Supplementary notes for "Uniqueness of Brownian motion on Sierpinski carpets"

M.T. Barlow, R.F. Bass, T. Kumagai, A. Teplyaev.

December 10, 2008

suppl6.tex, MB, 8 Dec 2008

Abstract

These are supplementary notes for [5]. Theorem 1.3 is a variant of the results in [13] and it gives a proof of Lemma 3.18 and Theorem 3.21 of [5]. Theorem 1.4 is a variant of Theorem 4.1 in [14] and it gives a proof of Theorem 3.22 of [5].

1 Framework and main results

These notes prove theorems which are used in [5] to prove the uniqueness of Brownian motion on Sieprinski carpets – see Theorems 4.30 and 4.31 of [5]. Many of these results are due to A. Grigoryan and A. Telcs, and are due to appear in [13].

Let (X, d) be a connected locally compact complete separable metric space. We assume that the metric d is *geodesic*: for each $x, y \in X$ there exists a (not necessarily unique) geodesic path $\gamma(x, y)$ such that for each $z \in \gamma(x, y)$, we have $d(x, z) + d(z, y) = d(x, y)$. Let μ be a Borel measure on X such that $0 < \mu(B) < \infty$ for every ball B in X. We write $B(x, r) = \{y : d(x, y) < r\}$, and $V(x, r) = \mu(B(x, r))$. We also assume that the closure of $B(x, r)$ is compact for all $x \in X$ and $0 < r \leq 1$. Since we work on bounded Sierpinski carpets in [5], in what follows we will assume that X has finite diameter (and for simplicity we take the diameter to be 1), but similar results (with obvious modifications to the statements and the proofs) hold when the diameter of X is infinite. We will call such a space a *metric measure space*, or a MM space.

Now let $(\mathcal{E}, \mathcal{F})$ be a regular, strong local Dirichlet form on $L^2(X, \mu)$: see [8] for details. We denote by Δ the corresponding (non-positive) self-adjoint operator; that is, we say h is in the domain of Δ and $\Delta h = f$ if $h \in \mathcal{F}$ and $\mathcal{E}(h, g) = - \int fg \, d\mu$ for every $g \in \mathcal{F}$. Let $\{P_t\}$ be the corresponding semigroup; $P_t = e^{t\Delta}$. We will often use the notation (f, g) for $\int fg \, d\mu$. $(\mathcal{E}, \mathcal{F})$ is called *conservative* (or *stochastically complete*) if $P_t 1 = 1$ for all $t > 0$. Throughout the paper, we assume that $(\mathcal{E}, \mathcal{F})$ is conservative. Since $\mathcal{E}$ is regular, $\mathcal{E}(f, g)$ can be written in terms of a signed measure $\Gamma(f, g)$. To be more precise, for $f \in \mathcal{F}_b$ (the collection $\mathcal{F}_b$ is the set of functions in $\mathcal{F}$ that are essentially bounded) $\Gamma(f, f)$ is the unique smooth Borel measure (called the energy measure) on X satisfying

$$\int_X \tilde{g} d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \qquad g \in \mathcal{F}_b,$$

where $\tilde{g}$ is the quasi-continuous modification of $g \in \mathcal{F}$. (Recall that $u : X \to \mathbb{R}$ is called quasi-continuous if for any $\varepsilon > 0$, there exists an open set $G \subset X$ such that $\text{Cap}(G) < \varepsilon$ and $u|_{X \setminus G}$ is

continuous. It is known that each $u \in \mathcal{F}$ admits a quasi-continuous modification $\tilde{u}$ – see [8], Theorem 2.1.3.) Throughout the paper, we will abuse notation and take the quasi-continuous modification of $g \in \mathcal{F}_b$ without writing $\tilde{g}$. We call $(X, d, \mu, \mathcal{E})$ a metric measure Dirichlet space, or a MMD space.

Let $Y = (Y_t, t \geq 0, P^x, x \in X\}$ be the Hunt process associated with the Dirichlet form $\mathcal{E}$ on $L^2(X, \mu)$ – see [8], Theorem 7.2.1. Note that there is an ambiguity of the starting point up to a set $\mathcal{N}$ of zero capacity. (However, such ambiguity can be removed when the process is sufficiently 'nice'.) In the following, we write $X \setminus \mathcal{N}$ as X if needed. Since $\mathcal{E}$ is strongly local, by [8], Theorem 7.2.2, Y is a diffusion.

We introduce the following assumptions.

Assumption 1.1 X *satisfies volume doubling* (VD) *if there exists a constant* C_1 *such that*

$$V(x, 2R) \leq C_1 V(x, R) \quad \text{for all } x \in X, \, 0 \leq R \leq 1. \tag{VD}$$

Let $H : [0, 2] \to [0, \infty)$ be a strictly increasing function which (for reasons which will be apparent later) we call the *time scaling function*. We introduce the following assumption on H:

Assumption 1.2 *There exist strictly positive constants* $C_2, \ldots C_5$, *and* $\beta_1 > 1$ *such that* $H(1) \in [C_2, C_3]$, *and*

(TD) $H(2R) \leq C_4 H(R)$ *for all* $0 < R \leq 1$.

(FTG) $H(R)/H(r) \geq C_5 (R/r)^{\beta_1}$ *for all* $0 < r < R \leq 2$.

Here (TD) refers to 'time doubling' and (FTG) to 'fast time growth'. It is well-known and easy to see that (VD), (TD) and (FTG) imply the existence of constants $D > 0, \beta_2 \geq \beta_1$ and $C_6, C_7 > 0$ such that for $x, y \in X$ and $0 < r < R \leq 2$,

$$\frac{V(x, R)}{V(y, r)} \leq C_6 \left(\frac{d(x, y) + R}{r} \right)^D, \qquad C_5 \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{H(R)}{H(r)} \leq C_7 \left(\frac{R}{r} \right)^{\beta_2}. \tag{1.1}$$

Note that we can take $\beta_2 = \log C_4 / \log 2$. Later, we assume $H(1) = 1$ instead of $H(1) \in [C_2, C_3]$ just for simplicity of notation.

We now mention various inequalities we will discuss in these notes.

(I) X satisfies the *Poincaré inequality* (PI(H)) if there exists a constant c_2 such that for any ball $B = B(x, R) \subset X, \, 0 < R \leq 1/3$, and $f \in \mathcal{F}$,

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \leq c_2 H(R) \int_B d\Gamma(f, f). \tag{PI(H)}$$

Here $\overline{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$.

(II) We say a function u is *harmonic* on a domain D if $u \in \mathcal{F}_{loc}$ and $\mathcal{E}(u, g) = 0$ for all $g \in \mathcal{F}$ with support in D. Here $u \in \mathcal{F}_{loc}$ if and only if for any relatively compact open set G, there exists a function $w \in \mathcal{F}$ such that $u = w$ μ-a.e. on G. See page 117 in [8] for the definition of $\mathcal{E}(u, g)$ for $u \in \mathcal{F}_{loc}$ when $(\mathcal{E}, \mathcal{F})$ is a regular, strong local Dirichlet form. Functions in $\mathcal{F}$ are only defined up to quasi-everywhere equivalence; we use a quasi-continuous modification of u. X satisfies the *elliptic Harnack inequality* (EHI) if there exists a constant c_3 such that, for any ball $B(x, R)$, whenever u is a

non-negative harmonic function on $B(x, R)$, $R \leq 1/3$, then there is a quasi-continuous modification $\tilde{u}$ of u that satisfies

$$\sup_{B(x,R/2)} \tilde{u} \leq c_3 \inf_{B(x,R/2)} \tilde{u}. \tag{EHI}$$

Note that (A.1) in the appendix is the natural definition of elliptic Harnack inequality, but it turns out (see subsection A.1) that (A.1) implies (EHI).

Note that by a standard argument (see subsection A.1) (EHI) implies that $\tilde{u}$ is Hölder continuous.

(III) Let A, B be disjoint subsets of X. We define the effective resistance $R(A, B)$ by

$$R(A, B)^{-1} = \inf\left\{ \int_X d\Gamma(f, f) : f = 0 \text{ on } A \text{ and } f = 1 \text{ on B}, \ f \in \mathcal{F} \right\}. \tag{1.2}$$

X satisfies the condition $(RES(H))$ if there exist constants c_1, c_2 such that for any $x_0 \in X$, $0 \leq R \leq 1/3$,

$$c_1 \frac{H(R)}{V(x_0, R)} \leq R(B(x_0, R), B(x_0, 2R)^c) \leq c_2 \frac{H(R)}{V(x_0, R)}. \tag{RES(H)}$$

(IV) We say X satisfies $(HK(H; \beta_1, \beta_2, c_0))$ if the heat kernel $p_t(x, y)$ on X exists and satisfies

$$\frac{c_0^{-1}}{V(x, H^{-1}(t))} \exp\left(-c_0 \left(\frac{H(d(x, y))}{t} \right)^{\frac{1}{\beta_1 - 1}} \right) \leq p_t(x, y)$$

$$\leq \frac{c_0}{V(x, H^{-1}(t))} \exp\left(-c_0^{-1} \left(\frac{H(d(x, y))}{t} \right)^{\frac{1}{\beta_2 - 1}} \right), \tag{1.3}$$

for all $x, y \in X$ and $t \in (0, 1]$. We sometimes refer to the first inequality of (1.3) as $(LHK(H))$ and the second inequality of (1.3) as $(UHK(H))$.

(V) X satisfies the condition $(E(H))$ if for any $x_0 \in X$, $0 \leq R \leq 1/3$,

$$c_1 H(R) \leq E^{x_0}[\tau_{B(x_0, R)}] \leq c_2 H(R), \tag{E(H)}$$

where $\tau_A = \inf\{t \geq 0 : Y_t \notin A\}$, Y_t is the strong Markov process associated to the Dirichlet form $(\mathcal{E}, \mathcal{F})$, and E^{x_0} denotes the expectation starting from the point x_0. The first inequality in $(E(H))$ is referred to as $(E(H)_\geq)$ and the second is referred to as $(E(H)_\leq)$.

Following the terminology used in number theory, we will say that a constant c which arises in the conclusion of a theorem is *effective* if it could in principle be given as an explicit function of the constants given in the 'input data'. See the remark after the next theorem for a more explicit statement.

Our first main theorem (cf. [13, 12]) is the following.

Theorem 1.3 *Let $(X, d, \mu, \mathcal{E}, \mathcal{F})$ be a MMD space. (Note that the assumption includes the facts that d is geodesic and $(\mathcal{E}, \mathcal{F})$ is conservative.) Let H satisfy Assumption 1.2, and β_i be as in (1.1). Then the following are equivalent, and the constants in each implication are effective:*
(a) X satisfies (VD), (EHI) and (RES(H)).
(b) X satisfies (VD), (EHI) and (E(H)).
(c) X satisfies (HK(H, β_1, β_2, c_0)).
The equivalence of the "global" version (i.e. each condition holds for $t \in (0, \infty), R \geq 0$) also holds.

The statement that the constants in the implication $(a) \Rightarrow (c)$ are effective means that the constant c_0 in $(\text{HK}(H, \beta_1, \beta_2, c_0))$ depends only on the constants $C_1, \ldots, C_7, \beta_1, \beta_2$ in Assumption 1.1 and 1.2, and the constants c_3 in (EHI) and c_1, c_2 in $\text{RES}(H)$.

One can show that if one of the above conditions holds, then there is no ambiguity of the starting point for the process, i.e. one can take $\mathcal{N} = \emptyset$.

Since we will only need $(a) \Rightarrow (c)$ in [5], we will only prove $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ We remark that in the proof of $(a) \Rightarrow (b)$, we do not need the assumption that H is strictly increasing and $H(1) = 1$.

Our second topic is the domains of Dirichlet forms. In the following, we will prove that if a heat kernel of a regular Dirichlet form satisfies suitable two-sided estimates, then the domain of the Dirichlet form is the so-called Besov-Lipschitz space. Let

$$\mathcal{E}^r(u) = \frac{1}{H(r)} \int_X \fint_{B(x,r)} [u(x) - u(y)]^2 \, \mu(dy) \, \mu(dx) \tag{1.4}$$

for all $u \in L^2(X, \mu)$. Here $\fint_A \ldots \mu(dy) := \mu(A)^{-1} \int_A \ldots \mu(dy)$ denotes the normalized integral.

We then have the following, which is a version of the result in [14], Section 4.

Theorem 1.4 *Let $(X, d, \mu, \mathcal{E}, \mathcal{F})$ be a MMD space. Let $H : [0, 2] \to [0, \infty)$ satisfy Assumption 1.2, and assume that $(\text{HK}(H; \beta_1, \beta_2, c_0))$ holds. Then, for all $\alpha > 1$ there exist $c_1(\alpha, \beta_1, \beta_2), c_2 > 0$ such that the following holds.*

$$c_1(\alpha, \beta_1, \beta_2) \mathcal{E}(f) \le \limsup_{m \to \infty} \mathcal{E}^{\alpha^{-m}}(f) \le \sup_{0 < r \le 1} \mathcal{E}^r(f) \le c_2 \mathcal{E}(f) \qquad \text{for } f \in \mathcal{F}. \tag{1.5}$$

Here $\mathcal{E}^r$ is the approximating Dirichlet form defined in (1.4). In particular,

$$\mathcal{F} = W_H(X) := \{u \in L^2 : \sup_{0 < r \le 1} \mathcal{E}^r(f) < \infty\}.$$

2 Proof of Theorem 1.3

2.1 Proof of Theorem 1.3: $(a) \Rightarrow (b)$

In this subsection, we will follow the argument in [4]. Note that we do not use the property that H is strictly increasing nor that $H(1) = 1$ in this subsection.

Recall from [8] Section 1.6 the definitions of invariant sets and irreducible Dirichlet forms.

Lemma 2.1 *Let X satisfy (EHI). Then $\mathcal{E}$ is irreducible.*

PROOF. Let A be an invariant set, and suppose both $\mu(A) > 0$ and $\mu(A^c) > 0$. Then there exists a ball $B = B(x, R)$ with $\mu(A \cap B') > 0$ and $\mu(A^c \cap B') > 0$, where $B' = B(x, R/2)$. Since $P_t 1_A = 1_A$ it follows that $u = 1_A$ and $v = 1_{A^c}$ are harmonic on B. So by (EHI) we have

$$\tilde{u}(x) \le C\tilde{u}(y), \quad x, y \in B'.$$

Since $u > 0$ on a set of positive measure in B', we have that there exists $x \in B'$ with $\tilde{u}(x) > 0$; hence by the (EHI), $\tilde{u} > 0$ on B'. But as $\tilde{u} = 1_A$ μ-a.e., we deduce that $\mu(A^c \cap B') = 0$, a contradiction. $\square$

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Proposition 2.2 *Let X satisfy (EHI), and $B = B(x, R)$. Then $Gg < \infty$ a.e. on B if $g \in L^1_+(B)$, where $L^1_+(B)$ is the set of non-negative L^1-functions on B.*

PROOF. Consider the Dirichlet form $\mathcal{E}_B$ with domain $\mathcal{F}_B = \{f \in \mathcal{F} : f|_{B^c} = 0\}$. Let $A = B(x, R/2)$ and $h(x) = P^x(T_A < \tau_B)$. Then h is excessive with respect to $\mathcal{E}_B$. If h were constant on B then we would have $h = 1$ on B, and the set B would be an invariant set for $\mathcal{E}$. Thus h is non-constant.

So by Ex. (4.22), p. 89 in [7], we deduce that the killed semigroup P^B_t is transient. Hence (see [8] Lemma 1.6.4) we have $Gg < \infty$ a.e. for any $g \in L^1_+(B, \mu)$. $\qquad\square$

Lemma 2.3 *Let D be a bounded domain in X. Then (EHI) implies that there exists a density $g^D(\cdot, \cdot)$ for the Green function which is continuous on $(X \times X) \setminus \Delta_g$ and $g_D(x, y) = g_D(y, x)$ for all $x, y \in (X \times X) \setminus \Delta_g$, where Δ_g is the diagonal. Further, there exists $C > 0$ such that for any $0 < r \leq 1/3$, if $y_0, y_1 \in X$ satisfy $d(y_0, y_1) \geq 2r$, then*

$$g_D(y_0, x) \leq C g_D(y_0, y) \qquad \forall x, y \in B(y_1, r). \tag{2.1}$$

PROOF.   Let $x_0, x_1 \in D$, Choose $r > 0$ such that $B(x_i, 2r) \subset D$, $B(x_0, 2r) \cap B(x_1, 2r) = \emptyset$. Write $B_i = B(x_i, 2r)$, $B'_i = B(x_i, r)$. Let $f, g \in \mathcal{F}$ with supports in B'_0 and B'_1, and $\int f = \int g = 1$. Let G_D be the Green operator for the process Y killed on exiting D. By Proposition 2.2 we have $G_D f < \infty$, $G_D g < \infty$.

Then if $u \in \mathcal{F}$ with $\mathrm{Supp}\, u \subset B(x_1, 2r)$,

$$\mathcal{E}(G_D f, u) = (f, u) = 0, \tag{2.2}$$

so $G_D f$ is harmonic on B_1. Similarly $G_D g$ is harmonic on B_0. By the (EHI) if $x \in B'_1$, then

$$G_D f(x) \leq C G_D f(y), \quad y \in B'_1. \tag{2.3}$$

Similarly

$$G_D g(x) \leq C G_D g(x_0), \quad x \in B'_0.$$

So

$$G_D f(x_1) \leq C(g, G_D f) = C(G_D g, f) \leq C^2 G_D g(x_0).$$

Now fix g such that $C_1 = G_D g(x_0) < \infty$; such a g exists by choosing $g \leq c_* h_0$, where $h_0(x) = P^x(T_{B(x, 2r)} < \tau_D)$. Then we have $G_D f(x_1) \leq c'_* ||f||_1$ for all f with support in B'_0. (Note that c_*, c'_* may not be effective, but this does not create a problem later on since we only use these constants for the existence of the Green kernel.) Therefore the kernel $G_D(x_1, dx)$ has a density $g_D(x_1, y)$ on B'_0. Since $(f, G_D g) = (G_D f, g)$ for $f, g \in L^2$, it follows that $g_D(x, y) = g_D(y, x)$ $\mu \times \mu$–a.e.

Now, take $y_0, y_1 \in X$ that satisfy $d(y_0, y_1) \geq 2r$. For any $\epsilon > 0$ and $f \in L^2$ with support in $B(y_0, \epsilon r)$, similarly to (2.2) we can show that $G_D f$ is harmonic on $B(y_1, (2 - \epsilon)r)$. Thus, by the same argument as (2.3), we have

$$G_D f(x) \leq C G_D f(y), \quad x, y \in B(y_1, r). \tag{2.4}$$

Now let $f_n(z) = V(y_0, r_n)^{-1} 1_{B(y_0, r_n)}(z)$ where $\epsilon r \geq r_n \downarrow 0$. Applying (2.4) to f_n and take $n \to \infty$, we obtain (2.1) for μ-a.e. y_0. By the usual oscillation argument (see subsection A.1), we can deduce that $g_D(x, y)$ is continuous on $(X \times X) \setminus \Delta_g$. In particular, $g_D(x, y) = g_D(y, x)$ for all $x, y \in (X \times X) \setminus \Delta_g$. We thus obtain (2.1) for all $y_0 \in X$. $\qquad\square$

Definition 2.4 $(\mathcal{E}, \mathcal{F})$ *satisfies (HG) if there exists a constant $c_1 > 0$ such that for any ball $B(x_0, R)$, $0 < R \leq 1/3$, there exists a Green kernel $g^{B_R}(x_0, y)$ and for any $0 < r \leq R/2$, we have*

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \leq c_1 \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y). \tag{HG}$$

Lemma 2.5 $(EHI) \Rightarrow (HG)$.

PROOF. We prove that if $d(x_0, x) = d(x_0, y) = R$, and $B(x_0, 2R) \subset D$ then

$$C_1^{-1} g_D(x_0, y) \leq g_D(x_0, x) \leq C_1 g_D(x_0, y). \tag{2.5}$$

Once (2.5) is proved, then (HG) holds by the maximum principle (which holds for $G_D f$ and so for g_D as well). By symmetry it is enough to prove the right hand inequality of (2.5).

Let x', y' be the midpoints of $\gamma(x_0, x)$, and $\gamma(x_0, y)$. Thus $d(x_0, x') = d(x_0, y') = R/2$. Clearly we have $d(x', y) \geq R/2$ and $d(x, y') \geq R/2$.

We now consider two cases.

CASE 1. $d(x', y') \leq R/3$. Let z be the midpoint of $\gamma(x', y')$. Then $d(z, x') \leq R/6 \leq R/4$. So applying (2.1) to $g_D(x_0, \cdot)$ in $B(x', R/4) \subset B(x', R/2)$, we deduce that

$$C_2^{-1} g_D(x_0, x') \leq g_D(x_0, z) \leq C_2 g_D(x_0, x').$$

Now apply (2.1) to $g_D(x_0, \cdot)$ in $B(x, R/2) \subset B(x, R)$, to deduce that

$$C_2^{-1} g_D(x_0, x) \leq g_D(x_0, x') \leq C_2 g_D(x_0, x).$$

Combining these inequalities we deduce that

$$C_2^{-2} g_D(x_0, x) \leq g_D(x_0, z) \leq C_2^2 g_D(x_0, x),$$

and this, with a similar inequality for $g_D(x_0, y)$, proves (2.5).

CASE 2. $d(x', y') > R/3$. Apply (2.1) to $g_D(y, \cdot)$ in $B(x_0, R/2) \subset B(x_0, R)$, to deduce that

$$C_2^{-1} g_D(y, x') \leq g_D(y, x_0) \leq C_2 g_D(y, x'). \tag{2.6}$$

Now look at $g_D(x', \cdot)$. If z' is on $\gamma(y', y)$ with $d(y', z') = s \in [0, R/2]$, then as $d(x', y') > R/3$ and $d(x', y) \geq R/2$, we have $d(x', z') \geq \max(R/3 - s, s)$. Hence we deduce $d(x', z') \geq R/6$. So applying (2.1) repeatedly to $g_D(x', \cdot)$ for a chain of balls $B(z', R/12) \subset B(z', R/6)$ we deduce that

$$C_2^{-6} g_D(x', y') \leq g_D(x', y) \leq C_2^6 g_D(x', y'). \tag{2.7}$$

So, we obtain from (2.6) and (2.7),

$$g_D(y, x_0) \leq C_2 g_D(y, x') \leq C_2^7 g_D(x', y'), \quad g_D(x', y') \leq C_2^6 g_D(y, x') \leq C_2^7 g_D(y, x_0).$$

We have similar inequalities relating $g_D(x, x_0)$ and $g_D(x', y')$, which proves (2.5). □

Lemma 2.6 *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (EHI) and let $0 < R \leq 1/3$.*
1) For any ball $B_R = B(x_0, R)$ and for any $0 < r \leq R/2$, we have

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \geq R(B_r, B_R^c) \geq \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y). \tag{2.8}$$

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2) There exist $C_1, C_2 > 0$ such that for any ball $B(x_0, R)$ and for any $0 < r \leq R/2$, we have

$$\sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y) \leq C_1 R(B_r, B_R^c) \leq C_2 \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y). \tag{2.9}$$

3) Let $B_k = B(x_0, 2^k r)$ for $k = 0, 1, \cdots$, such that $2^k r \leq 1/3$. Then, there exist $C_3, C_4 > 0$ such that for any integers $0 \leq m < n$,

$$\sup_{y \notin B_m} g^{B_n}(x_0, y) \leq C_3 \sum_{k=m}^{n-1} R(B_k, B_{k+1}^c) \leq C_4 \inf_{y \in B_m} g^{B_n}(x_0, y). \tag{2.10}$$

PROOF.   First, note that we have (HG) by using Lemma 2.5.

For 1), we modify the proof in Proposition 4.1 in [10]. Let us set

$$a = \sup_{y \notin B(x_0, r)} g^{B_R}(x_0, y), \quad b = \inf_{y \in B(x_0, r)} g^{B_R}(x_0, y),$$

and for any number c, define

$$A_c = \{x \in B_R : g^{B_R}(x_0, x) \geq c\}.$$

We claim that

$$A_a \subset \overline{B_r} \subset A_b. \tag{2.11}$$

Indeed, $g^{B_R}(x_0, \cdot)$ is harmonic in $B_R \setminus B_r$, and by the maximum principle, the supremum is attained on $\partial(B_R \setminus B_r) = \partial B_R \cap \partial B_r$. Since g^{B_R} vanishes on ∂B_R, we have $a = \sup_{y \in \partial B_r} g^{B_R}(x_0, y)$, so $A_a \subset \overline{B_r}$. Similarly, $g^{B_R}(x_0, \cdot)$ is super-harmonic in B_r, and by the minimum principle, $b = \inf_{y \in \partial B_r} g^{B_R}(x_0, y)$, so $A_b \supset \overline{B_r}$, and (2.11) is obtained. Next, (2.11) implies

$$\text{Cap}(A_a) \leq \text{Cap}(\overline{B_r}) \leq \text{Cap}(A_b),$$

where Cap is the 0-capacity with respect to $(\mathcal{E}, \mathcal{F}_{B_r})$, $\mathcal{F}_{B_r} = \{f \in \mathcal{F} : f|_{B_R^c} = 0\}$. Hence, noting that $\text{Cap}(B_r) = \text{Cap}(\overline{B_r}) = 1/R(B_r, B_R^c)$, (2.8) will follow if we show that for $c = a, b$,

$$\text{Cap}(A_c) = 1/c. \tag{2.12}$$

Now recall that for any compact set K,

$$\text{Cap}(K) = \sup\{\mu(K) : \mu \in S_{00}, \text{Supp}\,\mu \subset K, G\mu \leq 1 \; q.e.\},$$

where $S_{00} = \{\mu \in S_0 : \mu(X) < \infty, \|G\mu\|_\infty < \infty\}$, and S_0 is the family of all positive Radon measures of finite energy integrals (see Problem 2.2.2 in [FOT]; there Cap is the 1-capacity, but the 0-capacity version holds similarly). By the same argument as in Lemma 2.3 we have that $g^{B_R}(x_0, \cdot)$ is continuous on $X \setminus \{x_0\}$. This together with the superharmonicity of $g^{B_R}(x_0, \cdot)$ shows that A_c is compact. If μ is the capacitary measure for A_c, it will be supported on ∂A_c because the process has continuous paths, so

$$1 = G\mu(x_0) = \int g^{B_R}(x_0, y)\mu(dy) = \int_{\partial A_c} g^{B_R}(x_0, y)\mu(dy) = c\mu(A_c).$$

Here we used the fact $g^{B_R}(x_0, y) = c$ for $y \in \partial A_c$, which is due to the continuity of $g^{B_R}(x_0, \cdot)$. So $\mu(A_c) = 1/c$ and (2.12) is established.

For 2), using (2.8) and (HG), we obtain (2.9).

For 3), note first that the following holds:

$$\sum_{k=m}^{n-1} R(B_k, B_{k+1}^c) \leq R(B_m, B_n^c).$$

This and (2.9) imply the lower bound for $\inf g^{B_n}$ in (2.10). Next, we know that $g^{B_{k+1}}(x, \cdot) - g^{B_k}(x, \cdot)$ is a harmonic function in B_k. Thus,

$$g^{B_{k+1}}(x, y) - g^{B_k}(x, y) \leq \sup_{z \notin B_k} g^{B_{k+1}}(x, z) \leq cR(B_k, B_{k+1}), \quad \forall y \in X, \tag{2.13}$$

where the first inequality is by the maximum principle and the second inequality is by (2.9). For $y \notin B_m$, by (2.9)

$$g^{B_{m+1}}(x, y) \leq c' R(B_m, B_{m+1}). \tag{2.14}$$

For such y, adding up (2.14) with (2.13) for $m < k < n$, we obtain the upper bound of $\sup g^{B_n}$ in (2.10). □

Proof of $(VD) + (EHI) + (RES(H)) \Rightarrow (E(H))$.

$$E^{x_0}[\tau_{B_R}] = \int g^{B_R}(x_0, y) d\mu(y) \geq \int_{B(x_0, r)} g^{B_R}(x_0, y) d\mu(y) \geq cR(B_r, B_R^c) V(x_0, r) \geq cH(R),$$

where we used Lemma 2.6 1) in the second inequality and $(VD) + (RES(H))$ in the last inequality.

Now, for each $k \in \mathbb{Z}$, let $r_k = M^k$, $B_k = B(x_0, r_k)$ and let n_0 be the minimum number such that $R < r_{n_0}$. Then

$$
\begin{aligned}
E^{x_0}[\tau_{B_R}] &\leq E^{x_0}[\tau_{B(x_0, r_{n_0})}] = \int_{B_{n_0}} g^{B_{n_0}}(x_0, y) d\mu(y) \\
&= \sum_{m=-\infty}^{n_0-1} \int_{B_{m+1} \setminus B_m} g^{B_m}(x_0, y) d\mu(y) \leq c \sum_{m=-\infty}^{n_0-1} \Big(\sum_{k=m}^{n_0-1} R(B_k, B_{k+1}^c) \Big) \mu(B_{m+1} \setminus B_m) \\
&= c \sum_{k=-\infty}^{n_0-1} \Big(\sum_{m=-\infty}^{k} \mu(B_{m+1} \setminus B_m) \Big) R(B_k, B_{k+1}^c) = c \sum_{k=-\infty}^{n_0-1} \mu(B_{k+1}) R(B_k, B_{k+1}^c) \\
&\leq c' \sum_{k=-\infty}^{n_0-1} H(r_{k+1}) \leq c'' H(R) \Big(\sum_{l=0}^{\infty} M^{-\beta_1 l} \Big) \leq c''' H(R),
\end{aligned}
$$

where we used Lemma 2.6 2) in the second inequality and (VD), $(RES(H))$ and (FTG) in the third inequality. We thus obtain $(E(H))$. □

2.2 Proof of $(b) \Rightarrow (c)$

In this subsection, we fix a set of capacity zero $\mathcal{N}$ (the exceptional set) and write $X \setminus \mathcal{N}$ as X. There is an ambiguity of the starting point when $x \in \mathcal{N}$ at the beginning, but in the end one sees that one can take $\mathcal{N} = \emptyset$ due to the 'nice' properties of the process. Later on, we also consider the processes killed on exiting balls and the exceptional sets may depend on the choice of balls. However, one needs only a countable number of balls, so the union of the exceptional sets is still an exceptional set, which we denote by $\mathcal{N}$.

We first prove (HK(H)) for all $x, y \in X \setminus \mathcal{N}$ and then use continuity of the heat kernel to deduce (HK(H)) for all $x, y \in X$ (in other words, to show that $\mathcal{N} = \emptyset$). We first give some inequalities.

$$p_t(x, y) \leq \frac{C_1}{V(x, H^{-1}(t))}, \qquad \forall x, y \in X, 0 < t \leq 1. \tag{$DUHK(H)$}$$

$$P^x(\tau_{B(x,r)} \leq t) \leq C_2 \exp\left(- C_3\left(\frac{H(r)}{t}\right)^{\frac{1}{\beta_2 - 1}}\right), \quad \forall x \in X, 0 < r \leq 1/3, 0 < t \leq 1. \tag{$ELD(H)$}$$

$$p_t(x, x) \geq \frac{C_4}{V(x, H^{-1}(t))}, \qquad \forall x \in X, 0 < t \leq 1. \tag{$DLHK(H)$}$$

$$p_t(x, y) \geq \frac{C_5}{V(x, H^{-1}(t))}, \qquad \forall x, y \in X, 0 < t \leq 1 \ \text{ with } \ H(d(x, y)) \leq C_6 t. \tag{$NLHK(H)$}$$

Note that β_2 in this subsection (for example in $(ELD(H))$) is the one in (1.1).

In order to prove $(b) \Rightarrow (c)$, we first prove the following.

Proposition 2.7 $(E(H)) \Rightarrow (ELD(H))$.

To prove this proposition, we first give the following key lemma due to Barlow-Bass (see [1] for the proof).

Lemma 2.8 *Let $\{\xi_i\}$ be non-negative random variables. Suppose there exist $0 < p < 1$ and $a > 0$ such that*

$$P(\xi_i \leq t | \sigma(\xi_1, \cdots, \xi_{i-1})) \leq p + at, \qquad \forall t > 0.$$

Then,

$$\log P\left(\sum_{i=1}^{n} \xi_i \leq t\right) \leq 2\left(\frac{ant}{p}\right)^{1/2} - n \log \frac{1}{p}.$$

PROOF OF PROPOSITION 2.7. We first prove that there exists $0 < c_1 < 1$ and $c_2 > 0$ such that

$$P^x(\tau_{B(x,r)} \leq s) \leq 1 - c_1 + c_2 s / H(r) \qquad \text{for all } \ x \in X, \ 0 \leq s \leq 1. \tag{2.15}$$

Indeed, by the Markov property, for each $x \in X$ we have

$$E^x \tau_{B(x,r)} \leq s + E^x[1_{\{\tau_{B(x,r)} > s\}} E^{Y_s} \tau_{B(x,r)}] \leq s + E^x[1_{\{\tau_{B(x,r)} > s\}} E^{Y_s} \tau_{B(X_s, 2r)}]. \tag{2.16}$$

Applying $(E(H))$ and (TD), we have

$$c_3 H(r) \leq s + c_4 H(2r) P^x(\tau_{B(x,r)} > s) = s + c_5 H(r)(1 - P^x(\tau_{B(x,r)} \leq s)). \tag{2.17}$$

Rearranging gives (2.15).

Next, let $l \geq 1$, $b = r/l$, and define stopping times σ_i, $i \geq 0$ by

$$\sigma_0 = 0, \quad \sigma_{i+1} = \inf\{t \geq \sigma_i : d(Y_{\sigma_i}, Y_t) \geq b\}.$$

Let $\xi_i = \sigma_i - \sigma_{i-1}$, $i \geq 1$. Let $\mathcal{F}_t$ be the filtration generated by $\{Y_s : s \leq t\}$ and let $\mathcal{G}_m = \mathcal{F}_{\sigma_m}$. We have by (2.15)

$$P^x(\xi_{i+1} \leq t | \mathcal{G}_i) = P^{Y_{\sigma_i}}(\tau_{B(Y_{\sigma_i}, b)} \leq t) \leq p + c_2 t / H(b),$$

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where $0 < p < 1$. As $d(Y_{\sigma_i}, Y_{\sigma_{i+1}}) = b$, we have $d(Y_0, Y_{\sigma_l}) \leq r$, so that $\sigma_l = \sum_{i=1}^{l} \xi_i \leq \tau_{B(Y_0, r)}$. So, by Lemma 2.8,

$$\log P^x(\tau_{B(x,r)} \leq t) \leq 2p^{-1/2}\left(\frac{c_2 lt}{H(r/l)}\right)^{1/2} - l\log(1/p) = c_6\left(\frac{lt}{H(r/l)}\right)^{1/2} - c_7 l.$$

Now take $l_0 \in \mathbb{N}$ the largest integer l that satisfies

$$c_7 l/2 > c_6\left(\frac{lt}{H(r/l)}\right)^{1/2}. \tag{2.18}$$

This is equivalent to $H(r/l) > c_8 t/l$ for some $c_8 > 0$. Note that if $H(r) \leq c_8 t$, then $(ELD(H))$ clearly holds by taking $C_2 > 0$ large, so we may assume that (2.18) holds for small $l \in \mathbb{N}$. Then, by (1.1),

$$c_9\left(\frac{H(r)}{t}\right)^{\frac{1}{\beta_2 - 1}} < l_0 + 1, \quad \text{and} \quad \log P^x(\tau_{B(x,r)} \leq t) \leq -c_7 l_0/2.$$

We thus obtain $(ELD(H))$. $\qquad\square$

Corollary 2.9 *Assume $(E(H))$. Then the following holds.*
1) *For each $p > 0$, there exists $c_1 = c_1(p) > 0$ such that for any $x_0 \in X$, $0 \leq R \leq 1/3$,*

$$E^{x_0}[(\tau_{B(x_0, R)})^p] \leq c_1 H(R)^p.$$

2) *There exist $c_2, c_3 > 0$ such that for any $x_0 \in X$, $\lambda > 0$ and $0 \leq R \leq 1/3$,*

$$E^{x_0} e^{-\lambda \tau_{B(x_0, R)}} \leq c_2 \exp(-c_3(\lambda H(R))^{1/\beta_2})$$

PROOF. We first prove the following

$$P^x(\tau_{B(x,r)} \geq t) \leq C_1 \exp\left(-C_2\left(\frac{t}{H(r)}\right)\right), \quad \forall x \in X, 0 < r \leq 1/3, 0 < t \leq 1. \tag{2.19}$$

Indeed, by $(E(H))$, we have

$$P^x(\tau_{B(x,r)} \geq c_* H(r)) \leq \frac{E^x[\tau_{B(x,r)}]}{c_* H(r)} \leq \frac{1}{2},$$

by choosing $c_* > 0$ large. Iterating and using the strong Markov property, we deduce that $P^x(\tau_{B(x,r)} \geq c_* k H(r)) \leq 2^{-k} = e^{-c'k}$ and (2.19) follows easily.

1) is now obtained by computing the expectation using (2.19), and 2) is obtained by computing the expectation using $(ELD(H))$. $\qquad\square$

Our next goal is to prove the following.

Proposition 2.10 $(VD) + (EHI) + (E(H)) \Rightarrow (DUHK(H))$.

To prove this, we will compute higher order resolvents as in [2, Section 6]. We first make some preparations.

Lemma 2.11 *Assume* $(VD), (EHI)$ *and* $(E(H))$.

1) There exists $c_1 > 0$ *such that if* $x_0 \in X$, $0 < r \le 1/3$, *and* $A \subset B(x_0, r/8)$, *then*

$$E^y\left[\int_0^{\tau_{B(x_0,r)}} 1_A(Y_s)ds\right] \le c_1\mu(A)\frac{H(r)}{V(y,r)}, \quad \forall y \in B(x_0, 3r/4) \setminus B(x_0, r/4).$$

2) Let $p > 0$. *There exists* $c_2 = c_2(p) > 0$ *such that if* $x_0 \in X$, $0 < r \le 1/3$, *and* $A \subset B(x_0, r/8)$, *then*

$$E^y\left[\int_0^{\tau_{B(x_0,r)}} s^p 1_A(Y_s)ds\right] \le c_1\mu(A)\frac{H(r)^{1+p}}{V(y,r)}, \quad \forall y \in \partial B(x_0, r/2).$$

PROOF. For $y \in B(x_0, 3r/4) \setminus B(x_0, r/8)$, let

$$f(y) := E^y\left[\int_0^{\tau_{B(x_0,r)}} 1_A(Y_s)ds\right] = \int_0^\infty P^y(Y_s^B \in A)ds,$$

where Y_s^B is the process killed on exiting $B := B(x_0, r)$. Then, f is harmonic on $B(x_0, r) \setminus B(x_0, r/8)$ and it is 0 on $\partial B(x_0, r)$. So, for $y \in B(x_0, 3r/4) \setminus B(x_0, r/4)$, using $(EHI), (VD)$ and $(E(H))$,

$$\begin{aligned}
f(y) &\le \frac{c_1}{V(y, r/10)} \int_{B(y,r/10)} f(z)d\mu(z) \le \frac{c_2}{V(y, r)} \int_{B(x_0,r)} f(z)d\mu(z) \\
&= \frac{c_2}{V(y, r)} \int_0^\infty ds \int_B P^y(Y_s^B \in A)d\mu(y) = \frac{c_2}{V(y, r)} \int_0^\infty ds \int_A P^z(Y_s^B \in B)d\mu(z) \\
&= \frac{c_2}{V(y, r)} \int_A d\mu(z) \int_0^\infty P^z(Y_s^B \in B)ds = \frac{c_2}{V(y, r)} \int_A E^z[\tau_B]d\mu(z) \le \frac{c_3\mu(A)H(r)}{V(y, r)},
\end{aligned}$$

where the symmetry of Y_s^B is used in the second equality. This proves 1).

For 2), Let $g(y) := E^y[\int_0^{\tau_{B(x_0,r)}} s^p 1_A(Y_s)ds]$ and $T = \tau_{B(y,r/4)}$. Then, since $B(y, r/4) \cap A = \emptyset$,

$$\begin{aligned}
g(y) &= E^y\left[\int_T^{\tau_{B(x_0,r)}} s^p 1_A(Y_s)ds\right] = E^y\left[E^{Y_T}\left[\int_0^{\tau_{B(x_0,r)}} (T+s)^p 1_A(Y_s)ds\right]\right] \\
&\le c_1\left(E^y\left[T^p E^{Y_T}\left[\int_0^{\tau_{B(x_0,r)}} 1_A(Y_s)ds\right]\right] + E^y[g(Y_T)]\right) \\
&\le c_2\mu(A)\frac{H(r)^{1+p}}{V(y, r)} + c_1 E^y[g(Y_T)], \tag{2.20}
\end{aligned}$$

where 1), (VD) and Corollary 2.9 1) are used in the last inequality. Let $h(z) = E^z[g(Y_T)]$. Then

$$h(z) = E^z\left[E^{Y_T}\left[\int_0^{\tau_B} t^p 1_A(Y_t)dt\right]\right] = E^z\left[\int_T^{\tau_B} (t-T)^p 1_A(Y_t)dt\right] \le g(z), \tag{2.21}$$

where $\tau_B = \tau_{B(x_0,r)}$. Also, $h(z)$ is harmonic on $B(y, r/4)$. Thus, using $(EHI), (VD)$ and (2.21), we have

$$\begin{aligned}
h(y) &\le \frac{c_3}{V(y, r)} \int_{B(y,r/6)} g(x)d\mu(x) = \frac{c_3}{V(y, r)} \int_0^\infty t^p \int_{B(y,r/6)} P^x(Y_t^B \in A)dt\, d\mu(x) \\
&= \frac{c_3}{V(y, r)} \int_0^\infty t^p \int_A P^z(Y_t^B \in B(y, r/6))dt\, d\mu(z) \\
&\le \frac{c_3}{V(y, r)} \int_A d\mu(z) \int_0^\infty t^p P^z(\tau_B \ge t)dt = \frac{c_3}{V(y, r)}\mu(A)E^z[\tau_B^{p+1}] \le c_4\mu(A)\frac{H(r)^{1+p}}{V(y, r)},
\end{aligned}$$

where the symmetry of Y_s^B is used in the second equality and Corollary 2.9 1) is used in the last inequality. This together with (2.20) implies 2). □

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Proposition 2.12 *Assume* $(VD), (EHI)$ *and* $(E(H))$. *Let* $p > (\beta_1/D - 1) \vee 0$, *where* β_1, D *are given in (1.1). There exists* $c_1 = c_1(p) > 0$ *such that if* $x_0 \in X$, $\lambda \geq 1$, *and* $A \subset B(x_0, H^{-1}(\lambda^{-1})/8)$, *then*

$$E^{x_0}\left[\int_0^\infty 1_A(Y_s)s^p e^{-\lambda s}ds\right] \leq c_1 \frac{\mu(A)\lambda^{-1-p}}{V(x_0, H^{-1}(\lambda^{-1}))}.$$

PROOF. First, let $A_r := \{x \in X : d(x, A) < r/4\}$, $\hat{\tau}_0^r = 0$ and define inductively

$$\hat{\tau}_k^r = \inf\{t > \hat{\tau}_{k-1}^r : d(Y_t, Y_{\hat{\tau}_{k-1}^r}) \geq r \text{ and } Y_t \notin A_r\}, \quad \forall k \in \mathbb{N}.$$

Let $\xi_k^r = \hat{\tau}_k^r - \hat{\tau}_{k-1}^r$. Then, one can easily see

$$c_1 H(r) \leq E^x[\xi_k^r | \sigma(\xi_1^r, \cdots \xi_{i-1}^r)] \leq c_2 H(r), \quad \forall k \in \mathbb{N}, \tag{2.22}$$

and we can obtain

$$P^x(\hat{\tau}_k^r \leq t) \leq c_1 \exp\left(-c_2 k\left(\frac{H(r)}{t}\right)^{1/(\beta_2 - 1)}\right), \tag{2.23}$$

$$P^x(\hat{\tau}_k^r \geq t) \leq c_3 \exp\left(-c_4 \frac{t}{kH(r)}\right), \tag{2.24}$$

where $c_1, \cdots, c_4$ can be taken independently with k. Indeed, we can obtain $(ELD(H))$ for each $P(\xi_k^r \leq t | \sigma(\xi_1^r, \cdots \xi_{k-1}^r))$ thanks to (2.22). So, together with the estimate

$$P^x(\hat{\tau}_k^r \leq t) \leq \Pi_{i=1}^k P(\xi_i^r \leq t | \sigma(\xi_1^r, \cdots \xi_{i-1}^r)),$$

we obtain (2.23). (2.24) can be obtained in the same way as (2.19). Integrating these bounds, we have for any $x_0 \in X$, $0 \leq R \leq 1/3$,

$$E^{x_0}[(\hat{\tau}_k^r)^p] \leq c_1(kH(R))^p, \tag{2.25}$$

$$E^{x_0}e^{-\lambda\hat{\tau}_k^r} \leq c_2 \exp(-c_3(k^{\beta_2-1}\lambda H(R))^{1/\beta_2}), \tag{2.26}$$

similarly to Corollary 2.9, where $c_1 = c_1(p) > 0$. Using these estimates, we can obtain the following for each $A \subset B(x_0, r/8)$

$$E^{Y_{\hat{\tau}_k^r}}\left[\int_0^{\hat{\tau}_1^r} 1_A(Y_s)ds\right] \leq c_1\mu(A)\frac{H(r)}{V(Y_{\hat{\tau}_k^r}, r)} \leq c_2\mu(A)\frac{k^D H(r)}{V(x_0, r)}, \tag{2.27}$$

$$E^{Y_{\hat{\tau}_k^r}}\left[\int_0^{\hat{\tau}_1^r} s^p 1_A(Y_s)ds\right] \leq c_3\mu(A)\frac{H(r)^{1+p}}{V(Y_{\hat{\tau}_k^r}, r)} \leq c_4\mu(A)\frac{k^D H(r)^{1+p}}{V(x_0, r)}, \tag{2.28}$$

similarly to Lemma 2.11 where $c_3 = c_3(p), c_4 = c_4(p) > 0$ and $D > 0$. (Note that these inequalities are trivial when $A \cap B(Y_{\hat{\tau}_k^r}, r) = \emptyset$.) Here in the second inequalities in (2.27) and (2.28), we used (1.1).

We are now ready to estimate $E^{x_0}[\int_0^\infty 1_A(Y_s)s^p e^{-\lambda s}ds]$ which we denote by $V(A)$. For each $n \in \mathbb{N} \cup \{0\}$, let $r_n = 2^{-n}$ and let m_0 be such that $H(r_{m_0})^{-1} \leq \lambda < H(r_{m_0+1})^{-1}$. Since $\lambda \geq 1$, such $m_0 \in \mathbb{N} \cup \{0\}$ exists. Let $r := r_{m_0}$; then $\tau_r = \hat{\tau}_1^r$, $\hat{\tau}_k^r \to \infty$ as $k \to \infty$ and $\tau_{r_n} \to 0$ as $n \to \infty$. Thus,

$$V(A) = \sum_{n=m_0+1}^\infty E^{x_0}\left[\int_{\tau_{r_n}}^{\tau_{r_{n-1}}} 1_A(Y_s)s^p e^{-\lambda s}ds\right] + \sum_{k=1}^\infty E^{x_0}\left[\int_{\hat{\tau}_k^r}^{\hat{\tau}_{k+1}^r} 1_A(Y_s)s^p e^{-\lambda s}ds\right] =: \sum_{n=m_0+1}^\infty J_n + \sum_{k=1}^\infty L_k.$$

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Using the strong Markov property, (1.1), Corollary 2.9 and Lemma 2.11, we have

$$
\begin{aligned}
J_n &= E^x\left[e^{-\lambda\tau_{r_n}}E^{Y_{\tau_{r_n}}}\left[\int_0^{\tau_{r_{n-1}}}1_A(Y_s)(s+\tau_{r_n})^pe^{-\lambda s}ds\right]\right] \\
&\leq c_1\left\{E^x\left[e^{-\lambda\tau_{r_n}}\tau_{r_n}^pE^{Y_{\tau_{r_n}}}\left[\int_0^{\tau_{r_{n-1}}}1_A(Y_s)ds\right]\right]+E^x\left[e^{-\lambda\tau_{r_n}}E^{Y_{\tau_{r_n}}}\left[\int_0^{\tau_{r_{n-1}}}1_A(Y_s)s^pds\right]\right]\right\} \\
&\leq c_2\left\{E^x[e^{-2\lambda\tau_{r_n}}]^{1/2}E^x[\tau_{r_n}^{2p}]^{1/2}\mu(A)\frac{H(r_n)}{V(x_0,r_n)}+E^x[e^{-\lambda\tau_{r_n}}]\mu(A)\frac{H(r_n)^{1+p}}{V(x_0,r_n)}\right\} \\
&\leq c_3\mu(A)\frac{H(r_n)^{1+p}}{V(x_0,r_n)}\exp(-c_4(\lambda H(r_n))^{1/\beta_2}). \qquad (2.29)
\end{aligned}
$$

Let $\beta' := \beta_1(1+p) - D$ which is positive by our choice of p. Then, using (1.1) and (2.29),

$$
\sum_{n=m_0+1}^\infty J_n \leq c_3\mu(A)\sum_{n=m_0+1}^\infty\frac{H(r_n)^{1+p}}{V(x_0,r_n)}\leq c_4\mu(A)\frac{H(r_{m_0})^{1+p}}{V(x_0,r_{m_0})}\sum_{n=m_0+1}^\infty 2^{-\beta'n}\leq c_5\frac{\mu(A)\lambda^{-1-p}}{V(x_0,H^{-1}(\lambda^{-1}))}.
$$

We can estimate L_k similarly to (2.29) using (2.25), (2.26), (2.27), (2.28), (1.1) and obtain

$$
L_k \leq c_1\mu(A)\frac{H(r_{m_0})^{1+p}k^{D+p}}{V(x_0,r_{m_0})}\exp(-c_2k^{(\beta_2-1)/\beta_2}(\lambda H(r_{m_0}))^{1/\beta_2}).
$$

Since $H(r_{m_0})^{-1}\leq\lambda<H(r_{m_0+1})^{-1}$, we obtain

$$
\sum_{k=1}^\infty L_k\leq c_1\mu(A)\frac{H(r_{m_0})^{1+p}}{V(x_0,r_{m_0})}\sum_{k=1}^\infty k^{D+p}\exp(-c_3k^{(\beta_2-1)/\beta_2})\leq c_4\frac{\mu(A)\lambda^{-1-p}}{V(x_0,H^{-1}(\lambda^{-1}))}.
$$

We thus obtain the desired result. □

PROOF OF PROPOSITION 2.10. For a Borel function f, define

$$
U_{\lambda,p}f(x) := E^x\left[\int_0^\infty f(Y_s)s^pe^{-\lambda s}ds\right]\quad x\in X.
$$

Then, we see from Proposition 2.12 that $U_{\lambda,p}$ has a density $g_{\lambda,p}$ with respect to μ and

$$
g_{\lambda,p}(x,x)\leq c_1\frac{\lambda^{-1-p}}{V(x,H^{-1}(\lambda^{-1}))}. \qquad (2.30)
$$

Thus, since $\mu(X)<\infty$, we see that $U_{\lambda,p}$ is a Hilbert-Schmidt operator and therefore compact, and $U_{\lambda,p}L^2\subset L^\infty$. So, $g_{\lambda,p}$ has the Mercer expansion

$$
g_{\lambda,p}(x,y)=\sum_{i\in\mathbb{N}}\beta_i(\lambda)\varphi_i(x)\varphi_i(y),
$$

where $\{\beta_i(\lambda)\},\{\varphi_i\}$ are the eigenvalues and the eigenfunctions of $U_{\lambda,p}$. Furthermore $\{\varphi_i\}$ forms a complete orthonormal system of functions in L^2 that are also in L^∞, the convergence is absolute and takes place in $L^\infty(X\times X)$. Note that, if we denote the non-negative self-adjoint operator corresponding to Y_s as $-\Delta$, then $U_{\lambda,p}=p!\,(\lambda I-\Delta)^{-p-1}$ for each $p\in\mathbb{N}$. So, we see that $-\Delta$

has a compact resolvent, and there is a transition density $p_t(x,y) = \sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$ where $\lambda_i = (\beta_i(1)/(p!))^{-(p+1)^{-1}} - 1$. In particular, $p_t(x,x)$ is non-increasing for t. Now, since

$$g_{\lambda,p}(x,y) = \int_0^\infty p_t(x,y)t^p e^{-\lambda t}dt,$$

taking $\lambda = t^{-1}$ and using (2.30), we have

$$c_1 \frac{t^{1+p}}{V(x,H^{-1}(t))} \geq \int_0^\infty p_s(x,x)s^p e^{-1}ds \geq ct^p \int_{t/2}^t p_s(x,x)ds \geq c't^{p+1}p_t(x,x),$$

and $(DUHK(H))$ is obtained. $\qquad\square$

We are now ready to complete the proof of $(b) \Rightarrow (c)$.

Proposition 2.13

$$(VD) + (DUHK(H)) + (EHI) + (E(H)) \Rightarrow (HK(H)).$$

This proposition will be proved through several steps.

STEP 1: PROOF OF $(VD) + (DUHK(H)) + (ELD(H)) \Rightarrow (UHK(H))$. Here, for simplicity we will prove (UHK(H)) only for μ-a.e. x,y. One can prove the q.e. results by using the technique in subsection 2.1 of [6] and argue similarly to subsection 2.2 of [6]. –See also Theorem 6.2 of [13].

Fix $x \neq y$ and t and let $r := d(x,y)$, $\epsilon < r/6$. For $a \in X$, set $B_\epsilon(a) = \{b \in X : d(a,b) < \epsilon\}$. Let $\bar{\mu}_x = \mu|_{B_\epsilon(x)}$, $A_1 = \{z \in X : d(z,x) \leq d(z,y)\}$ and $A_2 = X - A_1$. Then

$$\begin{aligned}
P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) &= P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_1) \\
&\quad + P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_2) \equiv I_1 + I_2.
\end{aligned}$$

Now, letting $\tau := \tau_{B(x,r/2)}$, we have

$$\begin{aligned}
I_2 &\leq P^{\bar{\mu}_x}\left(Y_t \in B_\epsilon(y), \tau < \frac{t}{2}\right) = E^{\bar{\mu}_x}\left(1_{\tau < t/2}\int_{B_\epsilon(y)} p_{t-\tau}(Y_\tau, w)d\mu(w)\right) \\
&\leq P^{\bar{\mu}_x}(\tau < t/2)\sup_{z \in B(x,r/2)\cup B_\epsilon(y)} p_{t/2}(z,z)\mu(B_\epsilon(y)).
\end{aligned}$$

For $z \in B_\epsilon(x)$, by $(ELD(H))$,

$$P^z\left(\tau_{B(z,r/3)} < \frac{t}{2}\right) \leq c_1 \exp\left(-c_2\left(\frac{H(r)}{t}\right)^{\frac{1}{\beta_2-1}}\right).$$

Thus,

$$I_2 \leq c_1\left(\sup_{z \in B(x,r/2)\cup B_\epsilon(y)} p_{t/2}(z,z)\right)\mu(B_\epsilon(x))\mu(B_\epsilon(y))\exp\left(-c_2\left(\frac{H(r)}{t}\right)^{\frac{1}{\beta_2-1}}\right).$$

For I_1, by the symmetry of $p_t(x,y)$,

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y), Y_{\frac{t}{2}} \in A_1) = P^{\bar{\mu}_y}(Y_t \in B_\epsilon(x), Y_{\frac{t}{2}} \in A_1)$$

which is bounded in exactly the same way as I_2, where x and y are changed. Adding the bounds for I_1 and I_2,

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) \leq c_1\left(\sup_{z \in B(x,r/2)\cup B(y,r/2)} p_{t/2}(z,z)\right)\mu(B_\epsilon(x))\mu(B_\epsilon(y))\exp\left(-c_2\left(\frac{H(r)}{t}\right)^{\frac{1}{\beta_2-1}}\right).$$

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By $(DUHK(H))$ and (1.1),

$$\sup_{z \in B(x,r/2) \cup B(y,r/2)} p_{t/2}(z,z) \leq \frac{c_3}{V(x, H^{-1}(t))} \left(\frac{r + H^{-1}(t)}{H^{-1}(t)} \right)^D.$$

If $H(r) \leq t$, this is bounded by $c_4 V(x, H^{-1}(t))^{-1}$. If $H(r) > t$, then, for each $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\left(\frac{r + H^{-1}(t)}{H^{-1}(t)} \right)^D \exp\left(-\varepsilon \left(\frac{H(r)}{t} \right)^{\frac{1}{\beta_2 - 1}} \right) \leq c_\epsilon.$$

This is due to the following fact; $M = r/H^{-1}(t)$ is equivalent to $H(r/M) = t$, so using (1.1), $M < \left(\frac{H(r)}{t} \right)^{\beta_1}$. In any case, we obtain

$$P^{\bar{\mu}_x}(Y_t \in B_\epsilon(y)) \leq \frac{c_5}{V(x, H^{-1}(t))} \mu(B_\epsilon(x)) \mu(B_\epsilon(y)) \exp\left(-c_2 \left(\frac{H(r)}{t} \right)^{\frac{1}{\beta_2 - 1}} \right).$$

Dividing both sides by $\mu(B_\epsilon(x))$, $\mu(B_\epsilon(y))$, if x and y are Lebesgue points (i.e., points of density), we obtain $(UHK(H))$. Since μ-a.e. points are Lebesgue points, we obtain the result for μ-a.e. x, y. $\square$

STEP 2: PROOF OF $(VD) + (ELD(H)) \Rightarrow (DLHK(H))$. Using $(ELD(H))$ and the conservativeness of the process, we have that

$$P^x(Y_t \notin B(x,r)) \leq P(\tau_{B(x,r)} \leq t) \leq c_1 \exp\left(-c_2 \left(\frac{H(r)}{t} \right)^{\frac{1}{\beta_2 - 1}} \right).$$

Hence by choosing r such that $c_3 H(r) < t < c_4 H(r)$ for some $c_3, c_4 > 0$, we have

$$P^x(Y_t \notin B(x,r)) \leq c_5 < 1.$$

Thus $P^x(Y_t \in B(x,r)) \geq 1 - c_5 > 0$. By Cauchy-Schwarz,

$$(1 - c_5)^2 \leq P^x(Y_t \in B(x,r))^2 = \left(\int_{B(x,r)} p_t(x,z) d\mu(z) \right)^2 \leq V(x,r) p_{2t}(x,x).$$

Now, using the lower bound of our choice of t and (VD), we obtain the result. $\square$

Remark. By the same argument, we can obtain the following slightly stronger conclusion. Assume (VD) and $(ELD(H))$. Then there exist $c_1, c_2 > 0$ such that

$$p_t^{B(x,R)}(x,x) \geq \frac{c_1}{V(x, H^{-1}(t))}, \qquad \forall x \in X, 0 < R \leq 1/3, t \in (0, c_2 H(R)]. \tag{2.31}$$

STEP 3: PROOF OF $(VD) + (DUHK(H)) + (EHI) + (E(H)) \Rightarrow (NLHK(H))$. We follow the arguments in [13, 11]. Fix $x \in X$, $t > 0$ and set $R := H^{-1}(t/\varepsilon)$ where $\varepsilon > 0$ will be chosen later. We can assume $\varepsilon < c_2$ where c_2 is given in (2.31). Hence by (2.31)

$$p_t^B(x,x) \geq \frac{c_1}{V(x, H^{-1}(t))}, \tag{2.32}$$

where $B := B(x,R)$. Set $f(y) = \partial_t p_t^B(x,y)$. Applying Proposition A.7 to p_t^B, we have, for $y \in B$,

$$|f(y)| \leq \frac{2}{t} \sqrt{p_{t/2}^B(x,x) p_{t/2}^B(y,y)} \leq \frac{2}{t} \sqrt{p_{t/2}(x,x) p_{t/2}(y,y)}.$$

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By $(DUHK(H))$, we have

$$p_{t/2}(x,x) \leq \frac{c_1}{V(x, H^{-1}(t))},$$

and

$$
\begin{aligned}
p_{t/2}(y,y) &\leq \frac{c_1}{V(y, H^{-1}(t))} \leq \frac{c_1}{V(x, H^{-1}(t))} \frac{V(x, H^{-1}(t))}{V(y, H^{-1}(t))} \\
&\leq \frac{c_1}{V(x, H^{-1}(t))} \left(1 + \frac{d(x,y)}{H^{-1}(t)}\right)^D \leq \frac{c_1(1 + \varepsilon^{-\beta_1})^D}{V(x, H^{-1}(t))}, \qquad \forall y \in B,
\end{aligned}
$$

where we used (1.1) and the definition of R. Hence, we have

$$|f(y)| \leq \frac{c_2(1 + \varepsilon^{-1/\beta_1})^{D/2}}{t V(x, H^{-1}(t))}, \qquad \forall y \in B. \tag{2.33}$$

Define $u(y) = p_t^B(x,y)$. Note that $\partial_t u = \Delta_B u$ and the Green operator G_B is a bounded operator in $L^2(B)$ and $G_B = (-\Delta_B)^{-1}$. Thus, $u = -G_B(\partial_t u) = -G_B f$. Let $\gamma > D/(2\beta_1)$ and apply Proposition A.4 below with $\varepsilon^{\gamma+1}$ instead of ε. Then, there exists $\delta > 0$ such that for any $0 < r < R$,

$$\text{Osc}_{B(x,\delta r)} u \leq 2(\bar{E}(x,r) + \varepsilon^{\gamma+1} \bar{E}(x,R)) \|f\|_\infty,$$

where $\bar{E}(x,r) = \sup_z E^z(\tau_{B(x,r)})$. By $(E(H))$, we have $\bar{E}(x,r) \leq c_3 H(r)$ and $\bar{E}(x,R) \leq c_3 H(R)$. Estimating $\|f\|_\infty$ by (2.33), we obtain

$$\text{Osc}_{B(x,\delta r)} u \leq \frac{H(r) + \varepsilon^{\gamma+1} H(R)}{t} \cdot \frac{c_4(1 + \varepsilon^{-1/\beta_1})^{D/2}}{V(x, H^{-1}(t))}.$$

By definition of R, we have

$$\frac{\varepsilon^{\gamma+1} H(R)}{t} = \varepsilon^\gamma.$$

Choose r by the equation $H(r) = \varepsilon^{\gamma+1} H(R)$, which implies, by definition of H, $r \geq \delta' R$ for some $\delta' > 0$. Hence, we obtain

$$\text{Osc}_{y \in B(x, \delta\delta' R)} p_t^B(x,y) \leq \text{Osc}_{B(x, \delta r)} u \leq \frac{2c_4 \varepsilon^\gamma (1 + \varepsilon^{-1/\beta_1})^{D/2}}{V(x, H^{-1}(t))}. \tag{2.34}$$

By the choice of $\gamma > 0$, $\varepsilon^\gamma (1 + \varepsilon^{-1/\beta_1})^{D/2} \to 0$ as $\varepsilon \to 0$. So, choosing ε small enough and combining (2.34) with (2.32), we conclude that

$$p_t(x,y) \geq p_t^B(x,y) \geq \frac{c_1/2}{V(x, H^{-1}(t))}, \qquad \forall y \in B(x, \delta\delta' R),$$

which proves $(NLHK(H))$. $\qquad \square$

STEP 4: PROOF OF $(VD) + (NLHK(H)) \Rightarrow (LHK(H))$. Since there is nothing to prove when $H(d(x,y)) \leq C_6 t$ due to $(NLHK(H))$, we will consider the case $H(d(x,y)) > C_6 t$. Let $N \in \mathbb{N}$ be the smallest integer n that satisfies

$$c_0 t/n \geq H(d(x,y)/n), \tag{2.35}$$

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where $c_0 > 0$ is taken small enough so that (2.35) does not hold for $n = 1, 2$ and

$$t/N \geq H(3d(x,y)/(NC_6)).$$

Let $\varepsilon = d(x,y)/N$. Then, by the choice of c_0 and by (1.1), we have

$$H^{-1}\left(\frac{c_1 t}{N}\right) \leq \varepsilon \leq \frac{C_6}{3} H^{-1}\left(\frac{t}{N}\right). \tag{2.36}$$

Now, let $\{x_i\}_{i=0}^N$ be such that $x_0 = x, x_N = y$ and $d(x_i, x_{i+1}) \leq \varepsilon$ for $i = 0, 1, \cdots, N-1$. Such a sequence exists by the choice of N and by the fact that d is a geodesic metric. We then have

$$
\begin{aligned}
p_t(x,y) &= \int_X \cdots \int_X p_{t/N}(x, z_1) p_{t/N}(z_1, z_2) \cdots p_{t/N}(z_{N-1}, y) d\mu(z_1) \cdots d\mu(z_{N-1}) \\
&\geq \int_{B(x_1,\varepsilon)} \cdots \int_{B(x_{N-1},\varepsilon))} p_{t/N}(x, z_1) p_{t/N}(z_1, z_2) \cdots p_{t/N}(z_{N-1}, y) d\mu(z_1) \cdots d\mu(z_{N-1}) \tag{2.37}
\end{aligned}
$$

Clearly $d(z_i, z_{i+1}) \leq 3\varepsilon$. Hence, by $(NLHK(H))$, (VD) and (2.36), we have

$$p_{t/N}(z_i, z_{i+1}) \geq \frac{c_2}{V(z_i, H^{-1}(t/N))} \geq \frac{c_3}{V(x_i, H^{-1}(t/N))} \geq \frac{c_4}{V(x_i, \varepsilon)}.$$

Therefore, it follows form (2.37)

$$
\begin{aligned}
p_t(x,y) &\geq \frac{c_4}{V(x, H^{-1}(t/N))} \prod_{i=1}^{N-1} \frac{c_4}{V(x_i, \varepsilon)} \cdot V(x_i, \varepsilon) \geq \frac{c_4^N}{V(x, H^{-1}(t/N))} \\
&\geq c_5 \frac{\exp(-c_6(N-1))}{V(x, H^{-1}(t))}.
\end{aligned}
$$

On the other hand, by the choice of N in (2.35), we have $N - 1 < c_7(H(d(x,y))/t)^{1/(\beta_1-1)}$. We thus obtain $(LHK(H))$. $\qquad\square$

Combining Steps 1 –4, the proof of Proposition 2.13 is completed.

3 Proof of Theorem 1.4

We first prove $\sup_{0<r\leq 1} \mathcal{E}^r(f) \leq c_1 \mathcal{E}(f)$ which in turn immediately will imply $\mathcal{F} \subset W_H(X)$. For $t \leq 1$ and $f \in L^2(X,\mu)$, let $\mathcal{E}_t(f) := \frac{1}{t}(f - P_t f, f)_{L^2}$, where P_t is the semigroup corresponding to $(\mathcal{E}, \mathcal{F})$. Then, since $(\mathcal{E}, \mathcal{F})$ is conservative,

$$
\begin{aligned}
\mathcal{E}_t(f) &= \frac{1}{2t} \int\!\!\int_{X\times X} (f(x) - f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{1}{2t} \int\!\!\int_{d(x,y)\leq H^{-1}(t)} (f(x) - f(y))^2 p_t(x,y)\mu(dx)\mu(dy) \\
&\geq \frac{e^{-1}}{2t} \int\!\!\int_{d(x,y)\leq H^{-1}(t)} \frac{(f(x) - f(y))^2}{V(x, H^{-1}(t))} \mu(dx)\mu(dy), \tag{3.1}
\end{aligned}
$$

where we use the lower bound of $(HK(H; \beta_1, \beta_2, c_0))$ in the last inequality. Taking $t = h(r)$ for $r > 0$, we see that the RHS of (3.1) is equal to $\frac{1}{c_1}\mathcal{E}^r(f)$ for some $c_1 > 0$. It is well known that $\mathcal{E}_t(f) \nearrow \mathcal{E}(f)$ as $t \downarrow 0$ ([8], Lemma 1.3.4). Thus the claim follows.

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We next prove $c_2(\alpha, \beta_1, \beta_2)\mathcal{E}(f) \leq \sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(f)$ which then will imply $\mathcal{F} \supset W_H(X)$. In this proof, constants may depend on $\alpha > 1, \beta_2 \geq \beta_1 > 1$, but we will not write down this dependence explicitly. For each $t \leq 1$ and $g \in W_H(X)$, since $(\mathcal{E}, \mathcal{F})$ is conservative,

$$
\begin{aligned}
\mathcal{E}_t(g) &= \frac{1}{2t} \int \int_{X \times X} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy) \\
&= \frac{1}{2t} \int \int_{\substack{x, y \in X \\ d(x,y) > 1}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy) \\
&+ \frac{1}{2t} \int \int_{\substack{x, y \in X \\ d(x,y) \leq 1}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy) =: A(t) + B(t).
\end{aligned}
$$

We first estimate $A(t)$. (In fact, this part is zero since $diam(X) = 1$. Since we need to compute $A_k(t)$ in the end of this proof, we will make some estimates.) Since

$$
A(t) = \frac{1}{2t} \sum_{m=0}^{\infty} \int \int_{\alpha^m < d(x,y) \leq \alpha^{m+1}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy),
$$

for $\alpha > 1$ (note that $\int \int_{\alpha^m < d(x,y) \leq \alpha^{m+1}} \cdots = 0$ if $diam(X) < \alpha^m$), using the fact $(a+b)^2 \leq 2(a^2 + b^2)$ and the symmetry, we have

$$
A(t) \leq \frac{2}{t} \sum_{m=0}^{\infty} \int_X g(x)^2 \mu(dx) \int_{\alpha^m < d(x,y) \leq \alpha^{m+1}} p_t(x, y) \mu(dy).
$$

Set $L_m = \{y \in X : \alpha^m < d(x, y) \leq \alpha^{m+1}\}$. Let $\Phi_2(x) = c_3 \exp(-c_4 x^{1/(\beta_2 - 1)})$. By $(HK(H; \beta_1, \beta_2, c_0))$ we have

$$
\begin{aligned}
\int_{L_m} p_t(x, y) \mu(dy) &\leq \int_{L_m} \frac{1}{V(x, H^{-1}(t))} \Phi_2 \Big(\frac{H(\alpha^m)}{t}\Big) \mu(dy) \\
&\leq \frac{\mu(L_m)}{V(x, H^{-1}(t))} \Phi_2 \Big(\frac{c_3 \alpha^{m\beta_1}}{t}\Big) \leq \frac{V(x, \alpha^{m+1})}{V(x, H^{-1}(t))} \Phi_2 \Big(\frac{c_3 \alpha^{m\beta_1}}{t}\Big),
\end{aligned}
$$

where we use (FTG) and the fact $H(1) = 1$ in the second inequality. Using (1.1), we have

$$
V(x, \alpha^{m+1})/V(xH^{-1}(t)) \leq c_4(\alpha^m / H^{-1}(t))^D.
$$

Note that by (FTG), if t' is small we have $H(1)/H(t') = 1/H(t') \geq c_5/t'^{\beta_1}$. Taking $t = H(t')$, we have $1/H^{-1}(t) \leq c_6/t^{1/\beta_1}$. Combining these facts, we have

$$
A(t) \leq \frac{c_7}{t} \|g\|_{L^2}^2 \sum_{m=0}^{\infty} \Big(\frac{\alpha^{m\beta_1}}{t}\Big)^{D/\beta_1} \Phi_2 \Big(\frac{c_3 \alpha^{m\beta_1}}{t}\Big) \leq c_8 \|g\|_{L^2}^2 t^{-1-D/\beta_1} \Phi_2 \Big(\frac{c_3}{t}\Big) \sum_{m=0}^{\infty} \alpha^{-m\beta_2}
$$

for small $t \leq 1$. Here we used the fact $\alpha^{m(D+\beta_2)} \exp(-c(\frac{\alpha^{m\beta_1}}{t})^{1/(\beta_2-1)}) \leq c' \exp(-ct^{1/(\beta_2-1)})$ in the last inequality. Thus, we obtain

$$
A(t) \leq c_9 \|g\|_{L^2}^2 t^{-1-D/\beta_1} \Phi_2 \Big(\frac{c_3}{t}\Big) \tag{3.2}
$$

for small t and thus $A(t) \xrightarrow{t \to 0} 0$.

Next we estimate $B(t)$. By $(\mathrm{HK}(H; \beta_1, \beta_2, c_0))$ again, we have

$$B(t) \leq \frac{1}{2t} \sum_{m=1}^{\infty} \int \mu(dx) \int_{L_{-m}} \frac{1}{V(x, H^{-1}(t))} \Phi_2\Big(\frac{H(\alpha^{-m})}{t}\Big)(g(x) - g(y))^2 \mu(dy)$$

$$\leq \frac{c_9}{t} \sum_{m=1}^{\infty} \int \mu(dx) \frac{V(x, \alpha^{1-m})}{V(x, H^{-1}(t))} \int_{B(x, \alpha^{1-m})} \Phi_2\Big(\frac{H(\alpha^{-m})}{t}\Big)(g(x) - g(y))^2 \mu(dy)$$

$$\leq c_9 (\sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g)) \sum_{m=1}^{\infty} \{c_{10}(\frac{\alpha^{1-m}}{H^{-1}(t)})^D \vee 1\} \frac{H(\alpha^{1-m})}{t} \Phi_2\Big(\frac{H(\alpha^{-m})}{t}\Big) =: I_1,$$

where we use (1.1) in the last inequality. We now compute the sum in I_1. Let $t' = H^{-1}(t)$ and take $m_0 = m_0(t')$ so that $\alpha^{-m_0-1} < t' \leq \alpha^{-m_0}$. Then, by (1.1),

$$I_1 \leq c_{14} (\sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g)) \Big\{ \sum_{m=1}^{m_0 \vee 0} \Big(\frac{\alpha^{-m}}{t'}\Big)^{D+\beta_2} \Phi_2\Big(c_{15}\Big(\frac{\alpha^{-m}}{t'}\Big)^{\beta_1}\Big) + \sum_{m=(m_0+1) \vee 1}^{\infty} \Big(\frac{\alpha^{-m}}{t'}\Big)^{\beta_1} \Big\}$$

$$\leq c_{16}(\sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g)) \Big\{ \int_1^{\infty} \Phi_2(s)s^{\frac{D+\beta_2}{\beta_1}-1} ds + \sum_{m=0}^{\infty} \alpha^{-m\beta_1} \Big\}$$

$$\leq c_{17} \cdot \sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g),$$

where we use $\int_1^{\infty} \Phi_2(s)s^{\frac{D+\beta_2}{\beta_1}-1} ds \leq C_{\alpha,\beta_2} < \infty$ in the last inequality. Thus, together with (3.2), we obtain

$$\mathcal{E}(g) = \lim_{t \to 0} \mathcal{E}_t(g) \leq c_{17} \cdot \sup_{m \in \mathbb{N}} \mathcal{E}^{\alpha^{-m}}(g).$$

Now replacing $A(t)$ and $B(t)$ in the previous argument by

$$A_k(t) = \frac{1}{2t} \int \int_{d(x,y) > \alpha^{-k}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy)$$

and

$$B_k(t) = \frac{1}{2t} \int \int_{d(x,y) \leq \alpha^{-k}} (g(x) - g(y))^2 p_t(x, y) \mu(dx) \mu(dy)$$

yields $\mathcal{E}(g) \leq c_{17} \cdot \sup_{m \geq k} \mathcal{E}^{\alpha^{-m}}(g)$ for each $k \in \mathbb{N}$ and thus

$$\mathcal{E}(g) \leq c_{17} \cdot \limsup_{m \to \infty} \mathcal{E}^{\alpha^{-m}}(g).$$

$\square$

A   Appendix: Miscellaneous proofs

A.1   Oscillation inequalities and the Hölder continuity

In this subsection, we will assume (EHI) and deduce various Oscillation inequalities and Hölder continuity of harmonic functions.

Let u be nonnegative and harmonic in $B(x_0, R)$. To be precise, the definition of (EHI) in Section 1 should be,

$$\text{ess sup}_{B(x_0, R/2)} u \leq c_1 \text{ess inf}_{B(x_0, R/2)} u. \tag{A.1}$$

x_0 here is x in the definition of (EHI). We will show here that (A.1) implies the continuity of u inside the ball $B(x_0, R)$, so that (EHI) holds. Indeed, take x_1 and r such that $B(x_1, 3r) \subset B(x_0, R)$. By looking at $Cu + D$ for suitable constants C and D, we may suppose that ess $\text{sup}_{B(x_1, 2r)} u = 1$ and ess $\inf_{B(x_1, 2r)} u = 0$. Hence by (A.1), we have

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq (1 - c_1^{-1}) \text{ess sup}_{B(x_1, r)} u \leq (1 - c_1^{-1}).$$

So if $\rho = 1 - c_1^{-1}$ then

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq \rho \left[\text{ess sup}_{B(x_1, 2r)} u - \text{ess inf}_{B(x_1, 2r)} u \right].$$

It follows easily that

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq c_2 r^\gamma \tag{A.2}$$

for some $\gamma > 0$. Define $\widehat{u}(x_1) = \lim_{r \to 0} \text{ess sup}_{B(x_1, r)} u$. If one takes a countable basis $\{B_i\}$ for X and excludes those points $x \in B_i$ such that $u(x) \notin [\text{ess inf}_{B_i} u, \text{ess sup}_{B_i} u]$, then for every other x it is easy to see, using (A.2), that $u(x) = \widehat{u}(x)$. Thus, $\widehat{u}$ is equal to u for μ-almost every x. Moreover, from (A.2) we see that $\widehat{u}$ is continuous. Recall that in our definition of harmonic function we take a quasi-continuous modification as defined in [8]. We conclude $u = \widehat{u}$ quasi-everywhere, and so u has a quasi-continuous modification that is continuous. Using this modification and (A.1), we have

$$\sup_{B(x_0, R/2)} u \leq c_1 \inf_{B(x_0, R/2)} u,$$

which is the desired inequality.

Let $\mathcal{H}_{B(x_0, r)}$ be a space of harmonic functions on $B(x_0, r)$. Define the oscillation of a function f over B by $\text{Osc}_B f := \text{ess sup}_B f - \text{ess inf}_B f$. Then, the above arguments also show the following.

Lemma A.1 *Assume (EHI).*
1) For any $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\text{Osc}_{B(x_0, \delta r)} u \leq \varepsilon \text{Osc}_{B(x_0, r)} u, \qquad \forall u \in \mathcal{H}_{B(x_0, r)}.$$

2) There exist $c_1, \gamma > 0$ such that

$$\sup_{x, y \in B(x_0, \rho r)} |u(x) - u(y)| \leq c_1 \rho^\gamma \sup_{x \in B(x_0, r)} |u(x)|, \qquad \forall \rho \in (0, 1), \forall u \in \mathcal{H}_{B(x_0, r)}. \tag{A.3}$$

We can now prove the following Hölder continuity of harmonic functions.

Proposition A.2 *Assume (EHI). There exists $\gamma > 0$ with the property that for any $\delta \in (0, 1)$, there exists $C = C_\delta > 0$ such that*

$$\sup_{x, y \in B(x_0, \delta r)} \left\{ \frac{|u(x) - u(y)|}{d(x, y)^\gamma} \right\} \leq C r^{-\gamma} \sup_{x \in B(x_0, r)} |u(x)|, \qquad \forall u \in \mathcal{H}_{B(x_0, r)}.$$

PROOF. Denote $B_r := B(x_0, r)$. For $x, y \in B_{\delta r}$, we consider two cases. First, if $d(x, y) \geq (1 - \delta)r$, then

$$|u(x) - u(y)| \leq 2 \sup_{B_r} |u| \leq 2\{(1 - \delta)r\}^{-\gamma} d(x, y)^\gamma \sup_{B_r} |u|.$$

If $d(x, y) < (1 - \delta)r$, then $B(z, (1 - \delta)r) \subset B_r$ contains both x and y, where $z \in X$ is the mid point of x and y. Further $x, y \in B(z, d(x, y))$. Applying (A.3) with $\rho = d(x, y)/\{(1 - \delta)r\}$ yields

$$|u(x) - u(y)| \leq c_1 \{(1 - \delta)r\}^{-\gamma} d(x, y)^\gamma \sup_{B_r} |u|.$$

We thus obtain the result. □

We next discuss the oscillation of Green functions. Given an open set $\Omega \subset X$ and $f \in \mathcal{B}(\Omega)$, define the Green operator G_Ω as

$$G_\Omega f(x) = E^x \left[\int_0^{\tau_\Omega} f(Y_t) dt \right].$$

Denote $\bar{E}(\Omega) := \sup_z E^z[\tau_\Omega]$. When $\Omega = B(x, r)$, we will abbreviate $\bar{E}(B(x, r))$ as $\bar{E}(x, r)$. It is easy to see

$$\|G_\Omega\|_{L^\infty \to L^\infty} \leq \bar{E}(\Omega). \tag{A.4}$$

The following results are due to [13].

Lemma A.3 *Assume that $\bar{E}(\Omega) < \infty$. Then, for any $f \in C_0(\Omega)$, $G_\Omega f$ is harmonic in $\Omega \setminus \mathrm{Supp} f$. Also, for any open set $\Omega' \supset \Omega$, $G_{\Omega'} f - G_\Omega f$ is harmonic in Ω.*

PROOF. Let $u_f = G_\omega f$. Since $G_\omega = (-\Delta_\Omega)^{-1}$, we see that $u_f \in \mathcal{D}(\Delta_\Omega)$. So

$$\mathcal{E}(u_f, v) = -(\Delta_\Omega u_f, v) = (f, v) = 0, \qquad \forall v \in \mathcal{F}(\Omega \setminus \mathrm{Supp} f).$$

Thus, u_f is harmonic in $\Omega \setminus \mathrm{Supp} f$. Similarly, set $w_f = G_{\Omega'} f - G_\Omega f$, then

$$\mathcal{E}(w_f, v) = \mathcal{E}(G_{\Omega'} f, v) - \mathcal{E}(G_\Omega f, v) = (f, v)_{L^2(\Omega')} - (f, v)_{L^2(\Omega)} = 0,$$

for any $v \in \mathcal{F}(\Omega)$. □

Proposition A.4 *Assume (EHI). Let $f : B(x, r) \to \mathbb{R}$ be a bounded Borel function and set $u_f = G_{B(x,R)} f$. Then, for any $0 < r < R$,*

$$\mathrm{Osc}_{B(x,\delta r)} u_f \leq 2(\bar{E}(x, r) + \varepsilon \bar{E}(x, R)) \|f\|_\infty,$$

where ε and δ are the same as in Lemma A.1 1).

PROOF. If $\bar{E}(x, R) = \infty$, there is nothing to prove, so assume that $\bar{E}(x, R) < \infty$. Denote $B_r := B(x, r)$ and let $v_f = G_{B_r} f$. Then, by (A.4),

$$\|u_f\|_\infty \leq \bar{E}(x, R) \|f\|_\infty, \qquad \|v_f\|_\infty \leq \bar{E}(x, r) \|f\|_\infty. \tag{A.5}$$

By Lemma A.3, $w_f := u_f - v_f$ is harmonic in B_r. Using Lemma A.1 1) and $0 \leq w_f \leq u_f$, we obtain

$$\mathrm{Osc}_{B_{\delta r}} w_f \leq \varepsilon \mathrm{Osc}_{B_r} w_f \leq \varepsilon \|w_f\|_\infty \leq \varepsilon \|u_f\|_\infty.$$

Since $u_f = v_f + w_f$,

$$\mathrm{Osc}_{B_{\delta r}} u_f \leq \mathrm{Osc}_{B_{\delta r}} v_f + \mathrm{Osc}_{B_{\delta r}} w_f \leq \|v_f\|_\infty + \varepsilon \|u_f\|_\infty \leq (\bar{E}(x, r) + \varepsilon \bar{E}(x, R)) \|f\|_\infty,$$

where we used (A.5) in the last inequality. Thus we obtain the desired inequality for $f \geq 0$. For a general function f, write $f = f_+ - f_-$. Then $\mathrm{Osc}\, u_f = \mathrm{Osc}\, (u_{f_+} - u_{f_-}) \leq \mathrm{Osc}\, u_{f_+} + \mathrm{Osc}\, u_{f_-}$, and the desired inequality is obtained. □

A.2 Time derivative

We follow the arguments in [13, 11]. First, we show the following well-known fact from semigroup theory.

Lemma A.5 *For any $f \in L^2$, let $u_t = P_t f$. Then, we have*

$$\|\partial_t u_t\|_2 \leq \frac{1}{t-s}\|u_s\|_2, \qquad 0 < \forall s < t.$$

PROOF. Let $\{E_\lambda\}_{\lambda \geq 0}$ be spectral resolution of the operator $-\Delta$. Then we have

$$u_t = e^{t\Delta}f = \int_0^\infty e^{-t\lambda} dE_\lambda f, \quad \|u_t\|_2^2 = \int_0^\infty e^{-2t\lambda} d\|E_\lambda f\|^2.$$

Thus, we have

$$\partial_t u_t = \int_0^\infty (-\lambda)e^{-t\lambda} dE_\lambda f, \quad \|\partial_t u_t\|_2^2 = \int_0^\infty \lambda^2 e^{-2t\lambda} d\|E_\lambda f\|^2 = \int_0^\infty \lambda^2 e^{-2(t-s)\lambda} e^{-2s\lambda} d\|E_\lambda f\|^2.$$

Since $\lambda e^{-(t-s)\lambda} \leq (t-s)^{-1}$, we obtain

$$\|\partial_t u_t\|_2^2 \leq \frac{1}{(t-s)^2} \int_0^\infty e^{-2s\lambda} d\|E_\lambda f\|^2 = \frac{1}{(t-s)^2}\|u_s\|_2^2,$$

which is the desired estimate. $\qquad\square$

Corollary A.6 *For $t > 0$ and $z \in X$, the function $t \mapsto p_t(\cdot, z)$ is Frechet differentiable in L^2 and*

$$\|\partial_t p_t(\cdot, z)\|_2 \leq \frac{1}{t-s}\sqrt{p_{2s}(z, z)}, \qquad 0 < \forall s < t.$$

PROOF. Let $f = p_\varepsilon(\cdot, z)$ for some $\varepsilon > 0$. Then, $u_t = P_t f = p_{t+\varepsilon}(\cdot, z)$. Thus, by Lemma A.5,

$$\|\partial_t p_{t+\varepsilon}(\cdot, z)\|_2 \leq \frac{1}{t-s}\|p_{s+\varepsilon}(\cdot, z)\|_2 = \frac{1}{t-s}\sqrt{p_{2(s+\varepsilon)}(z, z)}.$$

Replacing $t + \varepsilon, s + \varepsilon$ by t, s respectively, we obtain the result. $\qquad\square$

Proposition A.7 *For any $x, y \in X$, the function $t \mapsto p_t(x, y)$ is differentiable in $t > 0$ and*

$$|\frac{\partial_t}{\partial t} p_t(x, y)| \leq \frac{2}{t}\sqrt{p_{t/2}(x, x)p_{t/2}(y, y)}.$$

PROOF. By the Chapman-Kolmogorov equation, $p_t(x, y) = (p_{t-s}(\cdot, x), p_s(\cdot, y))$ for any $s \in (0, t)$, so that $\partial_t p_t(x, y) = (\partial_t p_{t-s}(\cdot, x), p_s(\cdot, y))$. Thus, applying Corollary A.6,

$$|\frac{\partial_t}{\partial t} p_t(x, y)| \leq \|\partial_t p_{t-s}(\cdot, x)\|_2 \|p_s(\cdot, y)\|_2 \leq \frac{1}{t-s-r}\sqrt{p_{2r}(x, x)p_{2s}(y, y)}, \qquad 0 < \forall r < t - s.$$

Taking $s = r = t/4$, we obtain the result. $\qquad\square$

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