# Note on the equivalence of parabolic Harnack inequalities and heat kernel estimates

Martin T. Barlow, Richard F. Bass and Takashi Kumagai

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**Abstract.** The aim of this note is to prove the equivalence of parabolic Harnack inequalities and sub-Gaussian heat kernel estimates in a general metric measure space with a local regular Dirichlet form.

**1. Framework and the main theorem.**

Let $(X, d)$ be a connected locally compact complete separable metric space. We assume that the metric $d$ is geodesic: for each $x, y \in X$ there exists a (not necessarily unique) geodesic path $\gamma(x, y)$ such that for each $z \in \gamma(x, y)$, we have $d(x, z) + d(z, y) = d(x, y)$. Let $\mu$ be a Borel measure on $X$ such that $0 < \mu(B) < \infty$ for every ball $B$ in $X$. We write $B(x, r) = \{y : d(x, y) < r\}$, and $V(x, r) = \mu(B(x, r))$. Note that under the assumptions above, the closure of $B(x, r)$ is compact for all $x \in X$ and $0 < r < \infty$. For simplicity in what follows, we will also assume that $X$ has infinite diameter, but similar results (with obvious modifications to the statements and the proofs) hold when the diameter of $X$ is finite. We will call such a space a *metric measure space*, or a MM space.

Now let $(\mathcal{E}, \mathcal{F})$ be a regular, strong local Dirichlet form on $L^2(X, \mu)$: see [FOT] for details. We denote by $\Delta$ the corresponding self-adjoint operator; that is, we say $h$ is in the domain of $\Delta$ and $\Delta h = f$ if $h \in \mathcal{F}$ and $\mathcal{E}(h, g) = -\int fg\, d\mu$ for every $g \in \mathcal{F}$. Let $\{P_t\}$ be the corresponding semigroup. $(\mathcal{E}, \mathcal{F})$ is called *conservative* (or *stochastically complete*) if $P_t 1 = 1$ for all $t > 0$. Throughout the paper, we assume that $(\mathcal{E}, \mathcal{F})$ is conservative. Since $\mathcal{E}$ is regular, $\mathcal{E}(f, g)$ can be written in terms of a signed measure $\Gamma(f, g)$. To be more precise, for $f \in \mathcal{F}_b$ (the collection $\mathcal{F}_b$ is the set of functions in $\mathcal{F}$ that are essentially bounded) $\Gamma(f, f)$ is the unique smooth Borel measure (called the energy measure) on $X$ satisfying

$$\int_X \tilde{g}\, d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \qquad g \in \mathcal{F}_b, \tag{1.0}$$

where $\tilde{g}$ is the quasi-continuous modification of $g \in \mathcal{F}$. (Recall that $u : X \to \mathbb{R}$ is called quasi-continuous if for any $\varepsilon > 0$, there exists an open set $G \subset X$ such that $\mathrm{Cap}(G) < \varepsilon$ and $u|_{X \setminus G}$ is continuous. It is known that each $u \in \mathcal{F}$ admits a quasi-continuous modification $\tilde{u}$ – see [FOT], Theorem 2.1.3.) Throughout the paper, we will abuse notation and take the quasi-continuous modification of $g \in \mathcal{F}_b$ without writing $\tilde{g}$. $\Gamma(f, g)$ is defined by

$$\Gamma(f, g) = \frac{1}{2}(\Gamma(f + g, f + g) - \Gamma(f, f) - \Gamma(g, g)), \qquad f, g \in \mathcal{F}.$$

$\Gamma(f,g)$ is also local, linear in $f$ and $g$, and satisfies the Leibniz and chain rules – see [FOT], p. 115-116. That is, if $f_1, \ldots, f_m, g$, and $\varphi(f_1, \ldots, f_m)$ are in $\mathcal{F}_b$, and $\varphi_i$ denotes the partial derivative of $\varphi$ in the $i^{th}$ direction, we have:

$$d\Gamma(fg,h) = f d\Gamma(g,h) + g d\Gamma(f,h),$$

$$d\Gamma(\varphi(f_1,\ldots,f_m), g) = \sum_{i=1}^{m} \varphi_i(f_1,\ldots,f_m) d\Gamma(f_i, g).$$

We call $(X, d, \mu, \mathcal{E})$ a metric measure Dirichlet space, or a MMD space.

Let $Y = (Y_t, t \ge 0, \mathbb{P}^x, x \in X\}$ be the Hunt process associated with the Dirichlet form $\mathcal{E}$ on $L^2(X,\mu)$ – see [FOT, Theorem 7.2.1]. Since $\mathcal{E}$ is strongly local, by [FOT, Theorem 7.2.2] $Y$ is a diffusion.

Throughout the note, we let $\beta \ge 1$.

**Definition 1.1.** (a) We say a function $u$ is *harmonic* on a domain $D$ if $u \in \mathcal{F}_{loc}$ and $\mathcal{E}(u,g) = 0$ for all $g \in \mathcal{F}$ with support in $D$. Here $u \in \mathcal{F}_{loc}$ if and only if for any relatively compact open set $G$, there exists a function $w \in \mathcal{F}$ such that $u = w$ $\mu$-a.e. on $G$. See page 117 in [FOT] for the definition of $\mathcal{E}(u,g)$ for $u \in \mathcal{F}_{loc}$ when $(\mathcal{E},\mathcal{F})$ is a regular, strong local Dirichlet form. Functions in $\mathcal{F}$ are only defined up to quasi-everywhere equivalence; we use a quasi-continuous modification of $u$. $X$ satisfies the *elliptic Harnack inequality* EHI if there exists a constant $c_1$ such that, for any ball $B(x,R)$, whenever $u$ is a non-negative harmonic function on $B(x,R)$ then there is a quasi-continuous modification $\tilde{u}$ of $u$ that satisfies

$$\sup_{B(x,R/2)} \tilde{u} \le c_1 \inf_{B(x,R/2)} \tilde{u}. \tag{EHI}$$

Note that by a standard argument (see, e.g., [M], p. 571) EHI implies that $\tilde{u}$ is Hölder continuous.

(b) Let $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R) =: I \times B_{2R}$. Let $u(t,x) : Q \to \mathbb{R}$.

- We define $u_t = \frac{\partial u}{\partial t} \in L^2(dt \times \mu)$ as the derivative in the Schwartz' distribution sense. That is, we define $u_t$ to be the function $f$ in $L^2(dt \times \mu)$ so that for any function $g : Q \to \mathbb{R}$ such that $g(x,\cdot) \in C_K^\infty(0,4T)$ for each $x \in B(x_0, 2R)$, $g_t = \frac{\partial g}{\partial t} \in L^2(dt \times \mu)$ for each $t$, then
$$\int_Q (f(x,t)g(x,t) + u(x,t)g_t(x,t))\, dt\, \mu(dx) = 0.$$
- Let $H(I \to \mathcal{F}^*)$ be the space of functions $u \in L^2(I \to \mathcal{F}^*)$ with the distributional time derivative $u_t \in L^2(I \to \mathcal{F}^*)$ equipped with the norm
$$\Big( \int_I \|u(t,\cdot)\|^2_{\mathcal{F}^*} + \|u_t(t,\cdot)\|^2_{\mathcal{F}^*} dt \Big)^{1/2}.$$

  Here we identify $L^2(X,\mu)$ with its own dual and denote the dual of $\mathcal{F}$ by $\mathcal{F}^*$. So, $\mathcal{F} \subset L^2(X,\mu) \subset \mathcal{F}^*$ with continuous and dense embeddings.

Let $\mathcal{F}(I \times X) = L^2(I \to \mathcal{F}) \cap H(I \to \mathcal{F}^*)$ be a Hilbert space with norm

$$\|u\|_{\mathcal{F}(I\times X)} = \Big(\int_I \|u(t,\cdot)\|_{\mathcal{F}}^2 + \|u_t(t,\cdot)\|_{\mathcal{F}^*}^2 dt\Big)^{1/2}.$$

- We define $\mathcal{F}_{loc}(Q)$ to be the set of $dt \otimes d\mu$-measurable functions on $Q$ such that for every relatively compact open set $D \subset\subset B_{2R}$ and every open interval $I' \subset\subset I$, there exists a function $u' \in \mathcal{F}(I \times X)$ with $u = u'$ on $I' \times D$. We define

$$\mathcal{F}_c(Q) := \{u \in \mathcal{F}(I \times X) : u(t,\cdot) \text{ has compact support in } B_{2R} \text{ for a.e. } t \in I\}.$$

We say a function $u(t,x) : Q \to \mathbb{R}$ is a solution of the heat equation in $Q$ if $u \in \mathcal{F}_{loc}(Q)$ and

$$\int_J \Big[\int f(t,x)u_t(t,x)\mu(dx) + \mathcal{E}(f(t,\cdot),u(t,\cdot))\Big]dt = 0, \quad \forall J \subset\subset I,\ \forall f \in \mathcal{F}_c(Q). \tag{1.1}$$

$X$ satisfies the *parabolic Harnack inequality of order* $\beta$, PHI($\beta$), if there exists a constant $c_2$ such that the following holds. Let $x_0 \in X$, $R > 0$, $T = R^\beta$, and $u = u(t,x)$ be a non-negative solution of the heat equation in $Q(x_0,T,R)$. Write $Q_- = (T,2T) \times B(x_0,R)$ and $Q_+ = (3T,4T) \times B(x_0,R)$; then there exists $\tilde{u} = \tilde{u}(t,x)$ such that $\tilde{u}(t,\cdot)$ is a quasi-continuous modification of $u(t,\cdot)$ for each $t$ and

$$\sup_{Q_-} \tilde{u} \le c_2 \inf_{Q_+} \tilde{u}. \tag{PHI($\beta$)}$$

Given this PHI, a standard oscillation argument implies that $\tilde{u}$ is jointly continuous.

Let

$$h_\beta(r,t) = \exp\Big(-\Big(\frac{r^\beta}{t}\Big)^{1/(\beta-1)}\Big). \tag{1.2}$$

**Definition 1.2.** We say $X$ satisfies HK($\beta$) if there exists a version of the heat kernel $p_t(x,y)$ on $X$ which satisfies

$$\frac{c_1 h_\beta(c_2 d(x,y),t)}{V(x,t^{1/\beta})} \le p_t(x,y) \le \frac{c_3 h_\beta(c_4 d(x,y),t)}{V(x,t^{1/\beta})}, \tag{1.3}$$

for $x,y \in X$ and $t \in (0,\infty)$.

**Theorem 1.3.** *The following are equivalent:*
*(a) $X$ satisfies PHI($\beta$).*
*(b) $X$ satisfies HK($\beta$).*

**Remark.** 1) This equivalence is well-known for manifolds when $\beta = 2$. For MMD with $\beta = 2$, it is indirectly proved in [St2]. (There $(a) \iff \mathrm{VD} + \mathrm{PI}$ and $(b) \iff \mathrm{VD} + \mathrm{PI}$ are proved.) For MMD with general time scaling, [HSC] proves the equivalence assuming apriori that solutions to the heat equation are sufficiently regular. (See also [GT2] for the case of an infinite connected weighted graph.) We will prove the equivalence without assuming any apriori condition for solutions to the heat equation.
2) In this note, we only discuss the case when the time scaling exponent is $\beta \ge 1$, but a similar argument gives the equivalence for more general time scalings, for example, when the time scaling function $\Psi$ is $\Psi(R) = R^{\beta_1}$ for $R \le 1$ and $\Psi(R) = R^{\beta_2}$ for $R \ge 1$ where $\beta_1, \beta_2 \ge 1$.

We give some more definitions for later use.

**Definition 1.4.** (a) $X$ satisfies volume doubling, VD, if there exists a constant $c_1$ such that

$$V(x,2R) \le c_1 V(x,R) \quad \text{for all } x \in X,\ R \ge 0. \tag{VD}$$

(b) $X$ satisfies the *Poincaré inequality*, PI($\beta$), if there exists a constant $c_2$ such that for any ball $B = B(x,R) \subset X$ and $f \in \mathcal{F}$,

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \le c_2 R^\beta \int_B d\Gamma(f,f). \tag{PI($\beta$)}$$

Here $\overline{f}_B = \mu(B)^{-1}\int_B f(x)d\mu(x)$.
(c) $X$ satisfies the condition RES($\beta$) if there exist constants $c_3, c_4$ such that for any $x_0 \in X$, $R \ge 0$,

$$c_3 \frac{R^\beta}{V(x_0,R)} \le R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c) \le c_4 \frac{R^\beta}{V(x_0,R)}. \tag{RES($\beta$)}$$

Here, for $A$, $B$ which are disjoint subsets of $X$, we define the effective resistance $R_{\text{eff}}(A,B)$ by

$$R_{\text{eff}}(A,B)^{-1} = \inf\Big\{\int_X d\Gamma(f,f) : f = 0 \text{ on } A \text{ and } f = 1 \text{ on B},\ f \in \mathcal{F}\Big\}.$$

(d) $X$ satisfies the condition E($\beta$) if for any $x_0 \in X$, $R \ge 0$,

$$c_5 R^\beta \le \mathbb{E}^{x_0}[\tau_{B(x_0,R)}] \le c_6 R^\beta, \tag{E($\beta$)}$$

where $\tau_A = \inf\{t \ge 0 : Y_t \notin A\}$, $Y_t$ is the strong Markov process associated to the Dirichlet form $(\mathcal{E},\mathcal{F})$, and $\mathbb{E}^{x_0}$ denotes the expectation starting from the point $x_0$.

## 2. Proof of $(b) \Rightarrow (a)$.

Throughout this section, we assume that (b) holds. Fix $x_0 \in X$ and for $R > 0$, let $B_R := B(x_0,R)$. Let $\mathcal{F}_{B_R} = \{u \in L^2(X,\mu) : u = 0\ \mu\text{-a.e. on } B_R^c\}$ and consider the part of the Dirichlet form $(\mathcal{E},\mathcal{F}_{B_R})$ (see [FOT], Section 4.4). Let $\{P_t^{B_R}\}$ be the corresponding semigroup.

**Lemma 2.1.** *There exists a version of the heat kernel $p_t^{B_R}(x,y)$ for $\{P_t^{B_R}\}$ and, for each $\varepsilon_1,\varepsilon_2 \in (0,1)$, there exists $c_{\varepsilon_1,\varepsilon_2} > 0$ such that*

$$p_t^{B_R}(x,y) \ge \frac{c_{\varepsilon_1,\varepsilon_2}}{V(x_0,\varepsilon_1 R)},$$

*for all $x,y \in B(x_0,\varepsilon_1 R)$ and $\varepsilon_2 R^\beta < t < R^\beta$.*

*Proof.* First, define

$$p_t^{B_R}(x,y) := p_t(x,y) - \mathbb{E}^x[p_{t-\tau_{B_R}}(Y_{\tau_{B_R}},y), \tau_{B_R} \le t], \tag{2.1}$$

where $Y_t$ is the diffusion process corresponding to $(\mathcal{E}, \mathcal{F})$ and $\tau_{B_R} = \inf\{t \geq 0 : Y_t \notin B(x_0, R)\}$. Then, it is easy to check, using the strong Markov property, that $p_t^{B_R}(x,y)$ is a version of the heat kernel for $\{P_t^{B_R}\}$. The proof of (2.1) is now a standard argument (see, for example, Lemma 5.1 in [FS]). $\square$

Let $d\nu = dt \otimes d\mu$, $\mathcal{H} = L^2(\mathbb{R}^1 \times X, d\nu)$ and $\tilde{\mathcal{F}} = \{u : \mathbb{R}^1 \to \mathcal{F} : \mathcal{A}(u,u) + \|u\|_{\mathcal{H}}^2 < \infty\}$ where $\mathcal{A}(u,u) = \int_{\mathbb{R}^1} \mathcal{E}(u(t,\cdot), u(t,\cdot))dt$. Let $\tilde{\mathcal{F}}^* = \{u : \mathbb{R}^1 \to \mathcal{F}^* : \int_{\mathbb{R}^1} \|u(t,\cdot)\|_{\mathcal{F}^*}^2 dt + \|u\|_{\mathcal{H}}^2 < \infty\}$, where $\mathcal{F}^*$ is the dual of $\mathcal{F}$ in the sense $\mathcal{F} \subset L^2(X,\mu) \subset \mathcal{F}^*$. Note that $\tilde{\mathcal{F}} \subset \mathcal{H} = \mathcal{H}^* \subset \tilde{\mathcal{F}}^*$. Let

$$\tilde{\mathcal{W}} = \{u \in \tilde{\mathcal{F}} : \frac{\partial u}{\partial t} \in \tilde{\mathcal{F}}^*\}$$

$$\tilde{\mathcal{E}}(u,v) = (u, \frac{\partial v}{\partial t})_\nu + \mathcal{A}(u,v) \quad \text{if} \quad u \in \tilde{\mathcal{F}}, v \in \tilde{\mathcal{W}},$$

where $(u,v)_\nu = \int_{\mathbb{R}^1} \int_X uv \, d\mu \, dt$. Let $\{Y_t(x)\}$ be the diffusion process corresponding to $(\mathcal{E}, \mathcal{F})$. Then the semigroup corresponding to $\tilde{\mathcal{E}}$ can be written as $P_t u(t_0, x_0) = \mathbb{E}[u(t_0 + t, Y_t(x_0))]$ so that the corresponding generator is $\frac{\partial}{\partial t} + \mathcal{L}$ (the corresponding diffusion is $Z_t = (t, Y_t)$), whereas the dual semigroup $\{\hat{P}_t\}$ can be written as $\hat{P}_t u(t_0, x_0) = \mathbb{E}[u(t_0 - t, Y_t(x_0))]$ and the corresponding generator is $-\frac{\partial}{\partial t} + \mathcal{L}$. (See [O] for details.)

**Lemma 2.2.** *Let $u$ be a non-negative solution of the heat equation on $Q := I \times G$, where $I = (a,b)$ and $G$ is an open connected subset of $X$. Then $u(t,x) \geq \int p_{t-s}^B(x,y) u(s,y) d\mu(y)$ $\mu$-a.e. $x$ and all $0 < s < t$ where $B \subset\subset G$.*

*Proof.* The claim is equivalent to $(u - \hat{P}_{t-s}^Q u)(t,x) \geq 0$ for all $(t,x) \in Q$ and all $0 < s < t$.

Let $\alpha > 0$. Then, $\tilde{\mathcal{E}}_\alpha(u,g) \geq 0$ for all non-negative $g \in \tilde{\mathcal{F}}_Q$. So, for any non-negative $\alpha$-excessive function (w.r.t. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}_Q)$ –see [O], Section 4.3, for a discussion of excessive functions in the parabolic case) $v \in \mathcal{F}_Q$, we have

$$\begin{aligned}
(u - e^{-\alpha s}\hat{P}_s^Q u, v)_\nu &= (u, v - e^{-\alpha s} P_s^Q v)_\nu \\
&= (u_Q, v - e^{-\alpha s} P_s^Q v)_\nu + (H_Q^\alpha u, v - e^{-\alpha s} P_s^Q v)_\nu \\
&\geq (u_Q, v - e^{-\alpha s} P_s^Q v)_\nu = \tilde{\mathcal{E}}_\alpha(u, G_\alpha^Q v - e^{-\alpha s} P_s^Q G_\alpha^Q v)_\nu =: I_1,
\end{aligned}$$

where $u = u_Q + H_Q^\alpha u$ is the orthogonal decomposition of $u$ into $\mathcal{F}_Q \oplus \mathcal{H}_{Q^c}^\alpha$ (see p. 149 of [FOT] –the same proof works for the parabolic case). Here the inequality in the third line is because $H_Q^\alpha u(x) = E^x(e^{-\alpha \sigma_{Q^c}} u(Z_{\sigma_{Q^c}})) \geq 0$ (due to Lemma 5.1.3 in p. 105 of [O]) and the fact that $v$ is $\alpha$-excessive (the definition of excessive functions in [O] is different from that in [FOT], but the proof of Theorem 2.2.1 in [FOT] also establishes equivalent conditions for the parabolic case, too). Since $G_\alpha^Q v - e^{-\alpha s} P_s^Q G_\alpha^Q v = \int_0^s e^{-\alpha l} P_l^Q v dl \in \mathcal{F}_Q$ is non-negative on $Q$, $I_1 \geq 0$. Thus $u - e^{-\alpha s}\hat{P}_s^Q u \geq 0$ on $Q$. Since this holds for all $\alpha > 0$, we have $u \geq \hat{P}_s^Q u$ on $Q$. $\square$

Once these properties are established, then proving (a) is standard; prove the oscillation inequality first and then use the inequality to establish PHI. Indeed, the proof of Lemma 5.2 and Theorem 5.4 in [FS] work line by line, with suitable changes of the scaling exponents.

## 3. Proof of $(a) \Rightarrow (b)$.

There is a standard argument, given in [SC1] and Section 5.5 of [SC2] which proves that PHI($\beta$) implies VD, PI($\beta$), and HK($\beta$). See also [HSC] for the case $\beta \neq 2$. However, as this argument uses existence and regularity of caloric and harmonic functions, we will give more complete details of the initial stages of this argument.

First, if $f \in L^2(X,\mu)$ we have that $P_t f \in \mathcal{D}(\mathcal{L})$, and $v(t,x) = P_t f(x)$ is a solution to the heat equation in $X \times (0,\infty)$. Let $x \in X$, $t > 0$, $r = t^\beta$ and $f \geq 0$ with $\int f = 1$. Then applying PHI($\beta$) in $Q = (0,4t) \times B(x,2r)$ we obtain

$$\sup_{Q_-} \tilde{v} \leq C \inf_{Q_+} \tilde{v}.$$

Hence if $B = B(x,r)$ then since $\int P_s f = 1$

$$\mu(B) \sup_{Q_-} \tilde{v} \leq C \int_B v(2t,y)\mu(dy) \leq C.$$

Thus for each $x \in X$ we have

$$\widetilde{P_t f}(x) \leq c(t) ||f||_1. \tag{3.1}$$

Given (3.1), we can use the same arguments as in p. 52 of [B] (using the results in [Y]) to deduce the existence of a transition density $p_t(x,y)$.

**Lemma 3.1.** *There exist an exceptional set $N$ and a jointly measurable transition density $p_t(x,y)$, $t > 0$, $x, y \in (X \setminus N) \times (X \setminus N)$, such that*

$$\begin{aligned}
P_t(x,A) &= \int_A p_t(x,y)\mu(dy) \text{ for } x \in X \setminus N, \quad t > 0,\ A \in \mathcal{B}(X \setminus N),\\
p_t(x,y) &= p_t(y,x) \text{ for all } x, y, t,\\
p_{t+s}(x,z) &= \int p_s(x,y)p_t(y,z)\mu(dy) \quad \text{for all } x, z, t, s.
\end{aligned}$$

Since $p_t(x,y) = P_{t/2}p_{t/2}(\cdot,y)(x)$ it follows that $p_t(\cdot,y)$ is a solution of the heat equation. Now take a quasi continuous modification $\tilde{p}_t(x,y)$ w.r.t. $x$ and use it in the procedure of (4) in [Y]. Then, by Theorem 1 in [Y], there exists $p_t(x,y)$ which is quasi continuous and satisfies the three equalities in Lemma 3.1. (In fact, the uniqueness criteria in Theorem 1 in [Y] shows that this $p_t(x,y)$ is the same as the original one.) Thus it satisfies the PHI, and so can be extended to $(0,\infty) \times X \times X$ as a jointly continuous function.

We now sketch the argument that PHI($\beta$) implies VD, PI($\beta$), and HK($\beta$). We begin with VD, which also gives a key lower bound on the transition density for the killed process.

Applying the PHI to the function $u(t,x) = p_t(x_0,x)$ in the region $Q(x_0,0,R)$ we obtain (writing $T = R^\beta$)

$$p_{2T}(x_0,x_0) \leq c p_{4T}(x_0,y), \quad y \in B(x_0,R).$$

Integrating over $B(x_0,R)$ gives

$$p_{2T}(x_0,x_0)V(x_0,R) \le c\int_{B(x_0,R)} u(4T,y) \le c, \tag{3.2}$$

which gives an upper bound on $p_{2T}(x_0,x_0)$ in terms of the volume of balls.

To obtain a lower bound, write $B_\lambda = B(x_0,\lambda R)$, and let $\varphi \in \mathcal{F}$ be a cut-off function for $B_{5/2} \subset B_3$. Let $p_t^0(x,y)$ be the heat kernel for the process $Y$ killed on exiting $B_4$. Define

$$u(t,x) = \begin{cases} \varphi(x), & x \in B_2,\ 0 < t \le 2T, \\ \int_{B_3} p^0_{t-2T}(x,y)\varphi(y)\mu(dy), & x \in B_2,\ 2T < t \le 4T. \end{cases}$$

**Lemma 3.2.** *$u$ is a solution of the heat equation in $Q(x_0,T,R)$.*

*Proof.* The function $u_t(x,t) = \frac{\partial u}{\partial t}$ exists for $t > 2T$, and is zero for $t < 2T$. Since $u(x,t)$ is continuous at $t = 2T$ for $x \in B$, it is straightforward to check that $u_t$ is the derivative of $u$ in the Schwartz' distribution sense.

Since we have $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ for all $t > 2T$, we have for $f \in \mathcal{F} \cap C(X)$ with support in $B_2$ that

$$\int f u_t \, d\mu = -\mathcal{E}(f, u(t,\cdot)), \qquad t > 2T. \tag{3.3}$$

If $t < 2T$ then since $u = 1$ on $B_2$ (3.3) also holds for $t < 2T$. Thus it follows that (1.1) holds. $\square$

We can now, as in [SC1], [SC2], [HSC], use PHI($\beta$) in $Q(x_0,0,R)$ to obtain

$$1 = u(y,2T) \le cu(x_0,4T) \le c\int_{B_3} p^0_{2T}(x_0,y), \quad y \in B(x_0,R). \tag{3.4}$$

Using the PHI in a chain of regions $Q(y_i,t_i,r) \subset [0,4T] \times B(x_0,4R)$ we obtain

$$p^0_{2T}(x_0,y') \le cp^0_{4T}(x_0,y), \quad y' \in B(x_0,3R),\ y \in B(x_0,R). \tag{3.5}$$

Integrating (3.5) over $y' \in B_3$ gives

$$\int_{B_3} p^0_{2T}(x_0,y')\mu(dy') \le cp^0_{4T}(x_0,x_0)V(x_0,3R), \tag{3.6}$$

and combining (3.4) and (3.6) we deduce that

$$V(x_0,3R)^{-1} \le cp^0_{4T}(x_0,y), \quad y \in B(x_0,R). \tag{3.7}$$

The inequalities (3.2) and (3.7) control $p_t(x_0,x_0)$ from above and below in terms of the volume of balls, and since $t \to p_t(x_0,x_0)$ is decreasing one easily deduces, by the same arguments as in [SC2], that volume doubling holds.

Given the lower bound (3.7), the proof of HK($\beta$) now follows as in Section 5 of [HSC]. For the global lower bound one uses (3.7) and a standard chaining argument. (3.7) gives uniform control of the probability that $Y$ exits a ball radius $r$ before time $t = r^\beta$, and using this the upper bounds on $p_t(x,y)$ follow as in p. 1472–1475 of [HSC].

We remark that (3.7) also gives a lower bound on the transition density of the process $Y$ reflected at $\partial B$ (see [Ch]). Using this the argument of [SC1] can be used to obtain PI($\beta$).

## 4. Appendix: Proof of $\mathbf{VD}+\mathbf{EHI}+\mathbf{RES}(\beta) \Rightarrow \mathbf{VD}+\mathbf{EHI}+\mathbf{E}(\beta)$.

In this appendix, we modify the proof in [GT2] and prove VD + EHI + RES($\beta$) $\Rightarrow$ E($\beta$) in a general MMD framework. This fact is used in [BBK] Theorem 2.15.

Recall from [FOT, Section 1.6] the definition of invariant sets and an irreducible Dirichlet form.

**Lemma 4.1.** *Let $X$ satisfy EHI. Then $\mathcal{E}$ is irreducible.*

*Proof.* Let $A$ be an irreducible set, and suppose both $\mu(A) > 0$ and $\mu(A^c) > 0$. Then there exists a ball $B = B(x, R)$ with $\mu(A \cap B') > 0$ and $\mu(A^c \cap B') > 0$, where $B' = B(x, R/2)$. Since $P_t 1_A = 1_A$ it follows that $u = 1_A$ and $v = 1_{A^c}$ are harmonic on $B$. So by EHI we have

$$\tilde{u}(x) \leq C\tilde{u}(y), \quad x, y \in B'.$$

Since $u > 0$ on a set of positive measure, we have that there exists $x \in B'$ with $\tilde{u}(x) > 0$; hence by the EHI $\tilde{u} > 0$ on $B'$. But as $\tilde{u} = 1_A$ $\mu$-a.e., we deduce that $\mu(A^c \cap B') = 0$, a contradiction. □

**Proposition 4.2.** *Let $X$ satisfy EHI, and $B = B(x, R)$. Then $Gg < \infty$ on $B$ if $g \in L^1_+(B)$.*

*Proof.* (sketch). Consider the Dirichlet form $\mathcal{E}_B$ with domain $\mathcal{F}_B = \{f \in \mathcal{F} : f|_{B^c} = 0\}$. Let $A = B(x, R/2)$ and $h(x) = P^x(T_A < \tau_B)$. Then $h$ is excessive with respect to $\mathcal{E}_B$. If $h$ were constant on $B$ then we would have $h = 1$ on $B$, and the set $B$ would be an invariant set for $\mathcal{E}$. Thus $h$ is non-constant.

So by [BG, Ex. (4.22), p. 89] we deduce that the killed semigroup $P_t^B$ is transient. Hence (see [FOT, Section 1.6]) we have $Gg < \infty$ for any $g \in L^1_+(B, \mu)$. □

**Lemma 4.3.** *Let $D$ be a bounded domain in $X$. Then EHI implies that there exists the Green density $g^D(\cdot, \cdot)$ which is continuous on $(X \times X) \setminus \Delta$ and $g_D(x, y) = g_D(y, x)$ for all $x, y \in (X \times X) \setminus \Delta$, where $\Delta$ is the diagonal. Further, there exists $C > 0$ such that for any $r > 0$, if $y_0, y_1 \in X$ satisfy $d(y_0, y_1) \geq 2r$, then*

$$g_D(y_0, x) \leq C g_D(y_0, y) \qquad \forall x, y \in B(y_1, r). \tag{4.1}$$

*Proof.* Let $x_0, x_1 \in D$, Choose $r > 0$ such that $B(x_i, 2r) \subset D$, $B(x_0, 2r) \cap B(x_1, 2r) = \emptyset$. Write $B_i = B(x_i, 2r)$, $B_i' = B(x_i, r)$. Let $f, g \in \mathcal{F}$ with supports in $B_0'$ and $B_1'$, and $\int f = \int g = 1$. Let $G_D$ be the Green operator for the process $Y$ killed on exiting $D$. By Proposition 4.2 we have $G_D f < \infty$, $G_D g < \infty$.

Then if $u \in \mathcal{F}$ with $\operatorname{Supp} u \subset B(x_1, 2r)$,

$$\mathcal{E}(G_D f, u) = (f, u) = 0, \tag{4.2}$$

so $G_D f$ is harmonic on $B_1$. Similarly $G_D g$ is harmonic on $B_0$. By the EHI if $x \in B_1'$ then

$$G_D f(x) \leq C G_D f(y), \quad y \in B_1'. \tag{4.3}$$

Similarly

$$G_D g(x) \le C G_D g(x_0), \quad x \in B'_0.$$

So

$$G_D f(x_1) \le C(g, G_D f) = C(G_D g, f) \le C^2 G_D g(x_0).$$

Now fix $g$ such that $C_1 = G_D g(x_0) < \infty$ – such a $g$ exists by choosing $g \le ch_0$. Then we have $G_D f(x_1) \le c'||f||_1$ for all $f$ with support in $B'_0$. Therefore the kernel $G_D(x_1, dx)$ has a density $g_D(x_1, y)$ on $B'_0$. Since $(f, G_D g) = (G_D f, g)$ for $f, g \in L^2$, it follows that $g_D(x, y) = g_D(y, x)$ $\mu \times \mu$–a.e.

Now, take $y_0, y_1 \in X$ that satisfy $d(y_0, y_1) \ge 2r$. For any $\epsilon > 0$ and $f \in L^2$ with support in $B(y_0, \epsilon r)$, similarly to (4.2) we can show that $G_D f$ is harmonic on $B(y_1, (2-\epsilon)r)$. Thus, by the same way as (4.3), we have

$$G_D f(x) \le C G_D f(y), \quad x, y \in B(y_1, r). \tag{4.4}$$

Now let $f_n(z) = V(y_0, r_n)^{-1} 1_{B(y_0, r_n)}(z)$ where $\epsilon r \ge r_n \downarrow 0$. Applying (4.4) to $f_n$ and take $n \to \infty$, we obtain (4.1) for $\mu$-a.e. $y_0$. By the usual oscillation argument, we can deduce that $g_D(x, y)$ is continuous on $(X \times X) \setminus \Delta$. Especially, $g_D(x, y) = g_D(y, x)$ for all $x, y \in (X \times X) \setminus \Delta$. We thus obtain (4.1) for all $y_0 \in X$. □

Now let $M \ge 2$ be fixed. (In fact, we can take M=2.)

**Definition 4.4.** $(\mathcal{E}, \mathcal{F})$ satisfies (HG) if there exists a constant $c_1 > 0$ such that for any ball $B(x_0, R)$, there exists the Green kernel $g^{B_R}(x_0, y)$ and for any $0 < r \le R/M$, we have

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \le c_1 \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y). \tag{$HG$}$$

**Lemma 4.5.** *(EHI)* $\Rightarrow$ $(HG)$.

*Proof.* Given Lemma 4.3 this is the same argument as in [B]. We prove that if $d(x_0, x) = d(x_0, y) = R$, and $B(x_0, 2R) \subset D$ then

$$C_1^{-1} g_D(x_0, y) \le g_D(x_0, x) \le C_1 g_D(x_0, y). \tag{4.5}$$

Once (4.5) is proved, then (HG) holds by the maximum principle (which holds for $G_D f$ and so for $g_D$ as well). By symmetry it is enough to prove the right hand inequality of (4.5).

Let $x'$, $y'$ be the midpoints of $\gamma(x_0, x)$, and $\gamma(x_0, y)$. Thus $d(x_0, x') = d(x_0, y') = R/2$. Clearly we have $d(x', y) \ge R/2$ and $d(x, y') \ge R/2$.

We now consider two cases.

Case 1. $d(x', y') \le R/3$. Let $z$ be the midpoint of $\gamma(x', y')$. Then $d(z, x') \le R/6 \le R/4$. So applying (4.1) to $g_D(x_0, \cdot)$ in $B(x', R/4) \subset B(x', R/2)$, we deduce that

$$C_2^{-1} g_D(x_0, x') \le g_D(x_0, z) \le C_2 g_D(x_0, x'). \tag{4.6}$$

Now apply (4.1) to $g_D(x_0, \cdot)$ in $B(x, R/2) \subset B(x, R)$, to deduce that

$$C_2^{-1} g_D(x_0, x) \leq g_D(x_0, x') \leq C_2 g_D(x_0, x).$$

Combining these inequalities we deduce that

$$C_2^{-2} g_D(x_0, x) \leq g_D(x_0, z) \leq C_2^2 g_D(x_0, x),$$

and this, with a similar inequality for $g_D(x_0, y)$, proves (4.5).
Case 2. $d(x', y') > R/3$. Apply (4.1) to $g_D(y, \cdot)$ in $B(x_0, R/2) \subset B(x_0, R)$, to deduce that

$$C_2^{-1} g_D(y, x') \leq g_D(y, x_0) \leq C_2 g_D(y, x'). \tag{4.7}$$

Now look at $g_D(x', \cdot)$. If $z'$ is on $\gamma(y', y)$ with $d(y', z') = s \in [0, R/2]$ then as $d(x', y') > R/3$ and $d(x', y) \geq R/2$ we have $d(x', z') \geq \max(R/3 - s, s)$. Hence we deduce $d(x', z') \geq R/6$. So applying (4.1) repeatedly to $g_D(x', \cdot)$ for a chain of balls $B(z', R/12) \subset B(z', R/6)$ we deduce that

$$C_2^{-6} g_D(x', y') \leq g_D(x', y) \leq C_2^6 g_D(x', y'). \tag{4.8}$$

So, we obtain from (4.7) and (4.8),

$$g_D(y, x_0) \leq C_2 g_D(y, x') \leq C_2^7 g_D(x', y'), \quad g_D(x', y') \leq C_2^6 g_D(y, x') \leq C_2^7 g_D(y, x_0).$$

We have similar inequalities relating $g_D(x, x_0)$ and $g_D(x', y')$, which proves (4.5). □

**Lemma 4.6.** *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (HG).*
*1) For any ball $B(x_0, R)$ and for any $0 < r \leq R/M$, we have*

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \asymp R(B_r, B_R^c) \asymp \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y). \tag{4.9}$$

*2) Let $B_k = B(x_0, M^k r)$ for $k = 0, 1, \cdots$. Then, for any integers $0 \leq m < n$,*

$$\sup_{y \notin B_m} g^{B_n}(x_0, y) \asymp \sum_{k=m}^{n-1} R(B_k, B_{k+1}^c) \asymp \inf_{y \in B_m} g^{B_n}(x_0, y). \tag{4.10}$$

*Proof.* For 1), first the following is standard (see for example (4.7) in [GT2]).

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \geq R(B_r, B_R^c) \geq \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y).$$

Thus, using (HG), we obtain (4.9).

For 2), note first that the following holds by the definition of resistance

$$\sum_{k=m}^{n-1} R(B_k, B_{k+1}^c) \le R(B_m, B_n^c).$$

This and (4.9) implies the lower bound for $\inf g^{B_n}$ in (4.10). Next, by the reproducing property of $g^{B_k}$, we know that $g^{B_{k+1}}(x,\cdot) - g^{B_k}(x,\cdot)$ is a harmonic function in $B_k$. Thus,

$$g^{B_{k+1}}(x,y) - g^{B_k}(x,y) \le \sup_{z \notin B_k} g^{B_{k+1}}(x,z) \le cR(B_k, B_{k+1}), \quad \forall y \in X, \tag{4.11}$$

where the first inequality is by the maximum principle and the second inequality is by (4.9). For $y \notin B_m$, by (4.9)

$$g^{B_{m+1}}(x,y) \le c' R(B_m, B_{m+1}). \tag{4.12}$$

For such $y$, adding up (4.12) with (4.11) for $m < k < n$, we obtain the upper bound of $\sup g^{B_n}$ in (4.10). $\square$

*Proof of VD+ EHI+ RES($\beta$) $\Rightarrow$ E($\beta$).*

$$\begin{aligned} E^{x_0}[\tau_{B_R}] &= \int g^{B_R}(x_0,y)d\mu(y) \ge \int_{B(x_0,r)} g^{B_R}(x_0,y)d\mu(y) \\ &\ge cR(B_r, B_R^c)V(x_0,r) \ge cR^\beta, \end{aligned}$$

where we used Lemma 4.6 1) in the second inequality and VD + RES($\beta$) in the last inequality.

Now, for each $k \in \mathbb{Z}$, let $r_k = M^k$, $B_k = B(x_0, r_k)$ and let $n_0$ be the minimum number such that $R < r_{n_0}$. Then

$$\begin{aligned} E^{x_0}[\tau_{B_R}] &\le E^{x_0}[\tau_{B(x_0,r_{n_0})}] = \int_{B_{n_0}} g^{B_{n_0}}(x_0,y)d\mu(y) \\ &= \sum_{m=-\infty}^{n_0-1} \int_{B_{m+1}\setminus B_m} g^{B_m}(x_0,y)d\mu(y) \le c \sum_{m=-\infty}^{n_0-1} \Big( \sum_{k=m}^{n_0-1} R(B_k, B_{k+1}^c) \Big) \mu(B_{m+1}\setminus B_m) \\ &= c \sum_{k=-\infty}^{n_0-1} \Big( \sum_{m=-\infty}^{k} \mu(B_{m+1}\setminus B_m) \Big) R(B_k, B_{k+1}^c) = c \sum_{k=-\infty}^{n_0-1} \mu(B_{k+1}) R(B_k, B_{k+1}^c) \\ &\le c' \sum_{k=-\infty}^{n_0-1} r_{k+1}^\beta \le c'' R^\beta, \end{aligned}$$

where we used Lemma 4.6 2) in the second inequality and VD + RES($\beta$) in the third inequality. We thus obtain E($\beta$). $\square$

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MTB: Department of Mathematics, University of British Columbia, Vancouver V6T 1Z2, Canada

RFB: Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

TK: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan