Some remarks for stable-like jump processes on fractals

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Abstract. We summarize recent work on non-local Dirichlet forms on fractals whose corresponding processes are stable-like jump processes. Especially, we introduce three natural non-local Dirichlet forms on d-sets and prove that these forms are equivalent.

1. Introduction

Since the late 80's of the last century, there has been a lot of development in the mathematical study of stochastic processes and the corresponding operators on fractals (see, for instance, [2], [19], [21]). The main object of attention was diffusion processes and local operators on them. On the other hand, there has been intensive study of Besov spaces, (which are roughly speaking, fractional extensions of Sobolev spaces) on d-sets, which correspond to regular fractals (see [18], [29], [30]). These two research areas seem to have developed rather independently, but since the domains of local regular Dirichlet forms on fractals were revealed to be Lipschitz (Besov) spaces $\mathrm{Lip}(\alpha, 2, \infty)(G)$ ([16], [20], [23]), several works which connect the two areas have appeared (see, for instance, [17], [20], [24]). It has become clear recently that the Besov spaces $B_\alpha^{2,2}(G)$, which are mainly treated by Jonsson-Wallin ([18]) and Triebel ([29],[30]), are in fact non-local regular Dirichlet spaces and the corresponding processes are stable-like jump processes. The aim of this note is to summarize several recent works in this direction (including some new results) and to try to make clear the connections between the two research areas, i.e., functional spaces and stochastic processes. We introduce three natural non-local Dirichlet forms on d-sets and prove that these forms are equivalent.

In Section 2.1, we give our first construction through subordination of a diffusion process on a fractal. When there is a Markov process on G, one can always construct a (jump-type) Markov process by using a one-dimensional α-stable process as a subordinator (see [6], [25]). We take this procedure starting from a fractional diffusion (in the sense of [2]) on a fractal. This and the next construction summarize the results of [27], where the two constructions are given and proved to be equivalent in the sense of forms. In Section 2.2, we give our second construction through a Besov space on d-sets G. In [18], [29] and [30], a

Besov space $B_{\alpha}^{2,2}(G)$ is defined (see Section 2.2 for the detailed definition of the space). Further, in [29] Section 25, a quadratic form induced from the Besov norm is given and the asymptotic behaviour of eigenvalues is studied using approximation numbers of operators. This quadratic form is revealed to be a non-local regular Dirichlet form ([12],[27]) and the on-diagonal heat kernel upper bound can be deduced from a version of Sobolev embeddings and the Varopoulos theorem ([12]). We will introduce them following [12]. In Section 2.3, we give our third construction through a time change of a stable process on $\mathbf{R}^D$. Let $\{B_t^{(\alpha)}\}_{t\geq 0}$ be a (2α)-stable process on $\mathbf{R}^D$. According to the general theory ([11]), if a measure μ on $G\subset \mathbf{R}^D$ is a smooth measure, there is a corresponding positive continuous additive functional A_t and one can time change $B_t^{(\alpha)}$ by the inverse of A_t. A_t increases only when $B_t^{(\alpha)}\in G$, so the resulting process is a process which moves on G and the trajectory of the process is given from the original one by erasing the part of the trajectory on $\mathbf{R}^D\setminus G$. We prove that when $d>D-2\alpha$, μ is a smooth measure so that we can construct such a time changed process. We note that when $A\subset\mathbf{R}^D$ is a bounded domain with smooth boundary ∂A, a similar time changed form (with exponential killing) is studied in [15], where the original form lies on A and the time changed form lies on ∂A. They also gave an orthogonal decomposition of the original form on A (the so-called Weyl decomposition) to divide the effect of the form on the interior and boundary. In [9], a similar problem is treated when A is a bounded (ϵ,δ)-domain with boundary ∂A which is a d-set with $D-1\leq d<D$ (though they do not explicitly construct the corresponding process, as they do not know whether μ is a smooth measure or not in general). In Section 3, we compare the three forms constructed in Section 2 and prove that by a suitable transformation of parameters, the three forms are equivalent. We also remark that the forms do not coincide in general, so that one cannot construct any one from the other by time changes by positive continuous additive functionals.

Throughout the note, we only consider compact fractals, but we believe that similar results hold for unbounded fractals.

Finally, we note that there are also recent works for non-local Dirichlet forms on d-sets through Riesz potentials ([32],[33]), and very recently, work that treats functional spaces systematically on p.c.f. self-similar sets, which roughly correspond to finitely ramified fractals ([28]).

2. Dirichlet forms and jump type processes on compact fractals

In this section, we will introduce three types of regular Dirichlet forms whose corresponding processes are stable-like jump type processes on compact fractals.

First of all, we will briefly review the definition of Dirichlet forms and the correspondence to processes following [11]. Let X be a locally compact separable metric space and ν de a positive Radon measure on X whose support is X. Let $\mathcal{E}$ be a symmetric bilinear closed form on $\mathbf{L}^2(X,\nu)$ with domain $\mathcal{F}$. $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form if it is Markovian, i.e., for each $u\in\mathcal{F}$, $v\equiv(0\vee u)\wedge 1\in\mathcal{F}$ and

$\mathcal{E}(v,v) \le \mathcal{E}(u,u)$. A Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular if there exists $C \subset \mathcal{F} \cap C_0(X)$ such that C is dense in $\mathcal{F}$ with $\mathcal{E}_1$-norm and C is dense in $C_0(X)$ under the uniform norm, where $C_0(X)$ is a space of continuous compact supported functions on X and $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{\mathbf{L}^2}$. $(\mathcal{E},\mathcal{F})$ is local if for each $u,v \in \mathcal{F}$ whose supports are disjoint compact sets, $\mathcal{E}(u,v) = 0$. There is a one to one correspondence between a regular Dirichlet form on $\mathbf{L}^2(X,\nu)$ and a ν-symmetric Hunt process (i.e., a strong Markov process whose paths are right continuous and quasi-left continuous w.r.t. some filtration) on X except for some exceptional set of starting points. Further, if the regular Dirichlet form is local, then the corresponding process is a diffusion process (i.e. Hunt process with continuous paths).

We now introduce our framework. (Note that for the construction of the first process, we need further assumptions on G, but for others, no further assumption is required.) Let G be a compact d-set in $\mathbf{R}^D$ $(D \ge 2, 0 < d \le D)$. That is, $G \subset \mathbf{R}^D$ and there exists $c_1, c_2 > 0$ such that

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d \qquad \text{for all } \ x \in G,\ 0 < r < 1, \tag{1}$$

where $B(x,r)$ is a ball centered at x and radius r w.r.t. the Euclidean metric. Thus d is the Hausdorff dimension of G and μ the Hausdorff measure on G. We normalize the size of G so that the diameter of G is 1.

2.1. Jump process as a subordination of a diffusion on a fractal

In this subsection, we assume further that there exists a *fractional diffusion* in the sense of Barlow [2]. That is, firstly there exists a metric ρ on G which has the midpoint property; for each $x, y \in G$, there exists $z \in G$ such that $\rho(x,y) = \rho(x,z)/2 = \rho(z,y)/2$. Secondly, there exists a μ-symmetric conservative Feller diffusion $\{B_t^G\}_{t\ge 0}$ on G which has a symmetric jointly continuous transition density $p_t(x,y)$ $(t > 0, x, y \in G)$ which has the following estimate,

$$\begin{aligned} & c_{2.1} t^{-d_s/2} \exp(-c_{2.2}(\rho(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \le p_t(x,y) \\ \le \quad & c_{2.3} t^{-d_s/2} \exp(-c_{2.4}(\rho(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \ \text{ for all } 0 < t < 1,\ x, y \in G, \end{aligned} \tag{2}$$

with some constant $d_s > 0, d_w \ge 2$. For simplicity, we further assume that $\rho(\cdot,\cdot)$ is equivalent to the Euclidean metric, i.e.,

$$\rho(x,y) \asymp |x-y| \qquad \text{for all } \ x, y \in G. \tag{3}$$

Here $f \asymp g$ means the ratio is bounded from above and below by some positive constants. In this case, $d_s/2 = d/d_w$ holds. It is known that typical Sierpinski gaskets and carpets satisfy the conditions. In general, (affine) nested fractals, (which is a class of fractals including Sierpinski gaskets) and Sierpinski carpets satisfies the conditions with $\rho(x,y) \asymp |x-y|^{d_c}$ for some $d_c \ge 1$ instead of (3).

For $0 < \alpha < 1$, let $\{\xi_t\}_{t>0}$ be the strictly α-stable subordinator, i.e., it is a one dimensional non-negative Lévy process independent of $\{B_t^G\}_{t\ge 0}$ with the generating function $E[\exp(-u\xi_t)] = \exp(-tu^\alpha)$. Let $\{\eta_t(u) : t > 0, u \ge 0\}$ be the distribution density of $\{\xi_t\}_{t>0}$. We define

$$q_t(x,y) = \int_0^\infty p_u(x,y)\eta_t(u)du \qquad \text{for all } t>0,\ x,y\in G.$$

Then, it is well known that $q_t(x,y)$ is a transition density of some Markov process which we denote by $\{X_t^{(\alpha)}\}_{t\geq 0}$, called the subordinate process (see [6], [25]). In our case, $\{X_t^{(\alpha)}\}_{t\geq 0}$ is a μ-symmetric Hunt process and we denote the corresponding Dirichlet form on $\mathbf{L}^2(G,\mu)$ as $(\mathcal{E}_{X^{(\alpha)}}, \mathcal{F}_{X^{(\alpha)}})$. Then, this form can be expressed using the Lévy measure of the subordinator. (The author learned this general fact from Professor Fukushima [10].) First, note that

$$u^\alpha = \int_0^\infty (1-\exp(-us))n(ds),$$

where $n(ds) \equiv \alpha ds/(\Gamma(1-\alpha)s^{\alpha+1})$ is the Lévy measure. Define

$$j(x,y) = \int_0^\infty p_s(x,y)n(ds). \tag{4}$$

Then,

$$\mathcal{E}_{X^{(\alpha)}}(f,f) = \int\int_{G\times G} (f(x)-f(y))^2 j(x,y)d\mu(x)d\mu(y). \tag{5}$$

2.2. Jump process as a Besov space on a d-set

We first introduce Besov spaces on G and their trace theory within the scope of our use (see [18], [29], [30] etc. for details).

For $0<\alpha<1$, we introduce a Besov space $B_\alpha^{2,2}(G)$ as follows,

$$\begin{aligned} \|u|B_\alpha^{2,2}(G)\| &= \|u\|_{\mathbf{L}^2(G,\mu)} + \Big(\int\int_{G\times G} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}\mu(dx)\mu(dy)\Big)^{1/2} \\ B_\alpha^{2,2}(G) &= \{u : u \text{ is measurable}, \|u|B_\alpha^{2,2}(G)\| < \infty\}. \end{aligned}$$

For each $f \in \mathbf{L}^1_{loc}(\mathbf{R}^D)$ and $x\in\mathbf{R}^D$, define

$$Rf(x) = \lim_{r\downarrow 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy,$$

if the limit exists where m is the D-dimensional Lebesgue measure and $B(x,r)$ is a ball w.r.t. Euclidean metric. It is well-known that the limit exists quasi everywhere w.r.t. $H^1(\mathbf{R}^D)$-norm and coincides with $f(x)$ almost everywhere. For each $\beta>0$, denote $H^\beta(\mathbf{R}^D) = \Lambda_\beta^{2,2}(\mathbf{R}^D)$ the Besov space on $\mathbf{R}^D$ (see Remark 2.2 below). Then the following trace theorem is known (see, for instance, Chapter V, VI in [18] or Section 20 in [29]).

Theorem 2.1. *Let $\alpha\in\mathbf{R}$ such that $0<\hat{\alpha}\equiv \alpha-(D-d)/2<1$. Then the trace operator $Tr_G : u\mapsto Rf$ is a bounded linear surjection on*

$$Tr_G : H^\alpha(\mathbf{R}^D) \to B_{\hat{\alpha}}^{2,2}$$

with a bounded linear right inverse E_G (the extension operator).

Let $\overset{\circ}{H}{}^{\alpha}_{G}(\mathbf{R}^D)$ be a closure of $C_0^\infty(\mathbf{R}^D \setminus G)$ by the norm of $H^\alpha(\mathbf{R}^D)$. Then the following spectral synthesis theorem is known (Proposition 19.5 in [30]); $\overset{\circ}{H}{}^{\alpha}_{G}(\mathbf{R}^D)$ is the kernel of Tr_G, i.e.,

$$\overset{\circ}{H}{}^{\alpha}_{G}(\mathbf{R}^D) = \{f \in H^\alpha(\mathbf{R}^D) : \mathrm{Tr}_G f = 0\}. \tag{6}$$

Remark 2.2. *1) Note that the classical Besov norm over* $\|u|H^\beta(\mathbf{R}^D)\|$ *is given by*

$$\|u\|_{\mathbf{L}^2(\mathbf{R}^D,m)} + \left(\int_{|h|\le 1} \frac{\|\Delta_h^l u\|^2_{\mathbf{L}^2(\mathbf{R}^D,m)}}{|h|^{D+2\beta}} dh\right)^{1/2},$$

where l is an integer greater than β and Δ_h^l is the iterated difference in $\mathbf{R}^D$; see, for instance, Definition 4.1.1 *in* [1]*, Chapter I* 1.5 *in* [18]*, Definition* 10.3 *in* [29]*. It is known that this norm is equivalent to* $\|u|B^{2,2}_\beta(\mathbf{R}^D)\|$*; see, for instance, Chapter V* 1.1 *in* [18]*,* 10.5 *in* [29]*. Further, this space coincides with the classical Bessel potential space (also called the fractional Sobolev space or the Liouville space); see, for instance, Chapter V* 1.5 *in* [18]*.*
2) It is known that the norm $\|u|B^{2,2}_{\hat\alpha}(G)\|$ *is equivalent to the norm*

$$\inf\{\|g|H^\alpha(\mathbf{R}^D)\| : g \in H^\alpha(\mathbf{R}^D) \text{ such that } Tr_G g = u\}, \tag{7}$$

where $\alpha, \hat\alpha$ and Tr_G are given in Theorem 2.1; see, for instance, [29] *p193,* [30] *p307,* [18] *p103. We further have the following decomposition,*

$$H^\alpha(\mathbf{R}^D) = \overset{\circ}{H}{}^{\alpha}_{G}(\mathbf{R}^D) \oplus B^{2,2}_{\hat\alpha}(G). \tag{8}$$

Now, for $0 < \alpha < 1$ and $u, v \in B^{2,2}_\alpha(G)$, define

$$\mathcal{E}_{Y^{(\alpha)}}(u,v) = \int\int_{G\times G} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\alpha}} \mu(dx)\mu(dy).$$

By standard properties of Besov spaces, it is easy to check that $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_\alpha(G))$ is a regular Dirichlet space on $\mathbf{L}^2(G,\mu)$ (a detailed proof is given, for instance, in Theorem 3 of [27]). We denote $\{Y_t^{(\alpha)}\}_{t\ge 0}$ the corresponding Hunt process on G. We note that when $G = \mathbf{R}^D$, this is a (2α)-stable process on $\mathbf{R}^D$.

When G is an open D-set, $\{Y_t^{(\alpha)}\}_{t\ge 0}$ is the so called reflected (2α)-stable process on G proposed in [7]. It is obtained from the censored (2α)-stable process in G through Silverstein's reflected Dirichlet form procedure [26]. The censored (2α)-stable process in G, which is studied in [7], is obtained from the killed symmetric (2α)-stable process on G (the (2α)-stable process killed upon exiting G) by the so-called piecing together procedure, which is a procedure to glue an independent copy of the killed process from where it exits G from the interior (see [7]). Its corresponding form is $(\mathcal{E}_{Y^{(\alpha)}}, \overset{\circ}{B}{}^{2,2}_{\alpha}(G))$, where $\overset{\circ}{B}{}^{2,2}_{\alpha}(G)$ is a closure of $C_0^\infty(G)$ by the norm $\|\cdot|B^{2,2}_\alpha(G)\|$. The space $(\mathcal{E}_{Y^{(\alpha)}}, B^{2,2}_\alpha(G))$ is the reflected Dirichlet form of $(\mathcal{E}_{Y^{(\alpha)}}, \overset{\circ}{B}{}^{2,2}_{\alpha}(G))$ in the sense of [26]. In [7], various properties including conditions for the censored stable process to coincide with its reflected stable process are studied. For instance, when ∂G is a $\bar{d}$ set with $\bar{d} \in [D-1, D)$, then $B_\alpha^{2,2}(G) = \overset{\circ}{B_\alpha^{2,2}}(G)$ (thus the two processes are the same) if and only if $2\alpha \leq D - \bar{d}$.

2.3. Jump process as a time change of a stable process on $\mathbf{R}^D$

Let $\{B_t\}_{t\geq 0}$ be Brownian motion on $\mathbf{R}^D$ and $(\mathcal{E}, H^1(\mathbf{R}^D))$ the corresponding Dirichlet form. Following Section 2.2 in [11], we set

$$S_{00} = \{\mu \text{ is a positive finite Radon measure on } \mathbf{R}^D : \int_{\mathbf{R}^D} |v(x)|\mu(dx) \leq c_1 \sqrt{\mathcal{E}_1(v,v)} \ \forall v \in H^1(\mathbf{R}^D) \cap C_0(\mathbf{R}^D),\ \|U_1\mu\|_\infty < \infty\}, \quad (9)$$

where for $\beta \geq 0$, $\mathcal{E}_\beta(u,u) = \mathcal{E}(u,u) + \beta\|u\|^2_{\mathbf{L}^2(\mathbf{R}^D,m)}$ and $U_1\mu \in H^1(\mathbf{R}^D)$ is a unique function such that

$$\mathcal{E}_1(U_1\mu, v) = \int_{\mathbf{R}^D} v(x)\mu(dx) \qquad \text{for all } v \in H^1(\mathbf{R}^D) \cap C_0(\mathbf{R}^D).$$

Note that S_{00} is a subclass of smooth measures on $\mathbf{R}^D$. (In our setting, a smooth measure is a measure which charges no set of zero capacity.) Later, we will prove that when $d > D-2$ the Hausdorff measure μ is in S_{00}. Then, there exists a unique positive continuous additive functional $\{A_t\}_{t\geq 0}$ which is in Revuz correspondence with μ and satisfies

$$E^x[\int_0^\infty \exp(-t)dA_t] = \int_G g_1(x,y)\mu(dy) \qquad \text{for all } x \in G,$$

where $g_1(\cdot,\cdot)$ is a 1-order Green density of $\mathcal{E}$ (Theorem 5.1.6 of [11]). A_t increases only when $B_t \in G$. Set $\tau_t = \inf\{s > 0 : A_s > t\}$ and define $Z_t = B_{\tau_t}$. Then, $\{Z_t\}_{t\geq 0}$ is a μ-symmetric Hunt process which we are interested in and the corresponding regular Dirichlet form can be expressed as

$$\begin{aligned} \mathcal{F}_Z &= \{\phi \in \mathbf{L}^2(G,\mu) : \phi = u \ \mu-\text{a.e. for some } u \in H^1_e(\mathbf{R}^D)\}, && (10)\\ \mathcal{E}_Z(\phi,\phi) &= \mathcal{E}(H_\mathcal{S}u, H_\mathcal{S}u),\ \phi \in \mathcal{F}_Z, \phi = u \ \mu-\text{a.e.}, u \in H^1_e(\mathbf{R}^D). && (11)\end{aligned}$$

Here $\mathcal{S}$ is a quasi-support of μ (see p168 of [11]), $H_\mathcal{S}u(x) = E^x[u(B_{\sigma_\mathcal{S}})]$ where $\sigma_\mathcal{S} = \inf\{t \geq 0 : B_t \in \mathcal{S}\}$ and $H^1_e(\mathbf{R}^D)$ is the extended Dirichlet space. See Theorem 6.2.1 of [11] for details. Note that $\mathcal{S} \subset G$ and $\mu(G \setminus \mathcal{S}) = 0$. $\{Z_t\}_{t\geq 0}$ is a time changed process of Brownian motion (on $\mathbf{R}^D$) which moves only on G.

We now give a proof of $\mu \in S_{00}$ when $d > D-2$. For that, we recall the definition of capacities. For an open set $B \subset G$, define the (1-)capacity of B by

$$\mathrm{Cap}(B) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{F}, u \geq 1 \text{ on } B\}.$$

The capacity of any set $F \subset G$ is defined as the infimum of the capacity of open sets which contain F.

Proposition 2.3. *Assume $d > D-2$. Then $\mu \in S_{00}$.*

Proof. We follow the idea of Lemma 2.5 in [3]. We will prove the result for $D > 2$: the proof for $D = 2$ is similar. It is well-known that for each compact set $M \subset G$,

$$\begin{aligned}\mathrm{Cap}(M) = \sup\{\nu(M) \quad : \quad & \nu \text{ is a positive finite Radon measure, } \mathrm{Supp}[\nu] \subset M,\\ & G_1\nu(x) \equiv \int_M g_1(x,y)\nu(dy) \le 1 \ \text{ for all } \ x \in G\}. \end{aligned} \tag{12}$$

As $g_1(\cdot,\cdot)$ is the Green density w.r.t. the Dirichlet integral, we see

$$g_1(x,y) \asymp |x-y|^{2-D}. \tag{13}$$

Using this,

$$\begin{aligned}\int_M g_1(x,y)\mu(dy) &\le \int_G g_1(x,y)\mu(dy)\\ &\le \sum_{n=0}^{\infty}\int_{2^{-n-1}\le|x-y|_{\max}<2^{-n}} g_1(x,y)\mu(dy)\\ &\le c_1\sum_n 2^{n(D-2)}\mu(2^{-n-1}\le|x-y|\mathrm{max}<2^{-n})\\ &\le c_2\sum_n 2^{n(D-2)}2^{-dn}\equiv c_3<\infty, \end{aligned} \tag{14}$$

because of the assumption $d > D-2$. Here $|y|\mathrm{max} = \max_i |y_i|$ for $y = (y_1, \cdots, y_d) \in \mathbf{R}^D$. Thus, setting $\mu_M(\cdot) \equiv \mu(\cdot \cap M)$, we have $G_1\mu_M \le c_3$. Using (12),

$$\mathrm{Cap}(M) \ge \mu(M)/c_3 \tag{15}$$

for each compact set M, thus for each Borel set. Using Lemma 2.2.8 in [11], we have checked the first condition of (9). As $U_1\mu(x) = \int_{\mathbf{R}^D} g_1(x,y)\mu(dy)$, where we regard μ as a measure on $\mathbf{R}^D$ supported on G, (14) ensures the second condition of (9). This completes the proof. □

We can further prove the following.

Proposition 2.4. *Assume $d > D-2$. Then,*

$$P^x(\sigma_{\mathcal{S}} = 0) = 1 \qquad \textit{for all} \quad x \in G.$$

Proof. We follow the idea of Proposition 2.6 in [3]. As before, we will prove the result for $D > 2$: the proof for $D = 2$ is similar. Let $A_n(x) = \{y \in G : 2^{-n-1} \le |x-y|_{\max} < 2^{-n}\}$, and $\mathcal{S}^n(x) = \mathcal{S} \cap A_n(x)$. Note that $\mu(\mathcal{S}^n(x)) \neq 0$ for $n \ge 0$. To prove the assertion, it is enough to prove that

$$\sum_{n=1}^{\infty} 2^{n(D-2)}\,\mathrm{Cap}(\mathcal{S}^n(x_0)) = \infty \qquad \text{for all} \quad x_0 \in G. \tag{16}$$

If $K = [0,1]^d$ this is just the classical Wiener test (see [14]); the result used here follows, using (13), by exactly the same arguments. Using Kelvin's principle (see

Section 2.2 of [11]), we have for each compact set $M \subset G$,

$$\{\mathrm{Cap}(M)\}^{-1} = \inf \int_{M \times M} g_1(x,y)\nu(dx)\nu(dy), \tag{17}$$

where the infimum is taken over the positive Radon measures ν with $\nu(M) = 1$. Now, take an arbitrary compact set $M \subset \mathcal{S}^n(x_0)$ such that $\mu(M) \geq \mu(\mathcal{S}^n(x_0))/2$. Then $\mu(M \cap A_k(x)) \leq \mu(\mathcal{S}^n(x_0) \cap A_k(x)) \leq c_1 \mu(\mathcal{S}^n(x_0)) 2^{(n-k)d} \leq 2c_1 \mu(M) 2^{(n-k)d}$ for $k \geq n$. Thus, using (13) as before, we have

$$\begin{aligned}
& \mu(M)^{-2} \int_{M \times M} g_1(x,y)\mu(dx)\mu(dy) \\
\leq\ & \mu(M)^{-2} \int_M \mu(dx) \int_M c_2 |x-y|_{\max}^{2-D} \mu(dy) \\
\leq\ & c_3 \mu(M)^{-2} \int_M \mu(dx) \sum_{k=n}^{\infty} 2^{k(D-2)} \mu(M \cap A_k(x)) \\
\leq\ & c_4 \mu(M)^{-2} \int_M \mu(dx) \sum_{k=n}^{\infty} 2^{k(D-2)} \mu(M) 2^{(n-k)d} \\
=\ & c_4 2^{n(D-2)} \sum_{j=0}^{\infty} 2^{j(D-2-d)} \leq c_5 2^{n(D-2)},
\end{aligned}$$

because of the assumption $d > D-2$. Taking $\nu(\cdot) = \mu(\cdot \cap M)/\mu(M)$ in (17), we have $\mathrm{Cap}(M) \geq c_5^{-1} 2^{-n(D-2)}$. As this holds for all compact sets $M \subset \mathcal{S}^n(x_0)$ with $\mu(M) \geq \mu(\mathcal{S}^n(x_0))/2$, we have $\mathrm{Cap}(\mathcal{S}^n(x_0)) \geq c_5^{-1} 2^{-n(D-2)}$, which proves (16). $\square$

We can further construct one parameter family of jump processes by time changes. Instead of starting from Brownian motion on $\mathbf{R}^D$, we start with the rotation invariant (2α)-stable process $\{B_t^{(\alpha)}\}_{t\geq 0}$ $(0 < \alpha \leq 1)$, which can be constructed through subordination of Brownian motion on $\mathbf{R}^D$ by a strictly (2α)-stable subordinator for $0 < \alpha < 1$ (see Example 1.5.2 in [11], Example 30.6 in [25]). We denote the corresponding Dirichlet form $(\mathcal{E}^{(\alpha)}, H^\alpha(\mathbf{R}^D))$. When $\alpha = 1$, it is the Brownian motion. For $0 < \alpha < 1$, $\mathcal{E}^{(\alpha)}$ is expressed as

$$\mathcal{E}^{(\alpha)}(u,v) = \int\int_{\mathbf{R}^D \times \mathbf{R}^D} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{D+2\alpha}} m(dx)m(dy),$$

for each $u, v \in H^\alpha(\mathbf{R}^D)$ where m is a D-dimensional Lebesgue measure (for simplicity, we normalize the constant in front of the integrals). Then, we have a similar bound for the (1-order) Green density $g_1^{(\alpha)}(\cdot,\cdot)$ (see Example 37.19 in [25], note that $0 < 2\alpha < 2 \leq D$ in our case), so that

$$g_1^{(\alpha)}(x,y) \asymp |x-y|^{2\alpha - D}. \tag{18}$$

Thus, by the same argument as before, we see that when $2\alpha > D-d$, then $\mu \in S_{00}^{\alpha}$ (S_{00}^{α} is defined similarly to (9) w.r.t. $\mathcal{E}^{(\alpha)}$). Thus, under the condition we have a jump process on G which we denote by $\{Z_t^{(\alpha)}\}_{t\geq 0}$.

Remark 2.5. *For each $\lambda > 0$, set $\mathcal{E}_{\lambda}^{(\alpha)}(\cdot,\cdot) \equiv \mathcal{E}^{(\alpha)}(\cdot,\cdot) + \lambda\|\cdot\|_{\mathbf{L}^2}^2$; $(\mathcal{E}_{\lambda}^{(\alpha)}, H^{\alpha}(\mathbf{R}^D))$ is then a regular Dirichlet form on $\mathbf{L}^2(\mathbf{R}^D, m)$ with exponential killing. In [15], they consider a similar form on $\mathbf{L}^2(A,m)$ when A is a bounded domain with smooth boundary and in Theorem 4.1 of [15], they give a Weyl decomposition of $H^{\alpha}(A)$ w.r.t. $\mathcal{E}_{\lambda}^{(\alpha)}$. In Theorem 4.6 of [9], similar decomposition is given when A is a bounded (ϵ,δ)-domain with boundary ∂A which is a d-set with $D-1 \leq d < D$. We note here that we can obtain such a Weyl decomposition regarding G as a "boundary" of $\mathbf{R}^D$. Define*

$$X_{\lambda}^{\alpha}(\mathbf{R}^D) = \{u \in H^{\alpha}(\mathbf{R}^D) : \mathcal{E}_{\lambda}^{(\alpha)}(u,v) = 0 \text{ for all } v \in \overset{\circ}{H}{}_G^{\alpha}(\mathbf{R}^D)\}.$$

Then we can state the fact as follows.

Let $\alpha \in \mathbf{R}$ such that $0 < \hat{\alpha} \equiv \alpha - (D-d)/2 < 1$. Then, for each λ, the following orthogonal decomposition w.r.t. $\mathcal{E}_{\lambda}^{(\alpha)}$ holds;

$$H^{\alpha}(\mathbf{R}^D) = \overset{\circ}{H}{}_G^{\alpha}(\mathbf{R}^D) \oplus X_{\lambda}^{\alpha}(\mathbf{R}^D). \tag{19}$$

Further, there is a canonical isomorphism

$$\Pi_{\lambda}^{(\alpha)} : B_{\hat{\alpha}}^{2,2}(G) \to X_{\lambda}^{\alpha}(\mathbf{R}^D),$$

especially the decomposition is non-trivial in the sense that $X_{\lambda}^{\alpha}(\mathbf{R}^D) \neq \{0\}$.

This decomposition tells us that the decomposition in (8) is an orthogonal decomposition w.r.t. $\mathcal{E}_{\lambda}^{(\alpha)}$ if we identify $B_{\hat{\alpha}}^{2,2}(G)$ and $X_{\lambda}^{\alpha}(\mathbf{R}^D)$. As the proof goes in the same way and we do not use this later, we omit the proof here.

3. Comparison of the forms

Define $\bar{\alpha} = \alpha d_w/2$ and $\hat{\alpha} = \alpha - (D-d)/2$. We then have the following.

Proposition 3.1. *For $0 < \alpha < 2/d_w$,*

$$c_{3.1}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f) \leq \mathcal{E}_{X^{(\alpha)}}(f,f) \leq c_{3.2}\mathcal{E}_{Y^{(\bar{\alpha})}}(f,f) \qquad \text{for all } f \in \mathbf{L}^2(G,\mu). \tag{20}$$

For $(D-d)/2 < \alpha < 1$ or $\alpha = 1, D-2 < d < D$,

$$c_{3.3}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f) \leq \mathcal{E}_{Z^{(\alpha)}}(f,f) \leq c_{3.4}\mathcal{E}_{Y^{(\hat{\alpha})}}(f,f) \qquad \text{for all } f \in \mathbf{L}^2(G,\mu). \tag{21}$$

In particular, under the conditions,

$$\mathcal{F}_{X^{(\alpha)}} = B_{\bar{\alpha}}^{2,2}(G),\ \mathcal{F}_{Z^{(\alpha)}} = B_{\hat{\alpha}}^{2,2}(G).$$

Proof. (20) is proved by Stós ([27], Theorem 2). But we can prove it simply thanks to the expression (5). Indeed, using (2), (3), (4) and the fact $p_t(x,y) \le c_0$ for all $t \ge 1, x, y \in G$ (c_0 is some positive constant), we have

$$
\begin{aligned}
& j(x,y) \\
\le\ & \frac{\alpha}{\Gamma(1-\alpha)}\{c_{2.3}\int_0^1 s^{-(d_s/2+1+\alpha)}e^{-c_{2.4}(\frac{|x-y|^{d_w}}{s})^{\frac{1}{d_w-1}}}ds + c_0\int_1^\infty s^{-(1+\alpha)}ds\} \\
=\ & \frac{\alpha}{\Gamma(1-\alpha)}\{\frac{c_1}{|x-y|^{d_w(\alpha+d_s/2)}}\int_{c_{2.4}|x-y|^{\frac{d_w}{d_w-1}}}^\infty v^{(d_s/2+\alpha)(d_w-1)-1}e^{-v}dv + \frac{c_0}{\alpha}\} \\
\le\ & \frac{1}{|x-y|^{\alpha d_w+d}}\frac{c_1\alpha\Gamma((d_s/2+\alpha)(d_w-1))+c_0}{\Gamma(1-\alpha)},
\end{aligned}
\tag{22}
$$

where $c_1 = c_{2.3}(d_w-1)c_{2.4}^{-(\alpha+d_s/2)(d_w-1)}$ and we use the fact $d_s/2 = d/d_w$ in the last inequality (note that $|x-y| \le 1$ for all $x,y \in G$ as the diameter of G is 1). By the similar argument using a trivial estimate $j(x,y) \ge \int_0^1 p_s(x,y)n(ds)$, we can obtain the lower bound of $j(x,y)$. We thus obtain (20).

We next prove (21). Indeed, by the definition of $\mathcal{E}_{Z^{(\alpha)}}$ ((11) with $\mathcal{E}^{(\alpha)}$ instead of $\mathcal{E}$) and the fact that $H_{\mathcal{S}}u$ equals to u on G (due to Proposition 2.4) and is harmonic outside, we see that $\mathcal{E}_{Z^{(\alpha)}}(\cdot,\cdot)+(\cdot,\cdot)_{\mathbf{L}^2}$ is equivalent to the norm (7). Thus the result holds from Remark 2.2 2) and Theorem 2.1. □

We note here that in general the three-type Dirichlet forms introduced are different and the corresponding processes cannot be obtained by time changes of others by positive continuous additive functionals. Indeed, for Brownian motion on the 2-dimensional Sierpinski gasket G, it is known that there are oscillations both in the on-diagonal and the off-diagonal heat kernel so that one cannot take neither $c_{2.1} = c_{2.3}$ nor $c_{2.2} = c_{2.4}$ in (2) (see, for instance, [13], [5]). Thus, in view of (22), $j(x,y)$ cannot be a constant time of $|x-y|^{-(\alpha\log 5+\log 3)/\log 2}$, which means $\mathcal{E}_{X^{(\alpha)}}$ is not a constant time of $\mathcal{E}_{Y^{(\alpha\log 5/(2\log 2))}}$ (note that in this case, $d = \log 3/\log 2$ and $d_w = \log 5/\log 2$). Now, consider $\mathcal{E}_{Z^{(1)}}$ on the same G and let Δ_1, Δ_2 be a small triangle in G such that all the continuous paths on $\mathbf{R}^2$ from Δ_1 to Δ_2 intersect $G\setminus(\Delta_1\cup\Delta_2)$ before they reach Δ_2. Then, as the process corresponding to $\mathcal{E}_{Z^{(1)}}$ is constructed from Brownian motion on $\mathbf{R}^2$ by a time change, the particle starting from Δ_1 cannot jump to Δ_2 before hitting $G\setminus(\Delta_1\cup\Delta_2)$. On the other hand, the particle of the processes corresponding to $\mathcal{E}_{X^{(\log 3/\log 5)}}$ and $\mathcal{E}_{Y^{(\log 3/(2\log 2))}}$ can, because of the shape of the forms. This tells us that $\mathcal{E}_{Z^{(1)}}$ is not a constant time of $\mathcal{E}_{X^{(\log 3/\log 5)}}$ nor $\mathcal{E}_{Y^{(\log 3/(2\log 2))}}$.

We next consider some estimates of heat semigroups. By a Sobolev type embedding for the Besov space, for $0<\alpha<1, 2\alpha<d$ we have,

$$
\|u\|_{\mathbf{L}^{2d/(d-2\alpha)}(G,\mu)} \le c\|u|B_\alpha^{2,2}(G)\| \qquad \text{for all } u \in B_\alpha^{2,2}(G). \tag{23}
$$

See Theorem 3.2 in [12] or Lemma 4.10 in [9] (the latter only treats the case $0<d<D$) for the proof. Using this and the Varopoulos theorem ([31] or Corollary 2.4.3

in [8]), there exists a non-negative bounded heat kernel $p_t^{Y(\alpha)}(x,y)$ on $(t,x,y) \in (0,\infty) \times G \times G$ with

$$P_t^{Y(\alpha)} f(x) = \int_G p_t^{Y(\alpha)}(x,y) f(y) \mu(dy) \qquad \text{for } \mu\text{-a.e. } x \in G,\ f \in \mathbf{L}^2(G,\mu),$$

(where $P_t^{Y(\alpha)}$ is the heat semigroup w.r.t. $\mathcal{E}_{Y(\alpha)}$), satisfying the following.

Theorem 3.2. *For $d \geq 2$ or $d < 2,\ 0 < \alpha < d/2$,*

$$p_t^{Y(\alpha)}(x,y) \leq c_{3.5} t^{-d/(2\alpha)} \qquad \textit{for } \mu\textit{-a.e. } x,y \in G,\ 0 < t < 1.$$

By Proposition 3.1, we see that similar estimates hold for heat semigroups of $\mathcal{E}_{X(\alpha)}$ and $\mathcal{E}_{Z(\alpha)}$ (with suitable changes of α depending on the transformations $\alpha \mapsto \bar{\alpha}$ and $\alpha \mapsto \hat{\alpha}$).

Acknowledgements. The author expresses his gratitude to Professor P. Grabner and Professor W. Woess for inviting him to the nice conference. He is also grateful to Professor Z.-Q. Chen, Professor M. Fukushima, Professor M. Takeda and Dr. T. Uemura for fruitful comments concerning the jump type Dirichlet forms.

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