.

We can summarize the above discussion by saying that Hypothesis **(R)** for $I = 0$ should imply that in the case $\boldsymbol{p}_1 \neq \boldsymbol{x}$ the following statement holds.

(R) Recursion hypothesis for $I = 1$**.** *Vertices and paths (all edge-disjoint) with the following properties exist:*

(i) $\boldsymbol{p}_1$ *and* $\boldsymbol{q}_1$ *such that*

$$\boldsymbol{p}_1 \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_1 \not\longrightarrow R, \tag{5.11}$$

where $(\boldsymbol{p}_1, \boldsymbol{q}_1)$ *is the first edge of the path* $\overline{\boldsymbol{p}_1\boldsymbol{x}}$*, and* $|\boldsymbol{p}_1| > |\boldsymbol{p}_0|$*;*

(ii) $\boldsymbol{u}_0$*,* $\boldsymbol{v}_0$*,* $\boldsymbol{z}_1$*, such that*

$$\mathbf{0} \longrightarrow \boldsymbol{u}_0,\ \boldsymbol{u}_0 \longrightarrow \boldsymbol{z}_1,\ \boldsymbol{v}_0 \longrightarrow R; \tag{5.12}$$

(iii) $\boldsymbol{u}_0 \longrightarrow \boldsymbol{p}_1$*;*

(iv) $\boldsymbol{v}_0$ *lies either on* $\overline{\boldsymbol{u}_0\boldsymbol{p}_1}$*, in which case* $\boldsymbol{p}_0 = \boldsymbol{v}_0$*, or on* $\overline{\boldsymbol{u}_0\boldsymbol{z}_1}$*, in which case* $\boldsymbol{p}_0 = \boldsymbol{u}_0$*;*

(v) $\boldsymbol{p}_0 \longrightarrow \boldsymbol{p}_1 \longrightarrow \boldsymbol{z}_1$*.*

Letting

$$\begin{aligned} A_1 &= \{\overline{\boldsymbol{p}_0\boldsymbol{p}_1}, \overline{\boldsymbol{p}_1\boldsymbol{z}_1}\}, \\ B_1 &= A_0 \cup \{\overline{\boldsymbol{r}_0\boldsymbol{z}_1}\} = \{\textit{paths in } (5.12)\} \cup \{\overline{\boldsymbol{u}_0\boldsymbol{p}_0}\}, \end{aligned}$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_1$*,* $\boldsymbol{q} = \boldsymbol{q}_1$*,* $A = A_1$ *and* $B = B_1$*.*

The next step of the construction is carried out similarly. An application of Lemma 5.1 gives the paths shown in Figure 5(b). Again, we chose $\boldsymbol{p}'$ so that $|\boldsymbol{p}'|$ is maximal, and set $\boldsymbol{p}_2 = \boldsymbol{p}'$, $\boldsymbol{z}_2 = \boldsymbol{z}$ and $\boldsymbol{r}_1 = \boldsymbol{r}$ for this choice of $\boldsymbol{p}'$. We define $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ depending on the location of $\boldsymbol{r}_1$, similarly to the previous step.

If $\boldsymbol{p}_2 = \boldsymbol{x}$, we can conclude similarly to the previous step that the lemma holds with $J = 2$. If $\boldsymbol{p}_2 \neq \boldsymbol{x}$, as in Figure 5(b), we advance the induction similarly to the previous step. This time, we use both the choice of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to conclude the necessary statement about A_2 and B_2.

Now we state the recursion hypothesis in general for $I \geq 1$.

(R) Recursion hypothesis for $I \geq 1$**.** *Vertices and paths (all edge-disjoint) with the following properties exist:*

(i) $\boldsymbol{p}_I$ *and* $\boldsymbol{q}_I$ *such that*

$$\boldsymbol{p}_I \longrightarrow \boldsymbol{w} \longrightarrow \boldsymbol{x}, \qquad \boldsymbol{q}_I \not\longrightarrow R, \tag{5.13}$$

where $(\boldsymbol{p}_I, \boldsymbol{q}_I)$ *is the first edge of the path* $\overline{\boldsymbol{p}_I\boldsymbol{x}}$*, and* $|\boldsymbol{p}_I| > |\boldsymbol{p}_{I-1}|$*;*

(ii) $\boldsymbol{u}_i$, $0 \le i < I$; $\boldsymbol{v}_i$, $0 \le i < I$; $\boldsymbol{z}_i$, $1 \le i \le I$, *such that*

$$\text{Lemma 5.2 (iii) holds with } i \text{ restricted to } 1 \le i < I, \tag{5.14}$$

$$\text{Lemma 5.2 (iv) holds with } i \text{ restricted to } 1 \le i \le I, \tag{5.15}$$

$$\text{Lemma 5.2 (v) holds with } i \text{ restricted to } 1 \le i < I, \tag{5.16}$$

$$\boldsymbol{v}_0 \longrightarrow R; \tag{5.17}$$

(iii) $\boldsymbol{u}_{I-1} \longrightarrow \boldsymbol{p}_I$;

(iv) $\boldsymbol{v}_{I-1}$ *lies either on* $\overline{\boldsymbol{u}_{I-1}\boldsymbol{p}_I}$, *in which case* $\boldsymbol{p}_{I-1} = \boldsymbol{v}_{I-1}$, *or on* $\overline{\boldsymbol{u}_{I-1}\boldsymbol{z}_I}$, *in which case* $\boldsymbol{p}_{I-1} = \boldsymbol{u}_{I-1}$;

(v) $\boldsymbol{p}_{I-1} \longrightarrow \boldsymbol{p}_I \longrightarrow \boldsymbol{z}_I$.

Letting

$$A_I = \{\overline{\boldsymbol{p}_{I-1}\boldsymbol{p}_I}, \overline{\boldsymbol{p}_I\boldsymbol{z}_I}\},$$
$$B_I = B_{I-1} \cup A_{I-1} \cup \{\overline{\boldsymbol{r}_{I-1}\boldsymbol{z}_I}\} = \{\textit{paths in } (5.14)\text{–}(5.17)\} \cup \{\overline{\boldsymbol{u}_{I-1}\boldsymbol{p}_{I-1}}\},$$

the hypotheses of Lemma 5.1 are satisfied with $\boldsymbol{p} = \boldsymbol{p}_I$, $\boldsymbol{q} = \boldsymbol{q}_I$, $A = A_I$ *and* $B = B_I$.

Figure 5 illustrates those paths of Figure 4 that have been constructed at the stages $I = 1$ and $I = 2$. Note that $\boldsymbol{p}_I$ receives either the label $\boldsymbol{u}_I$ or $\boldsymbol{v}_I$. Hence $\boldsymbol{p}_i$ will always equal either $\boldsymbol{u}_i$ or $\boldsymbol{v}_i$, depending on the location of $\boldsymbol{v}_i$ (by part (iv) of the hypothesis). Note also that (5.8) holds if **(R)** holds for all $I = 0, 1, \ldots, i$.

Consequence of (R): definition of $\boldsymbol{p}_{I+1}$, $\boldsymbol{u}_I$, $\boldsymbol{v}_I$ and $\boldsymbol{z}_{I+1}$. We now assume that **(R)** holds for some $I \ge 0$. An application of Lemma 5.1 with the data given in the hypothesis shows the existence of vertices $\boldsymbol{p}'$, $\boldsymbol{r}$ and $\boldsymbol{z}$ with certain properties. We now choose $\boldsymbol{p}'$ so that $|\boldsymbol{p}'|$ be maximal, and such that the properties claimed in Lemma 5.1 hold. We set $\boldsymbol{p}_{I+1} = \boldsymbol{p}'$, $\boldsymbol{z}_{I+1} = \boldsymbol{z}$ and $\boldsymbol{r}_I = \boldsymbol{r}$ for this choice.

Note that $\boldsymbol{r}_I \in A_I$, which is a union of two paths in both cases $I = 0$ and $I \ge 1$. In the case $I = 0$, if $\boldsymbol{r}_0 \in \overline{\boldsymbol{p}_0 R}$ then we set $\boldsymbol{v}_0 = \boldsymbol{r}_0$ and $\boldsymbol{u}_0 = \boldsymbol{p}_0$, and if $\boldsymbol{r}_0 \in \overline{\boldsymbol{0}\boldsymbol{p}_0}$ then we set $\boldsymbol{v}_0 = \boldsymbol{p}_0$, $\boldsymbol{u}_0 = \boldsymbol{r}_0$. Similarly, in the case $I \ge 1$, we set $\boldsymbol{v}_I = \boldsymbol{r}_I$ and $\boldsymbol{u}_I = \boldsymbol{p}_I$ if $\boldsymbol{r}_I \in \overline{\boldsymbol{p}_I\boldsymbol{z}_I}$, and we set $\boldsymbol{v}_I = \boldsymbol{p}_I$, $\boldsymbol{u}_I = \boldsymbol{r}_I$ if $\boldsymbol{r}_I \in \overline{\boldsymbol{p}_{I-1}\boldsymbol{p}_I}$. In both cases, it is clear that $|\boldsymbol{u}_I| \le |\boldsymbol{p}_I| < |\boldsymbol{z}_{I+1}| < R$, and hence (5.7) holds for $i = I + 1$.

It follows immediately from these definitions, and from the disjointness properties ensured by Lemma 5.1, that assumptions (ii)–(v) of **(R)** now hold with I replaced by $I + 1$.

Verification of Lemma 5.2 if $\boldsymbol{p}_{I+1} = \boldsymbol{x}$. We show that if $\boldsymbol{p}_{I+1} = \boldsymbol{x}$, then Lemma 5.2 holds with $J = I + 1$. For this, we define $\boldsymbol{u}_{I+1} = \boldsymbol{w}$ and $\boldsymbol{v}_{I+1} = \boldsymbol{x}$. It is immediate from these definitions, from the disjointness properties ensured by Lemma 5.1, and from the already established properties (ii)–(v) of hypothesis **(R)** for $I + 1 = J$, that (i)–(v) of Lemma 5.2 hold.

It remains to show that either Case (a) or Case (b) holds. Since $\boldsymbol{0} \longrightarrow \boldsymbol{y}$, there exists $\boldsymbol{v}_* \in \overline{\boldsymbol{0}\boldsymbol{u}_0}$, such that $\boldsymbol{v}_* \longrightarrow \boldsymbol{y}$ disjointly from $\overline{\boldsymbol{0}\boldsymbol{u}_0}$. If $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ is not disjoint from $\overline{\boldsymbol{v}_0 R}$, we are in Case (a), and we can ignore $\boldsymbol{v}_*$. If $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ intersects $\overline{\boldsymbol{u}_0\boldsymbol{p}_0}$ or $\overline{\boldsymbol{u}_0\boldsymbol{z}_1}$, let $\boldsymbol{v}_0'$ be the last such intersection. Note that $\overline{\boldsymbol{v}_*\boldsymbol{y}}$ must be disjoint from all other paths constructed, since those are subsets of $\mathcal{C}(\boldsymbol{q}_0)$, and $\boldsymbol{q}_0 \not\longrightarrow R$. Hence if the intersection $\boldsymbol{v}_0'$ exists, we can replace $\boldsymbol{v}_0$ by $\boldsymbol{v}_0'$ and we are in Case (a). If the intersection $\boldsymbol{v}_0'$ does not exist, we are in Case (b). This verifies the claims of Lemma 5.2.

We are left to show that if $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$, then **(R)** must hold for $I+1$.

Advancing the recursion $I \Longrightarrow I+1$ if $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$. Since $\boldsymbol{p}_{I+1} \in \overline{\boldsymbol{q}_I \boldsymbol{x}}$, but $\boldsymbol{p}_{I+1} \neq \boldsymbol{x}$, we have $|\boldsymbol{p}_{I+1}| > |\boldsymbol{p}_I|$, and $\boldsymbol{p}_{I+1} \longrightarrow \boldsymbol{w}$, showing (i) of hypothesis **(R)**. We have already seen that (ii)–(v) are guaranteed to hold.

We are left to show that the hypotheses of Lemma 5.1 hold with the data given. (i), (ii) and $\mathbf{0} \in A_{I+1} \cup B_{I+1}$ are clear from the definitions. By the definition of $\boldsymbol{q}_{I+1}$, $A_{I+1} \cup B_{I+1}$ is a subgraph of $\tilde{\mathcal{C}}^{(\boldsymbol{q}_{I+1})}$.

Assume, for a contradiction, that we have $\boldsymbol{z}_* \in \mathcal{C}(\boldsymbol{q}_{I+1}) \cap (A_{I+1} \cup B_{I+1})$. Without loss of generality, assume that $\boldsymbol{z}_*$ is the first visit of an occupied path $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ to $A_{I+1} \cup B_{I+1}$. In particular, $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ is edge-disjoint from $A_{I+1} \cup B_{I+1}$. Observe that

$$A_{I+1} \cup B_{I+1} = A_{I+1} \cup A_I \cup B_I \cup \{\overline{\boldsymbol{r}_I \boldsymbol{z}_{I+1}}\}.$$

If we had $\boldsymbol{z}_* \in A_{I+1}$, then the disjoint paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*\boldsymbol{z}_{I+1}}$ and $\overline{\boldsymbol{r}_I \boldsymbol{z}_{I+1}}$ would satisfy the conclusions of Lemma 5.1 for $\boldsymbol{p} = \boldsymbol{p}_I$, $\boldsymbol{q} = \boldsymbol{q}_I$, etc. This contradicts the choice of $\boldsymbol{p}_{I+1}$ (the maximality of $|\boldsymbol{p}_{I+1}|$), since $|\boldsymbol{q}_{I+1}| > |\boldsymbol{p}_{I+1}|$. If we had $\boldsymbol{z}_* \in \overline{\boldsymbol{r}_I \boldsymbol{z}_{I+1}}$, we get a similar contradiction due to the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{r}_I \boldsymbol{z}_*}$. Finally, we can rule out $\boldsymbol{z}_* \in A_I \cup B_I$, since $\mathcal{C}(\boldsymbol{q}_{I+1}) \subset \mathcal{C}(\boldsymbol{q}_I)$, and the latter is disjoint from $A_I \cup B_I$.

We are left to show that every occupied path from B_{I+1} to $\mathcal{C}(\boldsymbol{q}_{I+1})$ has to pass through A_{I+1}. Assume, for a contradiction, that there exists $\boldsymbol{z}_* \in \mathcal{C}(\boldsymbol{q}_{I+1})$, and $\boldsymbol{z}'_* \in B_{I+1}$ such that $\boldsymbol{z}'_* \longrightarrow \boldsymbol{z}_*$ disjointly from A_{I+1}. By considering the last visit, we may also assume that $\boldsymbol{z}'_*$ is the only vertex of $\overline{\boldsymbol{z}'_*\boldsymbol{z}_*}$ in $A_{I+1} \cup B_{I+1}$. We may also assume that $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{z}'_*\boldsymbol{z}_*}$ are edge-disjoint. We already saw $\mathcal{C}(\boldsymbol{q}_{I+1}) \cap (A_{I+1} \cup B_{I+1}) = \varnothing$, in particular, $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ is edge-disjoint from $A_{I+1} \cup B_{I+1}$. Observe that

$$B_{I+1} = A_I \cup B_I \cup \{\overline{\boldsymbol{r}_I \boldsymbol{z}_{I+1}}\} = \bigcup_{i=0}^{I} (A_i \cup \{\overline{\boldsymbol{r}_i \boldsymbol{z}_{i+1}}\}). \tag{5.18}$$

If we had $\boldsymbol{z}'_* \in \overline{\boldsymbol{r}_i \boldsymbol{z}_{i+1}}$, then the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{r}_i \boldsymbol{z}'_* \boldsymbol{z}_*}$ would contradict the choice of $\boldsymbol{p}_{i+1}$. Finally, if we had $\boldsymbol{z}'_* \in A_i$, then the paths $\overline{\boldsymbol{q}_{I+1}\boldsymbol{z}_*}$ and $\overline{\boldsymbol{z}'_*\boldsymbol{z}_*}$ would contradict the choice of $\boldsymbol{p}_{i+1}$. This completes the verification of hypothesis **(R)** for $I+1$.

This completes the proof of Lemma 5.2. □

5.3 A diagrammatic bound

In this section, we use Lemma 5.2 and the BK inequality [8] to bound $\mathbb{P}_{p_c}[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})]$. For this, we need the following preliminaries.

The *critical survival probability* is defined by

$$\theta_N = \mathbb{P}_{p_c}(\mathbf{0} \longrightarrow N). \tag{5.19}$$

The two papers [22, 23] show that for $d > 4$ and $L \geq L_0(d)$, we have $\theta_N \sim cN^{-1}$ as $N \to \infty$, for some $c = c(d, L) = 2 + \mathcal{O}(L^{-d})$. Moreover,

$$\theta_N \leq \frac{K'}{N}, \qquad N \geq 0, \quad L \geq L_0, \tag{5.20}$$

with the constant $K' = 5$ which is of course independent of both d and L (see [22, Eqn. (1.11)]).

To abbreviate the notation, when $\boldsymbol{y}_1 = (y_1, m_1)$ and $\boldsymbol{y}_2 = (y_2, m_2)$ we write $\tau(\boldsymbol{y}_1, \boldsymbol{y}_2) = \tau_{m_2-m_1}(y_2 - y_1)$. We also introduce

$$\begin{aligned} U_1(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{z}_1) &= \tau(\boldsymbol{v}_0, \boldsymbol{u}_1)\, \tau(\boldsymbol{u}_1, \boldsymbol{v}_1)\, \tau(\boldsymbol{v}_1, \boldsymbol{z}_1)\, \tau(\boldsymbol{u}_0, \boldsymbol{z}_1) \\ U_2(\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{z}_1) &= \tau(\boldsymbol{u}_0, \boldsymbol{u}_1)\, \tau(\boldsymbol{u}_1, \boldsymbol{v}_1)\, \tau(\boldsymbol{v}_1, \boldsymbol{z}_1)\, \tau(\boldsymbol{v}_0, \boldsymbol{z}_1) \\ U &= U_1 + U_2. \end{aligned} \tag{5.21}$$

For $0 \leq |\boldsymbol{u}_0| < n$ and $|\boldsymbol{u}_0| \leq |\boldsymbol{v}_0| < R$ and $\boldsymbol{y} = (y, N)$, let

$$\begin{aligned} \varphi(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \sum_{y \in \mathbb{Z}^d} \tau(\boldsymbol{0}, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \tau(\boldsymbol{v}_0, \boldsymbol{y}), \\ \varphi_R(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \sum_{y \in \mathbb{Z}^d} \sum_{\boldsymbol{v}_* \in \mathbb{Z}^d \times \mathbb{Z}_+} \tau(\boldsymbol{0}, \boldsymbol{v}_*)\, \tau(\boldsymbol{v}_*, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \theta_{R-|\boldsymbol{v}_0|}\, \tau(\boldsymbol{v}_*, \boldsymbol{y}) \\ \psi^{(0)}(\boldsymbol{u}_0, \boldsymbol{v}_0) &= \varphi(\boldsymbol{u}_0, \boldsymbol{v}_0) + \varphi_R(\boldsymbol{u}_0, \boldsymbol{v}_0). \end{aligned} \tag{5.22}$$

For $I \geq 1$, $0 \leq |\boldsymbol{u}_I| < n$ and $|\boldsymbol{u}_I| \leq |\boldsymbol{v}_0| < R$, let

$$\begin{aligned} \psi^{(I)}(\boldsymbol{u}_I, \boldsymbol{v}_I) = & \sum_{\substack{\boldsymbol{u}_{I-1} \in \mathbb{Z}^d \times Z_+ \\ 0 \leq |\boldsymbol{u}_{I-1}| \leq |\boldsymbol{u}_I|}} \sum_{\substack{\boldsymbol{z}_I \in \mathbb{Z}^d \times Z_+ \\ |\boldsymbol{v}_I| < |\boldsymbol{z}_I| < R}} \sum_{\substack{\boldsymbol{v}_{I-1} \in \mathbb{Z}^d \times Z_+ \\ |\boldsymbol{u}_{I-1}| \leq |\boldsymbol{v}_{I-1}| \leq |\boldsymbol{z}_I|}} U(\boldsymbol{u}_{I-1}, \boldsymbol{v}_{I-1}, \boldsymbol{u}_I, \boldsymbol{v}_I, \boldsymbol{z}_I) \\ & \times \psi^{(I-1)}(\boldsymbol{u}_{I-1}, \boldsymbol{v}_{I-1}). \end{aligned} \tag{5.23}$$

Lemma 5.4. *For $J \geq 0$,*

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}_{p_c}\left[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})\right] \leq \psi^{(J)}(\boldsymbol{w}, \boldsymbol{x}). \tag{5.24}$$

Proof. Definition 5.3 guarantees that on the event $A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})$ certain disjoint paths exist. If we fix the vertices $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_J$, $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_J$ and $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_J$, then the probability of the existence of the disjoint paths is bounded by the product of the probabilities of the existence of the individual paths, by the BK inequality [8]. An individual path $\overline{\boldsymbol{y}_1 \boldsymbol{y}_2}$ contributes a factor $\tau(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Now summing the bound over all the vertices but $\boldsymbol{u}_J$ and $\boldsymbol{v}_J$, gives an upper bound on $\mathbb{P}_{p_c}[A_J(n, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y})]$. Further summing over $y \in \mathbb{Z}^d$ gives an upper bound for the left-hand side of (5.24).

Now it is merely a matter of bookkeeping to check that we get the expressions $\psi^{(J)}$. The terms φ and φ_R correspond to Cases (a) and (b) of Lemma 5.2, respectively, and their sum $\psi^{(0)}$ bounds the contribution of the paths constructed when we initialized the recursion, together with the path leading to $\boldsymbol{y}$. When $J = 0$, and we take $\boldsymbol{u}_0 = \boldsymbol{w}$ and $\boldsymbol{v}_0 = \boldsymbol{x}$, we get the bound in (5.24), with $J = 0$.

When $J \geq 1$, the recursive definition of $\psi^{(I)}$ reflects the recursion of Lemma 5.2. The factor $U = U_1 + U_2$ gives the contribution of the paths added in the I-th step: for U_1 these are $\overline{\boldsymbol{v}_{I-1} \boldsymbol{u}_I}$, $\overline{\boldsymbol{u}_I \boldsymbol{v}_I}$, $\overline{\boldsymbol{v}_I \boldsymbol{z}_I}$ and $\overline{\boldsymbol{u}_{I-1} \boldsymbol{z}_I}$ (when $\boldsymbol{v}_{I-1}$ lies on $\overline{\boldsymbol{u}_{I-1} \boldsymbol{u}_I}$), and for U_2 they are $\overline{\boldsymbol{u}_{I-1} \boldsymbol{u}_I}$, $\overline{\boldsymbol{u}_I \boldsymbol{v}_I}$, $\overline{\boldsymbol{v}_I \boldsymbol{z}_I}$ and $\overline{\boldsymbol{v}_{I-1} \boldsymbol{z}_I}$ (when $\boldsymbol{v}_{I-1}$ lies on $\overline{\boldsymbol{u}_{I-1} \boldsymbol{z}_I}$). Note that the path $\overline{\boldsymbol{u}_{I-1} \boldsymbol{v}_{I-1}}$ is not present in U, since it is taken care of inside $\psi^{(I-1)}$. $\square$

5.4 Estimation of diagrams

It follows from (5.3), (5.5) and Lemma 5.4 that

$$\begin{aligned}\mathbb{E}_\infty|D(n)| &\le \frac{1}{A}\limsup_{N\to\infty}\sum_{w,x,y\in\mathbb{Z}^d}\sum_{J=0}^{\infty}\mathbb{P}_{p_c}\left[A_J(n,\boldsymbol{w},\boldsymbol{x},\boldsymbol{y})\right] \\ &\le \frac{1}{A}\limsup_{N\to\infty}\left[\sum_{J=0}^{\infty}\sum_{w,x\in\mathbb{Z}^d}\psi^{(J)}(\boldsymbol{w},\boldsymbol{x})\right]. \end{aligned} \tag{5.25}$$

Fix $d>6$, $R\ge 1$, $0<a<1$ and $0<n\le\lfloor aR\rfloor$. To prove (5.2) and hence Proposition 3.3, it suffices to show that there exist $c_2=c_2(a)$ and a constant $0<c_3<\frac{1}{2}$ such that

$$\limsup_{N\to\infty}\sum_{w,x\in\mathbb{Z}^d}\psi^{(J)}(\boldsymbol{w},\boldsymbol{x})\le c_2c_3^J,\quad J\ge 0, \tag{5.26}$$

since (4.3) and (5.25)–(5.26) then imply that

$$\mathbb{E}_\infty|D(n)|\le \bar{K}c_2\sum_{J=0}^{\infty}c_3^J=\bar{K}\frac{c_2}{1-c_3}\le 2\bar{K}c_2=c_1(a).$$

We now state and prove two lemmas which imply (5.26). Their proofs use the bound

$$\tau_n\le\bar{K},\quad n\ge 0, \tag{5.27}$$

of (4.3), as well as (5.20). It is in Lemma 5.6, and only there, that we need to assume $d>6$ rather than $d>4$. The first lemma gives a bound on $\psi^{(0)}$.

Lemma 5.5. *Let $d>4$, $R\ge 1$, $0<a<1$, $0<n\le\lfloor aR\rfloor$, $\boldsymbol{w}=(w,n-1)$ and $\boldsymbol{x}=(x,n)$. Then*

$$\limsup_{N\to\infty}\sum_{w,x\in\mathbb{Z}^d}\psi^{(0)}(\boldsymbol{w},\boldsymbol{x})\le(\bar{K}^3+\bar{K}^4K'a/(1-a)). \tag{5.28}$$

Proof. By definition and (5.27),

$$\sum_{w,x\in\mathbb{Z}^d}\varphi(\boldsymbol{w},\boldsymbol{x})=\tau_{n-1}\tau_1\tau_{N-n}\le\bar{K}^3.$$

Similarly, writing $\boldsymbol{v}_*=(v_*,l_*)$,

$$\sum_{w,x\in\mathbb{Z}^d}\varphi_R(\boldsymbol{w},\boldsymbol{x})=\sum_{l_*=0}^{n-1}\tau_{l_*}\,\tau_{n-l_*-1}\,\tau_1\,\theta_{R-n}\,\tau_{N-l_*}\le\frac{\bar{K}^4K'n}{R-n}.$$

Since $n/(R-n)\le a/(1-a)$ because $n\le\lfloor aR\rfloor$, this gives (5.28). $\square$

For $J\ge 1$, we use a somewhat stronger formulation of the bound, in which $|\boldsymbol{u}_J|$ and $|\boldsymbol{v}_J|$ are not restricted to the values $n-1$ and n. This will allow us to prove a bound on $\psi^{(J)}$ by induction.

Lemma 5.6. *Let $d > 6$, $R \geq 1$, $0 < a < 1$, $0 < n \leq \lfloor aR \rfloor$. Suppose that $0 \leq k_J < n$, $k_J \leq l_J < R$, $\boldsymbol{u}_J = (u_J, k_J)$ and $\boldsymbol{v}_J = (v_J, l_J)$. Then*

$$\limsup_{N\to\infty} \sum_{u_J,v_J\in\mathbb{Z}^d} \psi^{(J)}(\boldsymbol{u}_J, \boldsymbol{v}_J) \leq (2\bar{K}^3K^3\beta)^J(\bar{K}^3 + 3\bar{K}^5K'a/(1-a)), \quad J \geq 1. \tag{5.29}$$

Proof. We start by inserting the definition of $\psi^{(J)}$ into the left-hand side of (5.29). With $\boldsymbol{z}_J = (z_J, s_J)$, $\boldsymbol{u}_{J-1} = (u_{J-1}, k_{J-1})$ and $\boldsymbol{v}_{J-1} = (v_{J-1}, l_{J-1})$, the left-hand side of (5.29) equals

$$\begin{aligned}\limsup_{N\to\infty} \sum_{u_J,v_J\in\mathbb{Z}^d} \sum_{z_J,u_{J-1},v_{J-1}\in\mathbb{Z}^d} \sum_{k_{J-1}=0}^{k_J} \sum_{s_J=l_J}^{R-1} \sum_{l_{J-1}=k_{J-1}}^{s_J} & U(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}, \boldsymbol{u}_J, \boldsymbol{v}_J, \boldsymbol{z}_J) \\ \times\, \psi^{(J-1)}(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}).&\end{aligned} \tag{5.30}$$

The vertices $\boldsymbol{u}_J$, $\boldsymbol{v}_J$ and $\boldsymbol{z}_J$ only appear in the factor U. We claim that

$$\sum_{u_J,v_J,z_J\in\mathbb{Z}^d} U(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}, \boldsymbol{u}_J, \boldsymbol{v}_J, \boldsymbol{z}_J) \leq 2\bar{K}^3K\beta(s_J - k_{J-1} + 1)^{-d/2}. \tag{5.31}$$

To see this, note that $s_J = |\boldsymbol{z}_J| > |\boldsymbol{u}_{J-1}| = k_{J-1}$, by (5.7). For the U_1 term, we use (4.2) to bound $\tau(\boldsymbol{u}_{J-1}, \boldsymbol{z}_J)$ by $K\beta(s_J - k_{J-1} + 1)^{-d/2}$. Then the sums over z_J, v_J and u_J contribute the factor $\bar{K}^3$, by using (5.27) for the other three factors in U_1. For the U_2 term, we apply (4.2) and $\tau_n \leq \bar{K}$ to see that

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \tau_n(y)\tau_m(x-y) \leq K\beta(n+m+1)^{-d/2}, \quad n+m \geq 1. \tag{5.32}$$

An application of (5.32) to the convolution of $\tau(\boldsymbol{u}_{J-1}, \boldsymbol{u}_J)$, $\tau(\boldsymbol{u}_J, \boldsymbol{v}_J)$ and $\tau(\boldsymbol{v}_J, \boldsymbol{z}_J)$, together with (5.27), yields an upper bound of the same form. This proves (5.31). Inserting (5.31) into (5.30) and rearranging, we get

$$\begin{aligned}(5.30) \leq 2\bar{K}^3K\beta \sum_{k_{J-1}=0}^{k_J} \sum_{s_J=l_J}^{R-1} & (s_J - k_{J-1} + 1)^{-d/2} \\ \times \sum_{l_{J-1}=k_{J-1}}^{s_J} & \limsup_{N\to\infty} \sum_{u_{J-1},v_{J-1}\in\mathbb{Z}^d} \psi^{(J-1)}(\boldsymbol{u}_{J-1}, \boldsymbol{v}_{J-1}).\end{aligned} \tag{5.33}$$

Now we prove (5.29) by induction on J. To start the induction, we verify (5.29) for $J = 1$. This is most of the work; advancing the induction is easy. When $J = 1$, the $\limsup$ in (5.33) consists of two terms, corresponding to φ and φ_R. The φ-term is bounded by

$$\limsup_{N\to\infty} \sum_{u_0,v_0\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \tau(\boldsymbol{0}, \boldsymbol{u}_0)\, \tau(\boldsymbol{u}_0, \boldsymbol{v}_0)\, \tau(\boldsymbol{v}_0, \boldsymbol{y}) = \limsup_{N\to\infty}\ \tau_{k_0}\tau_{l_0-k_0}\tau_{N-l_0} \leq \bar{K}^3. \tag{5.34}$$

Inserting this into (5.33), and assuming $d > 6$, we see that the φ contribution to (5.33) is bounded by

$$2\bar{K}^3K\beta\bar{K}^3 \sum_{k_0=0}^{k_1} \sum_{s_1=l_1}^{R-1} (s_1 - k_0 + 1)^{(2-d)/2} \leq (2\bar{K}^3K^2\beta)(\bar{K}^3). \tag{5.35}$$

The φ_R term is bounded as follows. First, the lim sup is bounded by

$$\begin{aligned}
&\limsup_{N\to\infty} \sum_{u_0,v_0\in\mathbb{Z}^d} \sum_{l_*=0}^{k_0} \sum_{y,v_*\in\mathbb{Z}^d} \tau(\mathbf{0},\boldsymbol{v}_*)\,\tau(\boldsymbol{v}_*,\boldsymbol{u}_0)\,\tau(\boldsymbol{u}_0,\boldsymbol{v}_0)\,\theta_{R-l_0}\,\tau(\boldsymbol{v}_*,\boldsymbol{y})\\
&\qquad \le \bar K^4 \sum_{l_*=0}^{k_0} \theta_{R-l_0} \le \bar K^4 (k_0+1)\frac{K'}{R-l_0} \le \bar K^4 K' n \frac{1}{R-l_0}.
\end{aligned} \tag{5.36}$$

We insert this bound into (5.33) to obtain

$$(2\bar K^3 K\beta)(\bar K^4 K')n \sum_{k_0=0}^{k_1} \sum_{s_1=l_1}^{R-1} (s_1-k_0+1)^{-d/2} \sum_{l_0=k_0}^{s_1} \frac{1}{R-l_0}. \tag{5.37}$$

We split the sum over s_1 into the cases: (1) $s_1 < n+(R-n)/2$; (2) $s_1 \ge n+(R-n)/2$. In case (1), we have

$$\frac{1}{R-l_0} \le \frac{1}{R-s_1} \le \frac{2}{R-n}.$$

Inserting this into (5.37), the contribution of case (1) to the expression in (5.37) is bounded by

$$\begin{aligned}
&(2\bar K^3 K\beta)(2\bar K^4 K')\frac{n}{R-n} \sum_{k_0=0}^{k_1} \sum_{s_1=l_1}^{n+(R-n)/2} (s_1-k_0+1)^{(2-d)/2}\\
&\qquad \le (2\bar K^3 K^2\beta)(2\bar K^4 K')\frac{n}{R-n} \le (2\bar K^3 K^2\beta)(2\bar K^4 K')\frac{a}{1-a}.
\end{aligned} \tag{5.38}$$

In case (2), since $n \ge k_1 \ge k_0$ we have

$$(s_1-k_0+1)^{-d/2} \le K(R-k_0+1)^{-d/2},$$

and the sum over l_0 in (5.37) is bounded by $\log(R-k_0+1) \le \bar K (R-k_0+1)^{\delta}$ for some fixed exponent δ (e.g., $\delta = 1/4$ suffices). Therefore the contribution of case (2) to the expression in (5.37) is bounded by

$$\begin{aligned}
&(2\bar K^3 K^2\beta)(\bar K^5 K')n\frac{R-n}{2} \sum_{k_0=0}^{k_1} (R-k_0+1)^{(2\delta-d)/2}\\
&\qquad \le (2\bar K^3 K^3\beta)(\bar K^5 K')n\frac{R-n}{2}(R-n)^{(2\delta+2-d)/2}\\
&\qquad \le (2\bar K^3 K^3\beta)(\bar K^5 K')\frac{n}{R-n}(R-n)^{(2\delta+6-d)/2}\\
&\qquad \le (2\bar K^3 K^3\beta)(\bar K^5 K')\frac{a}{1-a}.
\end{aligned} \tag{5.39}$$

Putting (5.38) and (5.39) together, we get that (5.37) is bounded by $(2\bar K^3 K^3\beta)(3\bar K^5 K' a/(1-a))$. Together with (5.35) this proves the $J=1$ case of (5.29).

To advance the induction, we assume now that (5.29) holds for an integer $J = M-1 \ge 1$, and prove that it holds for $J = M$. Using $d > 6$, we insert the bound (5.29) into (5.33) to get that the right-hand side of (5.33) is bounded by

$$(2\bar{K}^3K\beta)(2\bar{K}^3K^3\beta)^{M-1}(\bar{K}^3+3\bar{K}^5K'a/(1-a))\sum_{k_{M-1}=0}^{k_M}\sum_{s_M=l_M}^{R-1}(s_M-k_{M-1}+1)^{(2-d)/2}$$
$$\leq (2\bar{K}^3K^3\beta)^M(\bar{K}^3+3\bar{K}^5K'a/(1-a)). \tag{5.40}$$

This completes the proof of (5.29). □

Proof of (5.26). It follows immediately from Lemmas 5.5–5.6 that (5.26) holds with $c_2 = (\bar{K}^3 + 3\bar{K}^5K'a/(1-a))$ and $c_3 = 2\bar{K}^3K^3\beta$. Recall that the constant $K' = 5$ of (5.20) is independent of d and L. Choosing β small ensures that $0 < c_3 < \frac{1}{2}$. This proves (5.26), and thus completes the proof of Proposition 3.3. □

Acknowledgements

The work of MTB, AAJ and GS was supported in part by NSERC of Canada. The work of TK was supported in part by the Ministry of Education, Culture, Sports, Science and Technology of Japan, Grant-in-Aid 18654018 (Houga). We thank an anonymous referee for suggesting several improvements to the exposition.

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