Heat kernel estimates for strongly recurrent random walk on random media

Takashi Kumagai *†and Jun Misumi ‡§

July 29, 2008

Abstract

We establish general estimates for simple random walk on an arbitrary infinite random graph, assuming suitable bounds on volume and effective resistance for the graph. These are generalizations of the results in [6, Section 1,2], and in particular, imply the spectral dimension of the random graph. We will also give an application of the results to random walk on a long range percolation cluster.
Key Words: Random walk - Random media - Heat kernel estimates - Spectral dimension - Long range percolation
Running Head: Heat kernel estimates on random media

1 Introduction and Main results

1.1 Introduction

Recently, there are intensive study for detailed properties of random walk on a percolation cluster. For a random walk on a supercritical percolation cluster on $\mathbb{Z}^d\,(d \geq 2)$, detailed Gaussian heat kernel estimates and quenched invariance principle have been obtained ([4, 10, 17, 19]). This means, such a random walk behaves in a diffusive fashion similar to a random walk on $\mathbb{Z}^d$. On the other hand, it is generally believed that random walk on a large critical cluster behaves subdiffusively (see [3] and the references therein). Critical percolation clusters are believed (and for some cases proved) to be finite. So, it is natural to consider random walk on an incipient infinite cluster (IIC), namely a critical percolation cluster conditioned to be infinite. Random walk on IICs has been proved to be subdiffusive on $\mathbb{Z}^2$ ([13]), on trees ([14, 7]), and for the spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimension $d > 6$ ([6]).

In order to study detailed properties of the random walk, it is nice and useful if one can compute the long time behaviour of the transition density (heat kernel). Let $p_n(x, y)$ be its transition density

*Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan. `kumagai@math.kyoto-u.ac.jp`
†Research partially supported by the Grant-in-Aid for Scientific Research (B) 18340027.
‡Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo 153-8914, Japan. `misumi@ms.u-tokyo.ac.jp`
§Research partially supported by the 21 century COE program at Graduate School of Mathematical Sciences, the University of Tokyo.

(see (1.5) for a precise definition) of a random walk on an infinite (random) graph G. Define the *spectral dimension* of G by

$$d_s(G) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log n},$$

(if this limit exists). Let $\mathcal{C}_d$ be the IIC for percolation cluster on $\mathbb{Z}^d$. Alexander and Orbach [2] conjectured that, for any $d \geq 2$, $d_s(\mathcal{C}_d) = 4/3$. (To be precise, the original Alexander-Orbach conjecture was that the left side hand of (1.33) is equal to 2/3 for all dimensions on $\mathcal{C}_d$.) While it is now believed that this conjecture is unlikely to be true for small d ([3]), it is proved that the conjecture is true on trees ([14, 7]), and for the spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$, $d > 6$ ([6]). One very interesting open problem in this direction is to establish estimates of $d_s(\mathcal{C}_d)$ (or the corresponding IIC for the oriented percolation) for d small to disprove/prove the Alexander-Orbach conjecture.

In [6], general estimates are given for simple random walk on an arbitrary infinite random graph G, assuming suitable bounds on volume and effective resistance for the random graph. In particular, $d_s(G) = 4/3$. The main purpose of this paper is to extend this general estimates to the framework of strongly recurrent random walk on the random graph G. Here 'strongly recurrent' simply means $d_s(G) < 2$. (See [5] Section 1.1, for more precise meaning of 'strongly recurrent'.) Roughly saying our main results can be expressed as follows; if the volume of the ball of radius R on the random graph is of order R^D with high probability and the resistance between the center and the outside of the ball is of order R^α with high probability (precise statement is given in Assumption 1.2), then one can establish both quenched (i.e. almost sure with respect to the randomness of the graph) and annealed (i.e. averaged over the randomness of the graph) estimates for the exit time from the ball, on-diagonal heat kernel, the mean displacement, etc. (Propositions 1.3–1.4 and Theorem 1.5). In particular,

$$d_s(G) = \frac{2D}{D+\alpha}. \tag{1.1}$$

Note that the estimates given in Section 1–2 of [6] treat the case $D = 2, \alpha = 1$. Our results are general enough to allow logarithmic corrections for the volume and the resistance estimates. In fact, the volume growth $v(R)$ and the resistance growth $r(R)$ could be any functions satisfying (1.12).

Unfortunately, so far we could not establish any new results for random walk on the IIC for low dimensions. Though our results give 'simplest' conditions on volume and resistance growth (Assumption 1.2) to obtain the spectral dimension $d_s(G)$, significant work is required to prove them for such models, as was done in Section 3–5 of [6] for the spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$, $d > 6$. Instead, we apply our results to the long range percolation, which is another important random graph. We consider the following case of the long range percolation. On $\mathbb{Z}$, each pair of points $x, y \in \mathbb{Z}$, $|x-y| \geq 2$ is connected by an unoriented bond with probability $\beta|x-y|^{-s}$ for some $\beta > 0$ and $s > 2$, independently of each others. Each pair of nearest points is connected with probability 1. On this random graph, we can check the bounds on volume and effective resistance required for the general estimates and obtain, for example, (1.1) with $D = \alpha = 1$ (Theorem 2.2). In fact, it is quite likely that for $s > 2$, the transition density for simple random walk on the long range percolation is Gaussian-type, so there are other ways to establish (1.1). However, we can further observe an interesting discontinuity of the spectral dimension at $s = 2$ (Remark 2.3(1)).

Finally, we remark that after the first draft of this paper was submitted, some rigorous results were obtained concerning the Alexander-Orbach conjecture. In [16], it is proved that the conjecture is true when the two-point function behaves nicely; in particular the conjecture is true for simple random walk on the IIC of critical percolation on $\mathbb{Z}^d$ when d is large enough or when $d > 6$ and the lattice is sufficiently spread out. In [12], it is proved that the conjecture is false for simple random walk on a Galton-Watson tree conditioned to survive when the offspring distribution is in the domain of attraction of a stable law with index $\alpha \in (1, 2)$. (Similar result for the annealed case was already obtained in [14].) In both papers, the key ingredient is to obtain suitable bounds on volume and effective resistance mentioned above that are sufficient for general quenched and annealed estimates.

The organization of the paper is as follows. In the next subsection, we summarize the framework and the main results on random graphs. In Section 2, we give the application of our main results to the long range percolation. In Section 3, we give the full proof of the main results. Although the principal ideas of the proof is quite similar to the ones in Section 1–2 of [6], lots of additional careful computations are needed to obtain this general version. So, we think it would help readers to include the full proof.

1.2 Framework and Main results

Let $\Gamma = (G, E)$ be an infinite graph, with vertex set G and edge set E. The edges $e \in E$ are *not* oriented. We assume that Γ is connected. We write $x \sim y$ if $\{x, y\} \in E$, and assume that (G, E) is locally finite, i.e., $\mu_y < \infty$ for each $y \in G$, where μ_y is the number of bonds that contain y. Note that $\mu_x \geq 1$ since Γ is connected. We extend μ to a measure on G. Let $d(\cdot, \cdot)$ be a metric on G. (Note that d is not necessarily a graph distance. Any metric on G may be used in this section.) We write

$$B(x, r) = \{y : d(x, y) < r\}, \qquad V(x, r) = \mu(B(x, r)), \quad r \in (0, \infty). \tag{1.2}$$

We call $V(x, r)$ the *volume* of the ball $B(x, r)$. We will assume G contains a marked vertex, which we denote 0, and we write

$$B(R) = B(0, R), \qquad V(R) = V(0, R). \tag{1.3}$$

Let $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ be the discrete-time simple random walk on Γ. Then X has transition probabilities

$$P^x(X_1 = y) = \frac{1}{\mu_x}, \quad y \sim x. \tag{1.4}$$

We define the transition density (or discrete-time heat kernel) of X by

$$p_n(x, y) = P^x(X_n = y)\frac{1}{\mu_y}; \tag{1.5}$$

we have $p_n(x, y) = p_n(y, x)$. For $A \subset G$, we write

$$T_A = \inf\{n \geq 0 : X_n \in A\}, \qquad \tau_A = T_{A^c}, \tag{1.6}$$

and let

$$\tau_R = \tau_{B(0,R)} = \min\{n \geq 0 : X_n \notin B(0, R)\}. \tag{1.7}$$

We define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y \in G \\ x \sim y}} (f(x) - f(y))(g(x) - g(y)). \tag{1.8}$$

If we regard Γ as an electrical network with a unit resistor on each edge in E, then $\mathcal{E}(f,f)$ is the energy dissipation when the vertices of G are at a potential f. Set $H^2 = \{f \in \mathbb{R}^G : \mathcal{E}(f,f) < \infty\}$. Let A, B be disjoint subsets of G. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{1.9}$$

Let $R_{\text{eff}}(x,y) = R_{\text{eff}}(\{x\},\{y\})$, and $R_{\text{eff}}(x,x) = 0$.

It is known that $R_{\text{eff}}(\cdot,\cdot)$ is a metric on G (see [15, Section 2.3]), and the following holds.

$$|f(x) - f(y)|^2 \le R_{\text{eff}}(x,y)\mathcal{E}(f,f), \qquad \forall f \in L^2(G,\mu). \tag{1.10}$$

We now consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a family of random graphs $\Gamma(\omega) = (G(\omega), E(\omega), \omega \in \Omega)$. We assume that, for each $\omega \in \Omega$, the graph $\Gamma(\omega)$ is infinite, locally finite, connected, and contains a marked vertex $0 \in G$. Let $d(\cdot,\cdot) := d_\omega(\cdot,\cdot)$ be a metric on $G(\omega)$. (Again, d is any metric on $G(\omega)$, not necessarily the graph distance.) We denote balls in $\Gamma(\omega)$ by $B_\omega(x,r)$, their volume by $V_\omega(x,r)$, and write

$$B(R) = B_\omega(R) = B_\omega(0,R), \qquad V(R) = V_\omega(R) = V_\omega(0,R). \tag{1.11}$$

We write $X = (X_n, n \ge 0, P_\omega^x, x \in G(\omega))$ for the simple random walk on $\Gamma(\omega)$, and denote by $p_n^\omega(x,y)$ its transition density with respect to $\mu(\omega)$. To define X we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega \times \overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$.

Let $v, r : \mathbb{N} \to [0,\infty)$ be strictly increasing functions with $v(1) = r(1) = 1$ which satisfy

$$C_1^{-1}\Big(\frac{R}{R'}\Big)^{d_1} \le \frac{v(R)}{v(R')} \le C_1\Big(\frac{R}{R'}\Big)^{d_2}, \quad C_2^{-1}\Big(\frac{R}{R'}\Big)^{\alpha_1} \le \frac{r(R)}{r(R')} \le C_2\Big(\frac{R}{R'}\Big)^{\alpha_2} \tag{1.12}$$

for all $0 < R' \le R < \infty$, where $C_1, C_2 \ge 1$, $1 \le d_1 \le d_2$ and $0 < \alpha_1 \le \alpha_2 \le 1$. For convenience, we let $v(0) = r(0) = 0$, $v(\infty) = r(\infty) = \infty$ and extend them to $v, r : [0,\infty] \to [0,\infty]$ such that v, r are continuous, strictly increasing, and satisfy (1.12). One such extension is to extend them linearly.

The key ingredients in our analysis of the simple random walk are volume and resistance bounds. The following defines a random set $J(\lambda)$ of values of R for which we have 'good' volume and effective resistance estimates.

Definition 1.1. Let $\Gamma = (G,E)$ be as above. For $\lambda > 1$, define

$$\begin{aligned} J(\lambda) \;\; = \;\; & \{R \in [1,\infty] : \lambda^{-1}v(R) \le V(R) \le \lambda v(R), R_{\text{eff}}(0, B(R)^c) \ge \lambda^{-1}r(R), \\ & R_{\text{eff}}(0,y) \le \lambda r(d(0,y)),\ \forall y \in B(R)\}. \end{aligned}$$

As we see, $v(\cdot)$ gives the volume growth order and $r(\cdot)$ gives the resistance growth order.

We now make the following assumptions concerning the graphs $(\Gamma(\omega))$. This involves upper and lower bounds on the volume, as well as an estimate which says that R is likely to be in $J(\lambda)$ for large enough λ. Note that some assumption includes another assumption, because we assume part of them for each theorem and proposition.

Assumption 1.2. *(1) There exist* $\lambda_0 > 1$ *and* $p(\lambda)$ *which goes to* 0 *as* $\lambda \to \infty$ *such that,*

$$\mathbb{P}(R \in J(\lambda)) \geq 1 - p(\lambda) \qquad \textit{for each} \;\; R \geq 1, \lambda \geq \lambda_0. \tag{1.13}$$

(2) $\mathbb{E}[R_{\rm eff}(0, B(R)^c)V(R)] \leq c_1 v(R) r(R)$.
(3) There exist $q_0, c_2 > 0$ *such that*

$$p(\lambda) \leq \frac{c_2}{\lambda^{q_0}}. \tag{1.14}$$

(4) There exist $q_0, c_3 > 0$ *such that*

$$p(\lambda) \leq \exp(-c_3 \lambda^{q_0}). \tag{1.15}$$

We have the following consequences of Assumption 1.2 for random graphs. Some of the results apply also to the random walk started from an arbitrary point $x \in G(\omega)$. Some statements in the first proposition involve the annealed law

$$P^* = \mathbb{P} \times P_\omega^0. \tag{1.16}$$

Let $\mathcal{I}(\cdot)$ be the inverse function of $(v \cdot r)(\cdot)$.

Proposition 1.3. *Suppose Assumption 1.2(1) holds. Let* $n \geq 1$, $R \geq 1$. *Then*

$$\mathbb{P}(\theta^{-1} \leq \frac{E_\omega^0 \tau_R}{v(R) r(R)} \leq \theta) \to 1 \qquad \textit{as } \theta \to \infty, \tag{1.17}$$

$$\mathbb{P}(\theta^{-1} \leq v(\mathcal{I}(n)) p_{2n}^\omega(0,0) \leq \theta) \to 1 \qquad \textit{as } \theta \to \infty, \tag{1.18}$$

$$P^*(\frac{d(0, X_n)}{\mathcal{I}(n)} < \theta) \to 1 \qquad \textit{as } \theta \to \infty. \tag{1.19}$$

$$P^*(\theta^{-1} < \frac{1 + d(0, X_n)}{\mathcal{I}(n)}) \to 1 \qquad \textit{as } \theta \to \infty. \tag{1.20}$$

In each case the convergence is uniform in n *and* R.

Since $P_\omega^0(X_{2n} = 0) \approx 1/v(\mathcal{I}(n))$, we cannot replace $1 + d(0, X_n)$ by $d(0, X_n)$ in (1.20).

Proposition 1.4. *Suppose Assumption 1.2(1) and (2) hold. Then*

$$c_1 v(R) r(R) \leq \mathbb{E}(E_\omega^0 \tau_R) \leq c_2 v(R) r(R) \textit{ for all } R \geq 1, \tag{1.21}$$

$$\frac{c_3}{v(\mathcal{I}(n))} \leq \mathbb{E}(p_{2n}^\omega(0,0)) \textit{ for all } n \geq 1, \tag{1.22}$$

$$c_4 \mathcal{I}(n) \leq \mathbb{E}(E_\omega^0 d(0, X_n)) \textit{ for all } n \geq 1. \tag{1.23}$$

Assume in addition that there exist $c_5 > 0, \lambda_0 > 1$ *and* $q_0' > 2$ *such that*

$$\mathbb{P}(\lambda^{-1} v(R) \leq V(R), \; R_{\rm eff}(0,y) \leq \lambda r(d(0,y)), \; \forall y \in B(R)) \geq 1 - \frac{c_5}{\lambda^{q_0'}}, \tag{1.24}$$

for each $R \geq 1, \lambda \geq \lambda_0$. *Then*

$$\mathbb{E}(p_{2n}^\omega(0,0)) \leq \frac{c_6}{v(\mathcal{I}(n))} \textit{ for all } n \geq 1. \tag{1.25}$$

We do not have an upper bound in (1.23); see Example 2.6 in [6].

The additional assumption that $p(\lambda)$ decays either polynomially or exponentially enable us to obtain limit theorems. Both of the following theorems refer to the random walk started at an arbitrary point $x \in G(\omega)$.

Theorem 1.5. *(I) Suppose Assumption 1.2(1) and (3) hold. Then there exist* $\beta_1, \beta_2, \beta_3, \beta_4 < \infty$, *and a subset* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that the following statements hold.*
(a) For each $\omega \in \Omega_0$ *and* $x \in G(\omega)$ *there exists* $N_x(\omega) < \infty$ *such that*

$$\frac{(\log n)^{-\beta_1}}{v(\mathcal{I}(n))} \le p^{\omega}_{2n}(x,x) \le \frac{(\log n)^{\beta_1}}{v(\mathcal{I}(n))}, \quad n \ge N_x(\omega). \tag{1.26}$$

(b) For each $\omega \in \Omega_0$ *and* $x \in G(\omega)$ *there exists* $R_x(\omega) < \infty$ *such that*

$$(\log R)^{-\beta_2} v(R) r(R) \le E^x_{\omega}\tau_R \le (\log R)^{\beta_2} v(R) r(R), \quad R \ge R_x(\omega). \tag{1.27}$$

(c) Let $Y_n = \max_{0 \le k \le n} d(0, X_k)$. *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$ *there exist* $N_x(\omega, \varpi), R_x(\omega, \varpi)$ *such that* $P^x_{\omega}(N_x < \infty) = P^x_{\omega}(R_x < \infty) = 1$, *and such that*

$$(\log n)^{-\beta_3} \mathcal{I}(n) \le Y_n(\omega, \varpi) \le (\log n)^{\beta_3} \mathcal{I}(n), \quad n \ge N_x(\omega, \varpi), \tag{1.28}$$

$$(\log R)^{-\beta_4} v(R) r(R) \le \tau_R(\omega, \varpi) \le (\log R)^{\beta_4} v(R) r(R), \qquad R \ge R_x(\omega, \varpi). \tag{1.29}$$

(II) Suppose Assumption 1.2(1) and (4) hold. Then there exist $\beta_1, \beta_2 < \infty$, *and* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that (1.26) and (1.27) hold with* $\log\log n$ *(resp.* $\log\log R$*) instead of* $\log n$ *(resp.* $\log R$*).*
(III) Suppose Assumption 1.2(1) and (3) hold. Suppose further that v, r *satisfy the following in addition to (1.12):*

$$C_3^{-1} R^D (\log R)^{-m_1} \le v(R) \le C_3 R^D (\log R)^{m_1}, \;\; C_4^{-1} R^{\alpha} (\log R)^{-m_2} \le r(R) \le C_4 R^{\alpha} (\log R)^{m_2} \tag{1.30}$$

for all $0 < R$, *where* $C_3, C_4 \ge 1$, $1 \le D$, $0 < \alpha \le 1$ *and* $0 < m_1, m_2$. *Then the following statements hold.*
(a) $d_s(G) := -2\lim_{n\to\infty} \frac{\log p^{\omega}_{2n}(x,x)}{\log n} = \frac{2D}{D+\alpha}$, $\mathbb{P}$*–a.s., and the random walk is recurrent.*
(b) $\lim_{R\to\infty} \frac{\log E^x_{\omega}\tau_R}{\log R} = D + \alpha$.
(c) Let $W_n = \{X_0, X_1, \ldots, X_n\}$ *and let* $S_n = \mu(W_n) = \sum_{x \in W_n} \mu_x$. *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$,

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{D}{D+\alpha}, \quad P^x_{\omega}\text{–a.s.}. \tag{1.31}$$

Remark 1.6. *1. Let*

$$\begin{aligned} \hat{J}(\lambda) \;\; = \;\; & \{R \in [1,\infty] : \lambda^{-1} v(R) \le V(R) \le \lambda v(R), R_{\text{eff}}(0, B(R)^c) \ge \lambda^{-1} r(R), \\ & R_{\text{eff}}(0,y) \le r(d(0,y)), \; \forall y \in B(R)\}. \end{aligned}$$

(Note that $J(\lambda)$ *contains* λ *in the last inequality whereas* $\hat{J}(\lambda)$ *does not.) Assume that Assumption 1.2(1) holds w.r.t.* $\hat{J}(\lambda)$ *and further the following holds:*

$$\mathbb{E}[1/V(R)] \le c_1/v(R). \tag{1.32}$$

(Note that this condition is a bit weaker than (1.24).) Then, (1.22) and (1.25) hold.
2. If one chooses the resistance metric $R_{\rm eff}(\cdot,\cdot)$ as the metric $d(\cdot,\cdot)$, then clearly $R_{\rm eff}(0,y) \le \lambda r(d(0,y))$ holds with $r(x)=x$ and $\lambda=1$.
3. If all the vertices in $G(\omega)$ have degree bounded by a constant c_0 for all $\omega \in \Omega_0$, then $|W_n| \le S_n \le c_0|W_n|$. Hence, under the assumption (1.30), Theorem 1.5(III)(c) implies also that

$$\lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{D}{D+\alpha}, \quad P_\omega^x\text{-a.s.} \tag{1.33}$$

2 Application: Long range percolation

In this section, we will apply the theorem in Section 1 to the long range percolation. Let $\mathbf{p} = \{p(n)\}_{n=1}^\infty$ be a sequence of real numbers. Each unoriented pair of distinct points $x, y \in \mathbb{Z}^d$ is connected by an unoriented bond with probability $p(x,y) = p(y,x) = p(|x-y|)$, independently of other pairs. Here, $|x-y| = \sum_{i=1}^d |x_i - y_i|$. We consider the situation that $\mathbf{p}$ satisfies

$$\lim_{n\to\infty} \frac{p(n)}{\beta n^{-s}} = 1, \tag{2.1}$$

for some $s>0$, $\beta>0$.

Let μ_{xy} be $\{0,1\}$-valued random variable, which takes 1 if x and y are connected by a bond and takes 0 if there is no bond between x, y. ($\mu_{xy} = \mu_{yx}$, and $\mu_{xx}=0$.) $\mu_x = \sum_{y\in\mathbb{Z}^d} \mu_{xy}$ stands for the number of bonds which have x as an endpoint. $G = \mathbb{Z}^d$ is the vertex set and $E = \{\langle x,y\rangle | \mu_{xy}=1\}$ is the edge set of the corresponding random graph. (We identify $\langle x,y\rangle = \langle y,x\rangle$.)

Here, we give some comments on the backgrounds. Random walks on long-range percolation clusters is discussed in [9]. Let $\mathbf{p}$ be a sequence satisfying (2.1) and $p(n) \in [0,1)$ for $n \ge 1$. The random graphs are locally finite if and only if $s > d$, and we can define random walks in such a case. We choose $\mathbf{p}$ for which there exists a unique ∞-cluster with probability 1. Then, the main results in [9] are the following:

(1) For $d=1$, random walks are transient if $1<s<2$, and recurrent if $s=2$.
(2) For $d=2$, random walks are transient if $2<s<4$, and recurrent if $s\ge 4$.

In the above, the case $d=1$, $s>2$ is not mentioned because there is no ∞-cluster in such a case. From now on, we explore the case $d=1$, $p(1)=1$, and for $n\ge 2$, $p(n)\in[0,1)$ satisfies (2.1) for some $s>2$. In this case, the effects of long bonds are not so strong, and as we will see later, behaviours of random walks are similar to the random walk on $\mathbb{Z}$. Also, we will refer to some kind of discontinuity on $s=2$.

Let $d(x,y) := |x-y|$ and define $B(R), V(R)$ with respect to this metric. Then, since $V(R) \ge \sharp B(R) = 2R-1$,

$$\mathbb{P}(V(R) \le \lambda^{-1}R) = 0 \qquad \text{if} \quad \lambda > 1.$$

Further, we have

$$\mathbb{P}(V(R) \ge \lambda R) \le \frac{\mathbb{E}V(R)}{\lambda R} \le \frac{c}{\lambda}.$$

The upper bound on the resistance is obvious by comparing percolation clusters with $\mathbb{Z}$;

$$\mathbb{P}\left[\bigcup_{y\in B(R)}\{R_{\text{eff}}(0,y)>\lambda d(0,y)\}\right]=0 \qquad \text{if} \quad \lambda>1, \tag{2.2}$$

and

$$\mathbb{E}[R_{\text{eff}}(0,B(R)^c)V(R)]\le R\mathbb{E}[V(R)]\le c_1R^2. \tag{2.3}$$

The remained work is the lower bound on the resistance. We have the following.

Proposition 2.1. *Let $q=1$ for $s>3$, and let q be an any value in $(0,s-2)$ for $2<s\le 3$. Then, there exists $c_1=c_1(\beta,s,q)>0$, such that for each $R\ge 1$,*

$$\mathbb{P}[R_{\text{eff}}(0,B(R)^c)<\lambda^{-1}R]\le c_1\lambda^{-q}. \tag{2.4}$$

Proof. First, we apply the "projecting long bonds" method in Lemma 3.8 in [9] to our case. For each ω, we construct a new weighted graph from the original one in the following way.

(1) If a bond $\langle x,y\rangle$ such that $x,y\in\mathbb{Z}$, $x+2\le y$ exists, then, divide the bond into $y-x$ short bonds.
(2) For each $i=1,\cdots,y-x$, replace the i-th short bond by a bond which has $x+i-1$ and $x+i$ as its endpoints and has weight $y-x$.
(3) Repeat (1),(2) for all bonds except nearest-neighbour bonds.

We use the notation $\tilde{R}_{\text{eff}}$ for the resistance on the new graph. By the *shorting law* in the terminology of the electrical network, the resistance does not increase in the above procedures. And from the way of construction, we can see that

$$\tilde{R}_{\text{eff}}(0,R)=\sum_{i=1}^{R}(\sum_{e\in A_i}|e|)^{-1}, \tag{2.5}$$

where A_i is the set of all $\langle u,v\rangle$ satisfying $u,v\in\mathbb{Z}$, $u<v$, $\mu_{uv}=1$, and $[u,v]\supset[i-1,i]$. (In other words, A_i is the collection of bonds crossing over $[i-1,i]$.) We denote $|e|=|u-v|$ for $e=\langle u,v\rangle$. It is easy to see that

$$\begin{aligned}
\mathbb{E}[R_{\text{eff}}(0,B(R)^c)^{-q}] &\le \mathbb{E}[\tilde{R}_{\text{eff}}(0,B(R)^c)^{-q}]\\
&= \mathbb{E}[\{\tilde{R}_{\text{eff}}(0,R)^{-1}+\tilde{R}_{\text{eff}}(0,-R)^{-1}\}^q]\\
&\le \mathbb{E}[\tilde{R}_{\text{eff}}(0,R)^{-q}+\tilde{R}_{\text{eff}}(0,-R)^{-q}]=2\mathbb{E}[\tilde{R}_{\text{eff}}(0,R)^{-q}],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\tilde{R}_{\text{eff}}(0,R)^{-q}] &= \mathbb{E}\left[\{\sum_{i=1}^{R}(\sum_{e\in A_i}|e|)^{-1}\}^{-q}\right]\\
&\le R^{-q-1}\sum_{i=1}^{R}\mathbb{E}\left[(\sum_{e\in A_i}|e|)^q\right].
\end{aligned}$$

We have used the Hölder inequality in the last estimate. The expectation in the right hand side is finite for each q ;

$$\begin{aligned}\mathbb{E}\left[(\sum_{e\in A_i}|e|)^q\right] &\le \mathbb{E}[\sum_{e\in A_i}|e|^q] \\ &= \mathbb{E}[\sum_{n=1}^{\infty}\sum_{k=1}^{n}1_{\{\mu_{k-n,k}=1\}}n^q] \\ &= \sum_{n=1}^{\infty}n^q\sum_{k=1}^{n}\mathbb{P}[\mu_{k-n,k}=1] \\ &\le c_2\sum_{n=1}^{\infty}n^{q-s+1}<\infty.\end{aligned}$$

Combining these calculations, we have

$$\mathbb{E}[R_{\mathrm{eff}}(0,B(R)^c)^{-q}]\le c_3R^{-q}. \tag{2.6}$$

So, by the Chebyshev inequality and (2.6),

$$\begin{aligned}\mathbb{P}[R_{\mathrm{eff}}(0,B(R)^c)\le\lambda^{-1}R] &\le \lambda^{-q}R^q\mathbb{E}[R_{\mathrm{eff}}(0,B(R)^c)^{-q}] \\ &\le c_3\lambda^{-q},\end{aligned}$$

which completes the proof. □

From the above estimates, we have the following. Below, $a_n \sim b_n$ stands for $\lim_{n\to\infty} a_n/b_n = 1$.

Theorem 2.2. *The long-range percolation on $\mathbb{Z}$ with $p(n) \sim \beta n^{-s}$ for $s > 2$, $\beta > 0$, $p(1) = 1$ satisfies Assumption 1.2(1),(2) and (3) with $v(x) = r(x) = x$.*

In this case, the additional assumption (1.24) also holds directly from the condition $p(1) = 1$. So, we obtain the conclusion of Proposition 1.3, 1.4 and Theorem 1.5(I,III) with $v(x) = r(x) = x$.

Remark 2.3. *(1) We have proved that, when $s > 2$, $p_{2n}(x,x)$ is the order of $n^{-\frac{1}{2}}$. On the other hand, from the transience result in [9], it is natural to see that, when $1 < s < 2$, $p_{2n}(x,x) \sim n^{-\xi(s)}$ ($\xi(s) > 1$) in some sense, though we do not have a rigorous proof. Hence, Theorem 2.2 implies that the order of the heat kernel is* discontinuous *at $s = 2$.*

(2) In the one-dimensional long-range percolation model, the phenomena at the point $s = 2$ are non-trivial. In [1], the discontinuity of the percolation density at $s = 2$ is shown. In the recent study in [8] , long range percolation mixing time is considered and it is shown that the order of the mixing time changes discontinuously when $s = 2$. In [18], estimates of effective resistance are given for general d, s, and the discontinuity mentioned in (1) is shown in a sense of effective resistance.

(3) Note that the proof of Theorem 2.2 can be also applied to some case where we do not assume the independence of open bonds. $s = 2$ is not necessarily critical in the case.

Remark 2.4. *There is a question whether the log-corrections in Theorem 1.5(I) can be weaken or not. To answer this question, it is crucial to study the fluctuation of the volume and resistance.*

First, let us consider the fluctuation of the volume. The following large deviation estimate holds for $s > 1$.

$$\mathbb{P}[V(R) \geq c_1 R] \leq \exp\{-c_2 R\}.$$

Since $\sum_{R=1}^{\infty} \exp\{-c_2 R\} < \infty$, by the Borel–Cantelli lemma, we have

$$\mathbb{P}[\limsup_{R\to\infty} \frac{V(R)}{R} \leq c_1] = 1.$$

The lower bound of $V(R)/R$ is trivial from $p(1) = 1$. Therefore, there is no fluctuation of the volume such as Proposition 2.8 in [7].

Next, we consider the fluctuation of the resistance. When $2 < s < 3$, by calculating $\mathbb{E}[\mathcal{E}(g,g)]$ for appropriate g, we can see that $\mathbb{E}[R_{\text{eff}}(0, B(R)^c)^{-1}] \leq cR^{2-s}$. It seems to us that this is the best estimate one can obtain. If so, there may be some fluctuation of the resistance.

We also give an example satisfying Assumption 1.2 (4) instead of (3).

Theorem 2.5. *We consider the long-range percolation on $\mathbb{Z}$ with $p(n) \sim e^{-cn}$ for $c > 0$, $p(1) = 1$. Then, Assumption 1.2(1),(2) and (4) are satisfied with $v(x) = r(x) = x$.*

Proof. The estimate for volume is easy, and the upper bound of resistance is trivial. We will show that $\mathbb{P}[R_{\text{eff}}(0, B(R)^c) \leq \lambda^{-1} R] \leq e^{-c_1 \lambda}$ for some $c_1 > 0$.

$$\begin{aligned}
\mathbb{P}[R_{\text{eff}}(0, B(R)^c) \leq \lambda^{-1} R] &= \mathbb{P}[\inf\{\mathcal{E}(f,f) : f(0) = 1, f|_{B(R)^c} \equiv 0\} \geq \lambda R^{-1}] \\
&\leq \mathbb{P}[\mathcal{E}(g,g) \geq \lambda R^{-1}] \\
&\leq e^{-c_2 \lambda} \mathbb{E}[\exp\{c_2 R \mathcal{E}(g,g)\}],
\end{aligned}$$

where $g(x) = 1_{B(R)}(x)(1 - R^{-1}|x|)$, and c_2 is a positive constant determined later. We see that

$$\begin{aligned}
\mathbb{E}[\exp\{c_2 R \mathcal{E}(g,g)\}] &= \mathbb{E}[\exp\{c_3 R \sum_{x,y\in\mathbb{Z}} |g(x) - g(y)|^2 \mu_{xy}\}] \\
&= \prod_{x,y\in\mathbb{Z}} \mathbb{E}[\exp\{c_3 R |g(x) - g(y)|^2 \mu_{xy}\}] \\
&= \prod_{x,y\in\mathbb{Z}} \left\{1 + (\exp\{c_3 R |g(x) - g(y)|^2\} - 1) e^{-c|x-y|}\right\} \equiv \prod_{x,y\in\mathbb{Z}} I_{xy},
\end{aligned}$$

where $c_3 = \frac{c_2}{2}$. Clearly, $\prod_{x,y\in B(R)^c} I_{xy} = 1$, and

$$\begin{aligned}
\prod_{x,y\in B(R)} I_{xy} &\leq \prod_{x,y\in B(R)} \left\{1 + (\exp\{c_3 R^{-1} |x-y|^2\} - 1) e^{-c|x-y|}\right\} \\
&\leq \prod_{n=1}^{2R} \{1 + (\exp(c_3 R^{-1} n^2) - 1) e^{-cn}\}^{2R} \equiv X.
\end{aligned}$$

Now

$$\begin{aligned}
\log X &= \sum_{n=1}^{2R} 2R \log\{1 + (\exp(c_3 R^{-1} n^2) - 1) e^{-cn}\} \\
&\leq 2R \sum_{n=1}^{2R} (\exp(c_3 R^{-1} n^2) - 1) e^{-cn} \\
&= 2R\{\sum_{n=1}^{[\sqrt{R}]} + \sum_{n=[\sqrt{R}]+1}^{2R}\} \equiv 2R(S_1 + S_2).
\end{aligned}$$

We first estimate S_1.

$$S_1 \le \sum_{n=1}^{[\sqrt{R}]} c_4 R^{-1} n^2 e^{-cn} \le c_4 R^{-1} \sum_{n=1}^{\infty} n^2 e^{-cn} \le c_5 R^{-1}.$$

For the estimate of S_2, we choose c_3 sufficiently small so that

$$S_2 \le \sum_{n=[\sqrt{R}]+1}^{2R} \exp\{c_3 R^{-1} n^2\} e^{-cn} \le \sum_{n=[\sqrt{R}]+1}^{2R} \exp\{-(c-2c_3)n\} \le e^{-c_6\sqrt{R}}.$$

Therefore, we have $\log X \le c_7$, and $X \le e^{c_7}$. Furthermore, we see that

$$\begin{aligned}\prod_{x\in B(R), y\in B(R)^c} I_{xy} &= \prod_{x\in B(R)} \prod_{y\in B(R)^c} \left\{1 + (\exp\{c_3 R^{-1}(R-|x|)^2\} - 1)e^{-c|x-y|}\right\} \\ &\equiv \prod_{x\in B(R)} a_x,\end{aligned}$$

and

$$\begin{aligned}\log a_x &= \sum_{y\in B(R)^c} \log\left\{1 + (\exp\{c_3 R^{-1}(R-|x|)^2\} - 1)e^{-c|x-y|}\right\} \\ &\le \sum_{y\in B(R)^c} (\exp\{c_3 R^{-1}(R-|x|)^2\} - 1)e^{-c|x-y|} \\ &\le c_8 \exp\{c_3 R^{-1}(R-|x|)^2\} e^{-c(R-|x|)} \\ &\le c_8 e^{c_3(R-|x|)} e^{-c(R-|x|)} = c_8 e^{-c_9(R-|x|)},\end{aligned}$$

for sufficiently small c_3. Thus,

$$\begin{aligned}\prod_{x\in B(R)} a_x &\le \prod_{x\in B(R)} \exp\{c_8 e^{-c_9(R-|x|)}\} \\ &= \exp\{c_8 \sum_{x\in B(R)} e^{-c_9(R-|x|)}\} \le \exp\{2c_8 \sum_{n=1}^{\infty} e^{-c_9 n}\} \le c_{10}.\end{aligned}$$

From these estimates, we obtain the result for suitable c_1 . $\square$

3 Proof of the main results

In Section 3.1, we prove several preliminary results for random walk on a fixed but general graph. Then in Section 3.2 we apply these results to prove Propositions 1.3–1.4 and Theorem 1.5. We adopt the convention that if we cite elsewhere the constant c_1 in Lemma 3.2 (for example), we denote it as $c_{3.2.1}$. C_1, C_2 stand for the constants in (1.12).

3.1 Estimates for general graphs

In this section, we fix an infinite locally-finite connected graph $\Gamma = (G, E)$, and use bounds on the quantities $V(R)$ and $R_{\text{eff}}(0, B(R)^c)$ to control $E^0\tau_R$, $p_n(0,0)$ and $E^0 d(0, X_n)$, where $d(\cdot,\cdot)$ is a metric on G. To deal with issues related to the possible bipartite structure of the graph, we will consider $p_n(x,y) + p_{n+1}(x,y)$.

Proposition 3.1. *Let* $0 \in G$ *and* $f_n(y) = p_n(0,y) + p_{n+1}(0,y)$.
(a) Let $R \geq 1$ *and assume that*

$$V(0,R) \geq \lambda^{-1} v(R), \; R_{\mathrm{eff}}(0,y) \leq \lambda r(d(0,y)) \qquad \text{for all } \; y \in B(R). \tag{3.1}$$

Then

$$f_n(0) \leq \frac{c_1 \lambda}{v(\mathcal{I}(n))} \qquad \text{for } \; \frac{1}{2} v(R) r(R) \leq n \leq v(R) r(R).$$

(b) We have

$$|f_n(y) - f_n(0)|^2 \leq \frac{c_2}{n} R_{\mathrm{eff}}(0,y) p_{2\lfloor n/2 \rfloor}(0,0).$$

Proof. (a) A natural modification of the third equation in [5, Proposition 3.3] using $R_{\mathrm{eff}}(0,y) \leq \lambda r(d(0,y)) \leq \lambda r(R)$ gives

$$f_{2n}(0)^2 \leq \frac{c}{V(0,R)^2} + \frac{c \lambda r(R) f_{2n}(0)}{n}, \qquad \text{for all } n \geq 1, \; R > 0.$$

Using $a+b \leq 2(a \vee b)$, we see that $f_{2n}(0) \leq (c'/V(0,R)) \vee (c' \lambda r(R)/n)$. So, by setting $v(R)r(R)/2 \leq n \leq v(R) r(R)$,

$$f_{2n}(0) \leq (c'/V(0,R)) \vee (2c'\lambda / v(R)) \leq 2c'\lambda/v(R) \leq c_1 \lambda / v(\mathcal{I}(n)),$$

where we used $V(0,R) \geq \lambda^{-1} v(R)$ in the second inequality.

(b) Using (1.10),

$$|f_n(y) - f_n(0)|^2 \leq R_{\mathrm{eff}}(0,y) \mathcal{E}(f_n, f_n).$$

We then use [5, Lemma 3.10] to bound $\mathcal{E}(f_n, f_n)$. □

Proposition 3.2. *Let* $R \geq 1$, $m \geq 1$, $\varepsilon^{\alpha_1} \leq 1/(4C_2 m \lambda)$. *Write* $B = B(0,R)$, $B' = B(0, \frac{1}{2}\varepsilon R)$, $V = V(0,R)$, $V' = V(0, \frac{1}{2}\varepsilon R)$ *and suppose* $R_{\mathrm{eff}}(0,y) \leq \lambda r(d(0,y))$ *for all* $y \in B(R)$.
(a) For $x \in B$,

$$E^x \tau_R \leq 2 \lambda r(R) V. \tag{3.2}$$

(b) Suppose further that

$$R_{\mathrm{eff}}(x, B^c) \geq r(R)/m \qquad \text{for } x \in B(0, \varepsilon R). \tag{3.3}$$

Then for $x \in B'$,

$$E^x \tau_R \geq \frac{r(R) V'}{4m}, \tag{3.4}$$

$$P^x(\tau_R > n) \geq \frac{V'}{8 m \lambda V} - \frac{n}{2 r(R) \lambda V} \qquad \text{for } n \geq 0, \tag{3.5}$$

$$p_{2n}(x,x) \geq \frac{c_1 (V')^2}{m^2 \lambda^2 V^3} \qquad \text{for } n \leq \frac{r(R) V'}{8m}. \tag{3.6}$$

Proof. For any $z \in B$ we have

$$E^z \tau_B = \sum_{y \in B} g_B(z,y)\mu_y,$$

where $g_B(\cdot,\cdot)$ is the Green kernel of the Markov chain killed on exiting B.

(a) Since $R_{\text{eff}}(z,B^c) \le R_{\text{eff}}(0,z) + R_{\text{eff}}(0,B^c) \le 2\lambda r(R)$ for any $z \in B$,

$$E^z \tau_B = \sum_{y \in B} g_B(z,y)\mu_y \le \sum_{y \in B} g_B(z,z)\mu_y = R_{\text{eff}}(z,B^c)V(0,R) \le 2\lambda r(R)V(0,R), \tag{3.7}$$

where we used the fact $g_B(z,z) = R_{\text{eff}}(z,B^c)$ in the second equality. (For the proof of this fact, see, for example, section 3.2 in [5].)

(b) Let $p_B^x(y) = g_B(x,y)/g_B(x,x)$. Since $\mathcal{E}(p_B^x,p_B^x) = R_{\text{eff}}(x,B^c)^{-1} = g_B(x,x)^{-1}$ and so if $x,y \in B'$

$$|1 - p_B^x(y)|^2 \le R_{\text{eff}}(x,y)R_{\text{eff}}(x,B^c)^{-1} \le \frac{2\lambda r(\varepsilon R)m}{r(R)} \le 2C_2 m\varepsilon^{\alpha_1}\lambda \le 1/2,$$

where (1.12) is used in the third inequality. Hence $p_B^x(y) \ge 1 - 2^{-1/2} \ge \frac{1}{4}$. So,

$$E^x \tau_R \ge \sum_{y \in B'} g_B(x,x)p_B^x(y)\mu_y \ge \frac{1}{4}\mu(B')R_{\text{eff}}(x,B^c) \ge r(R)\mu(B')/(4m).$$

By the Markov property, (3.2) and (3.4), for $x \in B'$,

$$\frac{r(R)V'}{4m} \le E^x[\tau_R] \le n + E^x[1_{\{\tau_R > n\}}E^{X_n}(\tau_R)] \le n + 2\lambda r(R)VP^x(\tau_R > n),$$

for all $n \ge 1$. Rearranging this gives (3.5).

By (3.5),

$$P^x(X_n \in B) \ge P^x(\tau_R > n) \ge \frac{(r(R)V'/4m) - n}{2\lambda r(R)V}.$$

So, if $n \le r(R)V'/(8m)$ then

$$P^x(X_n \in B) \ge \frac{c_2 V'}{m\lambda V}. \tag{3.8}$$

By the Chapman–Kolmogorov equation and the Cauchy-Schwarz inequality,

$$P^x(X_n \in B)^2 = (\sum_{y \in B} p_n(x,y)\mu_y)^2 \le \mu(B)\sum_{y \in B} p_n(x,y)^2\mu_y \le p_{2n}(x,x)V,$$

and using (3.8) gives (3.6). $\square$

Recall the set $J(\lambda)$ defined in Definition 1.1. We will need to know that bounds in the following are polynomial in λ. To indicate this, we write $c_i(\lambda)$ to denote positive constants of the form $c_i(\lambda) = c_i\lambda^{\pm q_i}$. The sign of q_i is such that statements become weaker as λ increases. The following proposition controls the mean escape times and transition probabilities.

Proposition 3.3. *Let $\lambda > 1$.*
(1) Suppose that $R \in J(\lambda)$. Then there exists $c_1(\lambda)$ such that

$$E^z \tau_R \le 2\lambda^2 v(R) r(R) \qquad \text{for } z \in B(R), \tag{3.9}$$

$$p_n(0,0) + p_{n+1}(0,0) \le \frac{c_1(\lambda)}{v(\mathcal{I}(n))} \quad \text{if } \frac{1}{2} v(R) r(R) \le n \le v(R) r(R), \tag{3.10}$$

$$p_n(0,y) + p_{n+1}(0,y) \le \frac{c_1(\lambda)}{v(\mathcal{I}(n))} \quad \text{for } y \in B(R) \text{ if } \frac{1}{2} v(R) r(R) \le n \le v(R) r(R). \tag{3.11}$$

(2) There exist $c_2(\lambda), \cdots, c_7(\lambda)$ such that, if $R, c_2(\lambda) R \in J(\lambda)$, then

$$c_3(\lambda) v(R) r(R) \le E^x \tau_R \quad \text{for } x \in B(c_2(\lambda) R), \tag{3.12}$$

$$P^0(\tau_R > c_4(\lambda) v(R) r(R)) \ge c_5(\lambda), \tag{3.13}$$

$$p_{2n}(0,0) \ge \frac{c_6(\lambda)}{v(\mathcal{I}(n))} \quad \text{for } \tfrac{1}{2} c_7(\lambda) v(R) r(R) \le n \le c_7(\lambda) v(R) r(R). \tag{3.14}$$

Proof. (1) (3.9) is immediate by Proposition 3.2(a), and (3.10) follows from Proposition 3.1(a). Using Proposition 3.1(b), and writing $f_n(y) = p_n(0,y) + p_{n+1}(0,y)$, $n' = 2\lfloor n/2 \rfloor$,

$$f_n(y) \le f_n(0) + |f_n(y) - f_n(0)| \le f_n(0) + (c R_{\text{eff}}(0,y) n^{-1} p_{n'}(0,0))^{1/2}. \tag{3.15}$$

So, by (3.10) and by the definition of $J(\lambda)$, if $y \in B(R)$ then we have (3.11), namely

$$f_n(y) \le \frac{c'(\lambda)}{v(\mathcal{I}(n))}. \tag{3.16}$$

(2) Set $m = 2\lambda$, $\varepsilon^{\alpha_1} = 1/(2C_2 m\lambda) = 1/(4C_2\lambda^2)$. Since $R \in J(\lambda)$, we have, for $x \in B(0, \varepsilon R)$,

$$\frac{r(R)}{\lambda} \le R_{\text{eff}}(0, B^c) \le R_{\text{eff}}(0,x) + R_{\text{eff}}(x, B^c) \le \lambda r(\varepsilon R) + R_{\text{eff}}(x, B^c) \le \lambda C_2 \varepsilon^{\alpha_1} r(R) + R_{\text{eff}}(x, B^c),$$

where we used (1.12) in the last inequality. Hence $R_{\text{eff}}(x, B^c) \ge r(R)/m$ if $x \in B(0, \varepsilon R)$, and so the assumption of Proposition 3.2(b) holds. Since $R \in J(\lambda)$, $V(R) \le \lambda v(R)$. Also $\frac{1}{2}\varepsilon R = R/(2^{1+2/\alpha_1} C_2^{1/\alpha_1} \lambda^{2/\alpha_1}) =: c_2(\lambda) R \in J(\lambda)$, so $V' \ge \lambda^{-1} v(c_2(\lambda) R) \ge c'(\lambda) v(R)$ for some $c'(\lambda) > 0$; the bounds now follow from Proposition 3.2(b). □

Next we apply similar arguments to control $d(0, X_n)$, beginning with a preliminary lemma. Recall that T_A was defined in (1.6) to be the hitting time of $A \subset G$.

Lemma 3.4. *Let $\lambda \ge 1$ and $0 < \varepsilon^{\alpha_1} \le 1/(2C_2\lambda^2)$. If $R \in J(\lambda)$, and $y \in B(\varepsilon R)$ then*

$$P^y(\tau_R < T_0) \le 2C_2 \varepsilon^{\alpha_1} \lambda^2, \tag{3.17}$$

$$P^0(\tau_R < T_y) \le C_2 \varepsilon^{\alpha_1} \lambda^2. \tag{3.18}$$

Proof. If A and B are disjoint subsets of G and $x \notin A \cup B$, then (see [11, (4)])

$$P^x(T_A < T_B) \le \frac{R_{\text{eff}}(x, B)}{R_{\text{eff}}(x, A)}.$$

Let $d(0,y) \le \varepsilon R$. Then $R_{\rm eff}(y,0) \le \lambda r(d(y,0)) \le \lambda r(\varepsilon R) \le \lambda C_2 \varepsilon^{\alpha_1} r(R)$, while

$$R_{\rm eff}(y, B(R)^c) \ge R_{\rm eff}(0, B(R)^c) - R_{\rm eff}(0,y) \ge r(R)/\lambda - \lambda C_2 \varepsilon^{\alpha_1} r(R) \ge r(R)/2\lambda.$$

So,

$$P^y(\tau_R < T_0) \le \frac{R_{\rm eff}(y,0)}{R_{\rm eff}(y,B(R)^c)} \le 2C_2 \varepsilon^{\alpha_1} \lambda^2.$$

Similarly,

$$P^0(\tau_R < T_y) \le \frac{R_{\rm eff}(0,y)}{R_{\rm eff}(0,B(R)^c)} \le C_2 \varepsilon^{\alpha_1} \lambda^2.$$

□

Proposition 3.5. *For each $\lambda > 1$, there exist $c_1(\lambda), \cdots, c_{10}(\lambda)$ such that the following hold.*
(a) Let $\varepsilon < c_1(\lambda)$ and $R, \varepsilon R, c_2(\lambda)\varepsilon R \in J(\lambda)$. Then

$$P^y\Big(\tau_R \le c_3(\lambda) v(\varepsilon R) r(\varepsilon R)\Big) \le c_4(\lambda) \varepsilon^{\alpha_1}, \qquad \text{for } y \in B(\varepsilon R). \tag{3.19}$$

(b) Let $n \ge 1$, $M \ge 1$, and set $R = M\mathcal{I}(n)$. If $R, c_5(\lambda)R/M, c_6(\lambda)R/M \in J(\lambda)$, then

$$P^0\Big(\frac{d(0,X_n)}{\mathcal{I}(n)} > M\Big) \le \frac{c_7(\lambda)}{M^{\alpha_1}}. \tag{3.20}$$

(c) Let $R = \mathcal{I}(n)$ and $\theta \in (0,1]$. If $R, \theta R \in J(\lambda)$ then

$$P^0\Big(X_n \in B(\theta R)\Big) \le c_8(\lambda) \theta^{d_1}. \tag{3.21}$$

(d) Let $R = \mathcal{I}(n)$. If $R, c_9(\lambda) R \in J(\lambda)$ then

$$P^0(\tau_{c_9(\lambda)R} \le n) \ge P^0\Big(X_n \notin B(0, c_9(\lambda)R)\Big) \ge \tfrac{1}{2}. \tag{3.22}$$

Hence

$$E^0 d(0,X_n) \ge c_{10}(\lambda) \mathcal{I}(n). \tag{3.23}$$

Proof. (a) Let $c_1(\lambda) = (2^{1+1/\alpha_1} C_2^{1/\alpha_1} \lambda^{2/\alpha_1})^{-1} \wedge 1$, $c_2(\lambda) = c_{3.3.2}(\lambda)$, and $c_3(\lambda) = c_{3.3.4}(\lambda) < 1$. Then the desired inequality is trivial when $\varepsilon R \le 1$, so assume that $\varepsilon R > 1$. Let $q(y) = P^y(\tau_R < T_0)$, so that, by substituting 2ε into ε in Lemma 3.4, if $d(0,y) \le 2\varepsilon R$ then $q(y) \le c_0 \varepsilon^{\alpha_1} \lambda^2$. Write $t_0 = c_3(\lambda) v(\varepsilon R) r(\varepsilon R)$ and $a = P^0(\tau_R \le t_0)$. Now if $y \in B(2\varepsilon R)$ then

$$\begin{aligned} P^y(\tau_R \le t_0) &= P^y(\tau_R \le t_0, \tau_R < T_0) + P^y(\tau_R \le t_0, \tau_R > T_0) \\ &\le P^y(\tau_R \le T_0) + P^y(T_0 < \tau_R, \tau_R - T_0 \le t_0) \\ &\le q(y) + (1 - q(y)) a \le c_0 \varepsilon^{\alpha_1} \lambda^2 + a, \end{aligned} \tag{3.24}$$

using the strong Markov property for the second inequality. So, by a second application of the strong Markov property, and (3.13),

$$\begin{aligned} a = P^0(\tau_R \le t_0) &\le E^0[1_{\{\tau_{\varepsilon R} \le t_0\}} P^{X_{\tau_{\varepsilon R}}}(\tau_R \le t_0)] \\ &\le (1 - c_{3.3.5}(\lambda))(c_0 \varepsilon^{\alpha_1} \lambda^2 + a), \end{aligned} \tag{3.25}$$

where we used the fact that $X_{\tau_{\varepsilon R}} \in B(\varepsilon R+1) \subset B(2\varepsilon R)$ in the last inequality. Rewriting this gives $a \le c_0 \varepsilon^{\alpha_1} \lambda^2 (1 - c_{3.3.5}(\lambda))/c_{3.3.5}(\lambda)$. Substituting in (3.24) gives (3.19) with $c_4(\lambda) = c_0 \lambda^2 / c_{3.3.5}(\lambda)$.
(b)

Let $c_5(\lambda) = c_* c_3(\lambda)^{-1/(d_1+\alpha_1)}$, where $c_* > 0$ large is chosen later. Let $c_6(\lambda) = c_2(\lambda) c_5(\lambda)$, $c_7(\lambda) = (c_5(\lambda)/c_1(\lambda))^{\alpha_1} (c_4(\lambda) \vee 1)$, $M' = M/c_5(\lambda)$, and $\varepsilon = (M')^{-1}$. The desired inequality is trivial when $c_7(\lambda)/M^{\alpha_1} \ge 1$, so assume that $c_7(\lambda)/M^{\alpha_1} < 1$. Then, $M > c_5(\lambda)/c_1(\lambda)$, so $\varepsilon = c_5(\lambda)/M < c_1(\lambda)$. Thus the assumption in (a) is satisfied. Using (1.12), we have $\mathcal{I}(n/c_3(\lambda)) \le \hat{c} c_3(\lambda)^{-1/(d_1+\alpha_1)} \mathcal{I}(n)$, so taking $c_* = \hat{c}$, we have $\mathcal{I}(n/c_3(\lambda)) \le \varepsilon R$, which is equivalent to

$$n \le c_3(\lambda) v(\varepsilon R) r(\varepsilon r). \tag{3.26}$$

Since

$$\begin{aligned} P^0(d(0,X_n)/\mathcal{I}(n) > M) &= P^0(d(0,X_n) > R) \\ &\le P^0(\tau_R \le n) \le P^0(\tau_R \le c_3(\lambda) v(\varepsilon R) r(\varepsilon R)), \end{aligned} \tag{3.27}$$

where (3.26) is used in the last inequality. Using (a) gives the desired estimate.
(c) By (3.11), writing $B' = B(0, \theta R) \subset B(0,R)$ and $f_n(0,y) = p_n(0,y) + p_{n+1}(0,y)$,

$$P^0(X_n \in B') = \sum_{y \in B'} p_n(0,y) \mu_y \le \sum_{y \in B'} f_n(0,y) \mu_y \le V(\theta R) c_{3.3.1}(\lambda)/v(R). \tag{3.28}$$

Since $\theta R \in J(\lambda)$, using (1.12), the right hand side of (3.28) is bounded from above by $c_8(\lambda)\theta^{d_1}$.
(d) Let $\theta = c_9(\lambda) \in (0,1]$ satisfy $c_8(\lambda)\theta^{d_1} = \frac{1}{2}$. Then, since $R, \theta R \in J(\lambda)$, applying (c),

$$P^0(X_n \in B(\theta R)) \le c_8(\lambda)\theta^{d_1} = \tfrac{1}{2}. \tag{3.29}$$

This proves the first assertion. Also,

$$E^0 d(0,X_n) \ge \theta R P^0(X_n \notin B') \ge \tfrac{1}{2}\theta R \ge c_{10}(\lambda) \mathcal{I}(n). \tag{3.30}$$

□

3.2 Proof of Propositions 1.3–1.4 and Theorem 1.5

We now consider a family of random graphs, as described in Section 1.2, and prove Propositions 1.3–1.4 and Theorem 1.5.

We begin by obtaining tightness of $E^0 \tau_R/(v(R)r(R))$, $v(\mathcal{I}(n)) p_{2n}(0,0)$, and $d(0,X_n)/\mathcal{I}(n)$. In the following, we set $l(\lambda) = c_{3.3.2}(\lambda)$.

Proof of Proposition 1.3. We begin with (1.17). Let $\varepsilon > 0$. Choose $\lambda \ge 1$ such that $2p(\lambda) < \varepsilon$ – here $p(\lambda)$ is the function given by Assumption 1.2. Let $R \ge 1$ and set $F_1 = \{R, l(\lambda)R \in J(\lambda)\}$.

Suppose first that $l(\lambda)R \ge 1$. Then, by Assumption 1.2(1), $\mathbb{P}(F_1) \ge 1 - 2p(\lambda)$. For $\omega \in F_1$, by Proposition 3.3, there exists $c_1 < \infty$, $q_1 \ge 0$ such that

$$(c_1 \lambda^{q_1})^{-1} \le E^x_\omega \tau_R/(v(R)r(R)) \le c_1 \lambda^{q_1} \text{ for } x \in B(l(\lambda)R). \tag{3.31}$$

So, if $\theta_0 = c_1 \lambda^{q_1}$ then for $\theta \ge \theta_0$,

$$\mathbb{P}\Big(\theta^{-1} \le E^0_\omega \tau_R/(v(R)r(R)) \le \theta\Big) \ge \mathbb{P}(F_1) \ge 1 - 2p(\lambda) \ge 1 - \varepsilon. \tag{3.32}$$

Now consider the case when $R \le 1/l(\lambda)$. For each graph $\Gamma(\omega)$ let

$$Y(\omega) = \sup_{1 \le s \le 1/l(\lambda)} E^0_\omega \tau_s/(v(s)r(s)).$$

Then $Y(\omega) < \infty$ for each ω, so there exists θ_1 such that

$$\mathbb{P}(E^0_\omega \tau_R/(v(R)r(R)) > \theta_1) \le \mathbb{P}(Y > \theta_1) \le \varepsilon.$$

If we take $\theta_1 > v(1/l(\lambda))r(1/l(\lambda))$ then since $E^0_\omega \tau_R \ge 1$, we have $E^0_\omega \tau_R/(v(R)r(R)) \ge \theta_1^{-1}$. So, for $\theta \ge \theta_1$, we also have $\mathbb{P}\Big(\theta^{-1} \le E^0_\omega \tau_R/(v(R)r(R)) \le \theta\Big) \ge 1-\varepsilon$, which completes the proof of (1.17).

We now turn to (1.18). Let $n \ge 1$, $\lambda \ge 1$, and let R_0, R_1 be defined by $n = c_{3.3.7}(\lambda)v(R_1)r(R_1) = v(R_0)r(R_0)$. Let $F_2 = \{R_0, R_1, l(\lambda)R_1 \in J(\lambda)\}$. Suppose first that R_0 and $l(\lambda)R_1$ are both greater than 1; then $\mathbb{P}(F_2) \ge 1 - 3p(\lambda)$. If $\omega \in F_2$ then by Proposition 3.3

$$(c_2\lambda^{q_2})^{-1} \le v(\mathcal{I}(n))p^\omega_{2n}(0,0) \le c_2\lambda^{q_2},$$

for some $c_2 > 0, q_2 > 0$. So,

$$\mathbb{P}\Big((c_2\lambda^{q_2})^{-1} \le v(\mathcal{I}(n))p^\omega_{2n}(0,0) \le c_2\lambda^{q_2}\Big) \ge \mathbb{P}(F_2) \ge 1 - 3p(\lambda). \tag{3.33}$$

The case when n is small is dealt with in the same way as in the proof of (1.17).

Next we prove (1.19). Let $n \ge 1$ and $\lambda \ge 1$. Let $M = (\lambda c_{3.5.7}(\lambda))^{1/\alpha_1} =: l_1(\lambda)$, and set

$$R_0 = M\mathcal{I}(n), \quad R_1 = c_{3.5.5}(\lambda)\mathcal{I}(n), \quad R_2 = c_{3.5.6}(\lambda)\mathcal{I}(n), \tag{3.34}$$

$F_3 = \{R_0, R_1, R_2 \in J(\lambda)\}$. Suppose first that n is large enough so that $R_i \ge 1$ for $0 \le i \le 2$. By Proposition 3.5(b), if $\omega \in F_3$ then

$$P^0_\omega\Big(\frac{d(0,X_n)}{\mathcal{I}(n)} > l_1(\lambda)\Big) \le \frac{c_{3.5.7(\lambda)}}{l_1(\lambda)^{\alpha_1}} = \frac{1}{\lambda}.$$

Taking $\theta = l_1(\lambda)$, we have

$$\begin{aligned} P^*\Big(\frac{d(0,X_n)}{\mathcal{I}(n)} > \theta\Big) &\le \mathbb{P}(F_3^c) + \mathbb{E}\Big(P^0_\omega\Big(\frac{d(0,X_n)}{\mathcal{I}(n)} > l_1(\lambda)\Big); F_3\Big) \\ &\le 3p(l_1^{-1}(\theta)) + \frac{1}{l_1^{-1}(\theta)}, \end{aligned} \tag{3.35}$$

where $l_1^{-1}(\theta)$ is the inverse of $l_1(\theta)$.

Now let $\varepsilon > 0$. Choose θ_0 so that the right side of (3.35) is less than ε. Let $l_1(\lambda) = \theta_0$. Then there exists $n_1 = n_1(\varepsilon)$ such that if $n \ge n_1$ then R_0, R_1, R_2 (given by (3.34)) are all greater than 1. If $n \ge n_1$ then (3.35) implies that $P^*(d(0,X_n)/\mathcal{I}(n) > \theta_0) < \varepsilon$.

To handle the case when $n \le n_1$, for each ω let

$$Z_\theta(\omega) = \max_{1 \le n \le n_1} P^0_\omega(d(0,X_n)/\mathcal{I}(n) > \theta).$$

Then Z_θ is non-increasing in θ, and $\lim_{\theta\to\infty} Z_\theta(\omega) = 0$ for each ω. So, by monotone convergence

$$\lim_{\theta\to\infty} \mathbb{E}Z_\theta(\omega) = 0.$$

Thus there exists θ_1 such that

$$P^*(d(0,X_n)/\mathcal{I}(n) > \theta_1) \le \mathbb{E}Z_{\theta_1} < \varepsilon \quad \text{for all } n \le n_1.$$

Taking $\theta = \theta_0 \vee \theta_1$, we obtain (1.19).

Finally, we prove (1.20). Let $\varepsilon > 0$. Choose λ so that $2p(\lambda) + 1/c_{3.5.8}(\lambda) < \varepsilon$, and let $\theta_0 = c_{3.5.8}(\lambda)^{2/d_1}$, $\delta = 1/\theta_0$. Choose R so that $v(R)r(R) = n$, and $n_0 = n_0(\varepsilon)$ such that $n \ge n_0$ implies $\delta R \ge 1$. Set $\theta_1 = 1 + \mathcal{I}(n_0)$, and $\theta = \theta_0 \vee \theta_1$. Suppose first that $n \ge n_0$, and set $F_4 = \{R, \delta R \in J(\lambda)\}$. If $\omega \in F_4$ then by Proposition 3.5(c), we have

$$P^0_\omega\Big(d(0,X_n)/\mathcal{I}(n) \le \delta\Big) \le c_{3.5.8}(\lambda)\delta^{d_1}.$$

So,

$$\begin{aligned} P^*\Big((1+d(0,X_n))/\mathcal{I}(n) < \theta^{-1}\Big) &\le P^*\Big(d(0,X_n)/\mathcal{I}(n) < \theta_0^{-1}\Big) \\ &\le \mathbb{P}(F_4^c) + \mathbb{E}\Big(P^0_\omega(d(0,X_n)/\mathcal{I}(n) < \theta_0^{-1}); F_4\Big) \\ &\le 2p(\lambda) + 1/c_{3.5.8}(\lambda) \le \varepsilon. \end{aligned} \tag{3.36}$$

If $n \le n_0$ then $(1+d(0,X_n))/\mathcal{I}(n) \ge 1/\mathcal{I}(n) \ge \theta_1^{-1}$, and so we deduce that, for all n,

$$P^*((1+d(0,X_n))/\mathcal{I}(n) < \theta^{-1}) \le \varepsilon,$$

which proves (1.20). □

Proof of Proposition 1.4. We begin with the upper bounds in (1.21). By (3.7) and Assumption 1.2(2),

$$\mathbb{E}(E^0_\omega \tau_R) \le \mathbb{E}(R_{\text{eff}}(0, B(R)^c)V(R)) \le cv(R)r(R).$$

For the lower bounds, it is sufficient to find a set $F \subset \Omega$ of 'good' graphs with $\mathbb{P}(F) \ge c > 0$ such that, for all $\omega \in F$ we have suitable lower bounds on $E^0_\omega \tau_R$, $p^\omega_{2n}(0,0)$ or $E^0_\omega d(0,X_n)$. For the lower bounds, we assume that $R \ge 1$ is large enough so that $l(\lambda_0)R \ge 1$, where λ_0 is chosen large enough that $p(\lambda_0) < 1/8$. We can then obtain the results for all n (chosen below to depend on R) and R by adjusting the constants $c_1, \cdots, c_4$ in (1.21)–(1.23).

Let $F = \{R, l(\lambda_0)R \in J(\lambda_0)\}$. Then $\mathbb{P}(F) \ge \frac{3}{4}$, and for $\omega \in F$, by (3.12), $E^0_\omega \tau_R \ge c_1(\lambda_0)v(R)r(R)$. So,

$$\mathbb{E}(E^0_\omega \tau_R) \ge \mathbb{E}(E^0_\omega \tau_R; F) \ge c_1(\lambda_0)v(R)r(R)\mathbb{P}(F) \ge c_2(\lambda_0)v(R)r(R).$$

Also, by (3.14), if $n \in [\frac{1}{2}c_{3.3.7}(\lambda_0)v(R)r(R), c_{3.3.7}(\lambda_0)v(R)r(R)]$ then

$$p^\omega_{2n}(0,0) \ge \frac{c_3(\lambda_0)}{v(\mathcal{I}(n))}.$$

Given $n \in \mathbb{N}$, choose R so that $n = c_{3.3.7}(\lambda_0)v(R)r(R)$ and let F be as above. Then

$$\mathbb{E}p^\omega_{2n}(0,0) \ge \mathbb{P}(F)\frac{c_3(\lambda_0)}{v(\mathcal{I}(n))} \ge \frac{c_4(\lambda_0)}{v(\mathcal{I}(n))},$$

giving the lower bound in (1.22).

A similar argument uses (3.23) to conclude (1.23).

Finally we prove (1.25). Let H_k be the event of the left hand side of (1.24) with $\lambda = k$. By Proposition 3.1(a), we see that $p^\omega_{2n}(0,0) \le c_1 k/v(\mathcal{I}(n))$ if $\omega \in H_k$, where R is chosen to satisfy $v(R)r(R)/2 \le n \le v(R)r(R)$. Since $\mathbb{P}(\cup_k H_k) = 1$, using (1.24), we have

$$\begin{aligned}\mathbb{E}p^\omega_{2n}(0,0) &\le \sum_k \frac{c_1(k+1)}{v(\mathcal{I}(n))}\mathbb{P}(H_{k+1}\setminus H_k) \le \sum_k \frac{c_1(k+1)}{v(\mathcal{I}(n))}\mathbb{P}(H_k^c)\\ &\le \frac{c_2}{v(\mathcal{I}(n))}\sum_k (k+1)k^{-q_0'} < \infty,\end{aligned}$$

since $q_0' > 2$. We thus obtain (1.25). □

Proof of Remark 1.6. 1. In this case, we have

$$\mathbb{P}(\{R_{\rm eff}(0,y) \le r(d(0,y)),\ \forall y \in B(R)\}) = 1,$$

so, similarly to the proof of Proposition 3.1, for $v(R)r(R)/2 \le n \le v(R)r(R)$, we have

$$f_{2n}(0) \le c_1\Big(\frac{1}{V(0,R)} \vee \frac{1}{v(R)}\Big) \le \frac{c_2}{v(\mathcal{I}(n))}\Big(\frac{v(R)}{V(0,R)} \vee 1\Big).$$

Using this and (1.32), we have

$$\mathbb{E}p^\omega_{2n}(0,0) \le \frac{c}{v(\mathcal{I}(n))}\mathbb{E}\Big(1 + \frac{v(R)}{V(0,R)}\Big) \le \frac{c'}{v(\mathcal{I}(n))},$$

so (1.25) is obtained. □

Proof of Theorem 1.5. (I) We will take $\Omega_0 = \Omega_a \cap \Omega_b \cap \Omega_c$ where the sets Ω_* are defined in the proofs of (a), (b) and (c). By Assumption 1.2(3), $p(\lambda) = \mathbb{P}(R \notin J(\lambda)) \le c_0\lambda^{-q_0}$.
(a) We begin with the case $x = 0$, and write $w(n) = p^\omega_{2n}(0,0)$. By (3.33) we have

$$\mathbb{P}((c_1\lambda^{q_1})^{-1} < v(\mathcal{I}(n))w_n \le c_1\lambda^{q_1}) \ge 1 - 3p(\lambda).$$

Let $n_k = \lfloor e^k \rfloor$ and $\lambda_k = k^{2/q_0}$. Then, since $\sum p(\lambda_k) < \infty$, by Borel–Cantelli there exists $K_0(\omega)$ with $\mathbb{P}(K_0 < \infty) = 1$ such that $c_1^{-1}k^{-2q_1/q_0} \le v(\mathcal{I}(n_k))w(n_k) \le c_1 k^{2q_1/q_0}$ for all $k \ge K_0(\omega)$. Let $\Omega_a = \{K_0 < \infty\}$. For $k \ge K_0$ we therefore have

$$c_2^{-1}\frac{(\log n_k)^{-2q_1/q_0}}{v(\mathcal{I}(n_k))} \le w(n_k) \le c_2\frac{(\log n_k)^{2q_1/q_0}}{v(\mathcal{I}(n_k))},$$

so that (1.26) holds for the subsequence n_k. The spectral decomposition gives that $p^\omega_{2n}(0,0)$ is monotone decreasing in n. So, if $n > N_0 = e^{K_0} + 1$, let $k \ge K_0$ be such that $n_k \le n < n_{k+1}$. Then

$$w(n) \le w(n_k) \le c_2\frac{(\log n_k)^{2q_1/q_0}}{v(\mathcal{I}(n_k))} \le c_2'\frac{(\log n)^{2q_1/q_0}}{v(\mathcal{I}(n))}.$$

Similarly $w(n) \geq w(n_{k+1}) \geq \frac{c_3}{v(\mathcal{I}(n))}(\log n)^{-2q_1/q_0}$. Taking $q_2 > 2q_1/q_0$, so that the constants c_2, c_3 can be absorbed into the $\log n$ term, we obtain

$$\frac{(\log n)^{-q_2}}{v(\mathcal{I}(n))} \leq p^\omega_{2n}(0,0) \leq \frac{(\log n)^{q_2}}{v(\mathcal{I}(n))} \quad \text{for all } n \geq N_0(\omega). \tag{3.37}$$

If $x, y \in \mathcal{C}(\omega)$ and $k = d_\omega(x,y)$, then using the Chapman–Kolmogorov equation

$$p^\omega_{2n}(x,x)(p^\omega_k(x,y)\mu_x(\omega))^2 \leq p^\omega_{2n+2k}(y,y).$$

Let $\omega \in \Omega_a$, $x \in \mathcal{C}(\omega)$, write $k = d_\omega(0,x)$, $h^\omega(0,x) = (p^\omega_k(x,0)\mu_x(\omega))^{-2}$, and let $n \geq N_0(\omega) + 2k$. Then

$$\begin{aligned} p^\omega_{2n}(x,x) &\leq h^\omega(0,x)p^\omega_{2n+2k}(0,0) \\ &\leq h^\omega(0,x)\frac{(\log(n+k))^{q_2}}{v(\mathcal{I}(n+k))} \\ &\leq h^\omega(0,x)\frac{(\log(2n))^{q_2}}{v(\mathcal{I}(n))} \leq \frac{(\log n)^{1+q_2}}{v(\mathcal{I}(n))} \end{aligned}$$

provided $\log n \geq 2^{q_2}h^\omega(0,x)$. Taking

$$N_x(\omega) = \exp(2^{q_2}h^\omega(0,x)) + 2d_\omega(0,x) + N_0(\omega), \tag{3.38}$$

and $\beta_1 = 1 + q_2$, this gives the upper bound in (1.26). The lower bound is obtained in the same way.

(b) Let $R_n = e^n$ and $\lambda_n = n^{2/q_0}$. Let $F_n = \{R_n, l(\lambda_n)R_n \in J(\lambda_n)\}$. Then (provided $l(\lambda_n)R_n \geq 1$) we have $\mathbb{P}(F_n^c) \leq 2p(\lambda_n) \leq 2n^{-2}$. So, by Borel–Cantelli, if $\Omega_b = \liminf F_n$, then $\mathbb{P}(\Omega_b) = 1$. Hence there exists M_0 with $M_0(\omega) < \infty$ on Ω_b, and such that $\omega \in F_n$ for all $n \geq M_0(\omega)$.

Now fix $\omega \in \Omega_b$, and let $x \in \mathcal{C}(\omega)$. Write $F(R) = E^x_\omega \tau_R$. By (3.31) there exist constants c_4, q_4 such that

$$(c_4\lambda_n^{q_4})^{-1} \leq \frac{F(R_n)}{v(R_n)r(R_n)} \leq c_4\lambda_n^{q_4}. \tag{3.39}$$

provided $n \geq M_0(\omega)$ and n is also large enough so that $x \in B(l(\lambda_n)R_n)$. Writing $M_x(\omega)$ for the smallest such n,

$$c_4^{-1}(\log R_n)^{-2q_4/q_0}v(R_n)r(R_n) \leq F(R_n) \leq c_4(\log R_n)^{2q_4/q_0}v(R_n)r(R_n), \quad \text{for all } n \geq M_x(\omega).$$

As $F(R)$ is monotone increasing, the same argument as in (a) enables us to replace $F(R_n)$ by $F(R)$, for all $R \geq R_x = 1 + e^{M_x}$. Taking $\beta_2 > 2q_4/q_0$ we obtain (1.27).

(c) Recall that $Y_n = \max_{0\leq k\leq n} d(0, X_k)$. We begin by noting that

$$\{Y_n \geq R\} = \{\tau_R \leq n\}. \tag{3.40}$$

Using this, (1.28) follows easily from (1.29).

It remains to prove (1.29). Since τ_R is monotone in R, as in (b) it is enough to prove the result for the subsequence $R_n = e^n$.

The estimates in (b) give the upper bound. In fact, if $\omega \in \Omega_b$, and $n \geq M_x(\omega)$, then by (3.39)

$$P_\omega^x(\tau_{R_n} \geq n^2 c_4 \lambda_n^{q_4} v(R_n) r(R_n)) \leq \frac{F(R_n)}{n^2 c_4 \lambda_n^{q_4} v(R_n) r(R_n)} \leq n^{-2}.$$

So, by Borel–Cantelli (with respect to the law P_ω^x), there exists $N_x'(\omega, \overline{\omega})$ with

$$P_\omega^x(N_x' < \infty) = P_\omega^x(\{\overline{\omega} : N_x'(\omega, \overline{\omega}) < \infty\}) = 1$$

such that

$$\tau_{R_n} \leq c_5 (\log R_n)^{q_5} v(R_n) r(R_n), \quad \text{for all } n \geq N_x'.$$

For the lower bound, write $c_{3.5.3}(\lambda) = c_6 \lambda^{-q_6}$, $c_{3.5.4}(\lambda) = c_7 \lambda^{q_7}$, where we choose $q_6 + q_7 \geq 2$. Let $\lambda_n = n^{2/q_0}$, and $\varepsilon_n^{\alpha_1} = cn^{-2} \lambda_n^{-q_6 - q_7}$. Here $c > 0$ is chosen small enough so that $\varepsilon_n \leq c_{3.5.1}(\lambda)$. Set $G_n = \{R_n, \varepsilon_n R_n, l(\lambda_n) \varepsilon_n R_n \in J(\lambda_n)\}$. Then, for n sufficiently large so that $l(\lambda_n)\varepsilon_n R_n \geq 1$, we have $\mathbb{P}(G_n^c) \leq 3p(\lambda_n) \leq 3c_0 n^{-2}$. Let $\Omega_c = \Omega_b \cap (\liminf G_n)$; then by Borel–Cantelli, $\mathbb{P}(\Omega_c) = 1$ and there exists M_1 with $M_1(\omega) < \infty$ for $\omega \in \Omega_c$ such that $\omega \in G_n$ whenever $n \geq M_1(\omega)$. By Proposition 3.5(a), if $n \geq M_1$ and $x \in B(\varepsilon_n R_n)$ then

$$P_\omega^x(\tau_{R_n} \leq c_6 \lambda_n^{-q_6} v(\varepsilon_n R_n) r(\varepsilon_n R_n)) \leq c_7 \lambda_n^{q_7} \varepsilon_n^{\alpha_1} \leq c_7' n^{-2}. \tag{3.41}$$

So, using Borel–Cantelli, we deduce that (for some q_8)

$$\tau_{R_n} \geq c_6 \lambda_n^{-q_6} v(\varepsilon_n R_n) r(\varepsilon_n R_n) \geq n^{-q_8} v(R_n) r(R_n) = (\log R_n)^{-q_8} v(R_n) r(R_n),$$

for all $n \geq N_x''(\omega, \overline{\omega})$. This completes the proof of (1.29).

The proof of (II) is similar by the following changes; take $\lambda_k = (e + (2/c_4) \log k)^{1/q_0}$ instead of $\lambda_k = k^{2/q_0}$, and take $N_x(\omega) = \exp(\exp(Ch^\omega(0,x))) + 2d_\omega(0,x) + N_0(\omega)$ in (3.38). Then, $\log n$ (resp. $\log n_k, \log R_n$) in the above proof are changed to $\log \log n$ (resp. $\log \log n_k, \log \log R_n$) and the proof of (a) and (b) goes through. Since the modifications are simple, we omit details.

We now prove (III). For (a), $\lim_n \log p_{2n}^\omega(0,0)/\log n = -D/(D+\alpha)$, $\mathbb{P}$–a.s. is easy from (1.26) and (1.30). Since $\sum_n p_{2n}^\omega(0,0) = \infty$, X is recurrent. (b) is also easy from (1.27) and (1.30).
(c) We first consider the case $x = 0$. Let $c_1 \in (0,1)$, $c_2 \geq 2$, $q_1 \geq 1$, $q_2 \geq 2$ be chosen so that

$$c_{3.5.3}(\lambda) \geq c_1 \lambda^{-q_1}, \quad c_{3.5.4}(\lambda) \leq c_2 \lambda^{q_2}.$$

Let $R_k = e^k$, and $\lambda_k = k^{q_3}$ where $q_3 \geq 2$ is chosen large enough so that $\sum p(\lambda_k) < \infty$. Let $\varepsilon_k^{\alpha_1} = c_2^{-1} \lambda_k^{-q_2} k^{-q_3}$. Set

$$F_k = \{R_k, \varepsilon_k R_k, c_{3.5.2}(\lambda) \varepsilon_k R_k \in J(\lambda_k)\}.$$

For $\omega \in F_k$ we have by Proposition 3.5(a)

$$P_\omega^0(\tau_{R_k} \leq c_1 \lambda_k^{-q_1} v(\varepsilon_k R_k) r(\varepsilon_k R_k)) \leq c_2 \lambda_k^{q_2} \varepsilon_k^{\alpha_1} = k^{-q_3}.$$

Set $n(k) = c_1 \lambda_k^{-q_1} v(\varepsilon_k R_k) r(\varepsilon_k R_k) \geq c_3 \lambda_k^{-q_1} (\varepsilon_k R_k)^{D+\alpha} (\log(\varepsilon_k R_k))^{-m_1 - m_2}$. Then

$$P^*(\{\tau_{R_k} \leq n(k)\} \cup F_k^c) \leq \mathbb{P}(F_k^c) + k^{-q_3} \leq 3p(\lambda_k) + k^{-q_3}. \tag{3.42}$$

Therefore by Borel–Cantelli, we deduce that, P^*–a.s., for all sufficiently large k, $\tau_{R_k} > n(k)$ and F_k holds. So, for large k,

$$S_{n(k)} \le S_{\tau_{R_k}} \le V(R_k) \le \lambda_k v(R_k) \le c_4 \lambda_k R_k^D (\log R_k)^{m_1}.$$

If n is sufficiently large, then choosing k so that $n(k-1) < n \le n(k)$,

$$\begin{aligned} \frac{\log S_n}{\log n} &\le \frac{\log S_{n(k)}}{\log n(k-1)} \le \frac{Dk + \log(c_4\lambda_k) + m_1 \log k}{(D+\alpha)(k-1) + \log(c_3 \varepsilon_{k-1}^{D+\alpha} \lambda_{k-1}^{-q_1}) - (m_1+m_2)\log\log(\varepsilon_k e^k)} \\ &\le \frac{k}{k-1}\frac{D}{D+\alpha} + \frac{c_5 \log k}{k}, \end{aligned}$$

and this gives the upper bound in (1.31) for the case $x = 0$.

For the lower bound, let $\xi(x,R) = 1_{\{T_x > \tau_R\}}$. If $R \in J(\lambda)$ and $\varepsilon^{\alpha_1} < 1/(2C_2\lambda^2)$ then by Lemma 3.4,

$$P^0_\omega(\xi(x,R) = 1) \le C_2 \varepsilon^{\alpha_1} \lambda^2, \qquad \text{for } x \in B(\varepsilon R).$$

Set

$$Y_k = V(\varepsilon_k R_k)^{-1} \sum_{x \in B(\varepsilon_k R_k)} \xi(x, R_k)\mu_x.$$

Then if $\omega \in F_k$,

$$P^0_\omega(Y_k \ge \tfrac{1}{2}) \le 2E^0_\omega Y_k \le 2C_2 \varepsilon_k^{\alpha_1} \lambda_k^2 \le c_6 k^{-q_3}.$$

Let $m(k) = k^{q_3}\lambda_k^2 v(R_k) r(R_k) \le c_7 k^{q_3} \lambda_k^2 R_k^{D+\alpha} (\log R_k)^{m_1+m_2}$. Then if $\omega \in F_k$, by (3.9),

$$P^0_\omega(\tau_{R_k} \ge m(k)) \le 2\lambda_k^2 v(R_k) r(R_k) m(k)^{-1} = 2k^{-q_3}.$$

Thus

$$P^*(F_k^c \cup \{Y_k \ge \tfrac{1}{2}\} \cup \{\tau_{R_k} \ge m(k)\}) \le 3p(\lambda_k) + (2 + c_6)k^{-q_3},$$

so by Borel–Cantelli, P^*–a.s. there exists a $k_0(\omega) < \infty$ such that, for all $k \ge k_0$, F_k holds, $\tau_{R_k} \le m(k)$, and $Y_k \le 1/2$. So, for $k \ge k_0$,

$$\begin{aligned} S_{m(k)} &\ge S_{\tau_{R_k}} = \sum_{x \in B(\varepsilon_k R_k)} (1 - \xi(x, R_k))\mu_x = V(\varepsilon_k R_k)(1 - Y_k) \\ &\ge \tfrac{1}{2}\lambda_k^{-1} v(\varepsilon_k R_k) \ge c_8 \lambda_k^{-1} (\varepsilon_k R_k)^D (\log(\varepsilon_k R_k))^{-m_1} \end{aligned}$$

Let n be large enough so that $m(k) \le n < m(k+1)$ for some $k \ge k_0$. Then

$$\frac{\log S_n}{\log n} \ge \frac{\log S_{m(k)}}{\log m(k+1)} \ge \frac{Dk - c\log k}{(D+\alpha)(k+1) + c' \log(k+1)},$$

and the lower bound in (1.31) follows. This proves (1.31) when $x = 0$.

Now let

$$\Omega_0 = \{\omega : G(\omega) \text{ is recurrent and } P^0_\omega(\lim_n (\log S_n / \log n) = \tfrac{D}{D+\alpha}) = 1\}.$$

We have $\mathbb{P}(\Omega_0) = 1$. If $\omega \in \Omega_0$, and $x \in G(\omega)$ then X hits 0 with P^x_ω–probability 1. Since the limit does not depend on the initial segment $X_0, \dots, X_{T_0}$, we obtain (1.31).

□

References

[1] Aizenman, M., and Newman, C. M. (1986). Discontinuity of the percolation density in one dimensional $1/|x-y|^2$ Percolation Models. *Commun. Math. Phys.* **107**, 611–647.

[2] Alexander, S., and Orbach, R. (1982). Density of states on fractals: "fractons". *J. Physique (Paris) Lett.* **43**, L625–L631.

[3] Ben-Avraham, D., and Havlin, S. (2000). Diffusion and Reactions in Fractals and Disordered Systems. Cambridge University Press, Cambridge.

[4] Barlow, M. T. (2004). Random walks on supercritical percolation clusters. *Ann. Probab.* **32**, 3024–3084.

[5] Barlow, M. T., Coulhon, T., and Kumagai, T. (2005). Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. *Comm. Pure Appl. Math.* **58**, 1642–1677.

[6] Barlow, M. T., Járai, A. A., Kumagai, T., and Slade, G. (2008). Random walk on the incipient infinite cluster for oriented percolation in high dimensions. *Commun. Math. Phys.* **278**, 385–431.

[7] Barlow, M. T., and Kumagai, T. (2006). Random walk on the incipient infinite cluster on trees. *Illinois J. Math.* **50**, 33–65 (Doob volume).

[8] Benjamini, I., Berger, N., and Yadin, A. (2007). Long range percolation mixing time. Preprint available at arXiv:math/0703872.

[9] Berger, N. (2002). Transience, recurrence and critical behavior for long-range percolation. *Commun. Math. Phys.* **226**, 531–558.

[10] Berger, N., and Biskup, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Prob. Theory Related Fields.* **137**, 83–120.

[11] Berger, N., Gantert, N., and Peres, Y. (2003). The speed of biased random walk on percolation clusters. *Probab. Theory Related Fields.* **126**, 221–242.

[12] Croydon, D., and Kumagai, T. (2007). Random walks on Galton-Watson trees with infinite variance offspring distribution conditioned to survive. Preprint available at http://www.math.kyoto-u.ac.jp/~kumagai/kumpre.html.

[13] Kesten, H. (1986). The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Related Fields.* **73**, 369–394.

[14] Kesten, H. (1986). Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, 425–487.

[15] Kigami, J. (2001). *Analysis on Fractals.* Cambridge University Press, Cambridge.

[16] Kozma, G., and Nachmias, A. (2008). The Alexander-Orbach conjecture holds in high dimensions. Preprint available at arXiv:0806.1442.

[17] Mathieu, P., and Piatnitski, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proceedings of the Royal Society A.* **463**, 2287–2307.

[18] Misumi, J. (2008). Estimates on the effective resistance in a long-range percolation on $\mathbb{Z}^d$. *J. Math. Kyoto Univ.*, to appear.

[19] Sidoravicius, V., and Sznitman, A. -S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields.* **129**, 219–244.