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An algebaric formulation of propositional calculi

By Kiyoshi ISEKI

As we know, in the present mathematics, many interesting algebraic systems are obtained from the theory of configurations in the foundation of geometry as shown by M. Hall, L. A. Skorniakov and I. Argunov etc. On the other hand, other interesting algebraic systems are obtained from the algebraic formulations of propositional calculi. In this lecture, I give a fundamental idea of algebraic formulations of propositional calculi and some results obtained in this field.

As an example of CN-systems of propositonal calculi, take an axiom system of the classical propositional calculus:

a) CpCqp,

b) CCpCqrCCpqCpr,

c) CCNpNqCqp.

In this calculus, as the rules of inferences we use the rules of substitution, detachment and replacement which enable us to make use of definitions.

Then we have a general algebra from this calculus by the following technique.

Let $\langle X, 0, *, \sim \rangle$ be an algebra consisting of a set X, where 0 is an element of X, $*$ is a binary operation and $\sim$ is a unary operation on X. The axioms given below hold in X. We write $x \leq y$ for $x * y = 0$. For any $x, y, z \in X$,

1 $x * y \leq x$,

2 $(x * z) * (y * z) \leq (x * y) * z$,

3 $x * y \leq (\sim y) * (\sim x)$,

4 $0 \leq x$,

5 $x \leq y$ and $y \leq x$ imply $x = y$.

Such an algebra is called <u>B-algebra</u>.

I explain how to get theorems in the B-algebra.

6 $\quad 0 * x = 0.$

Proposition 6 is obtained by the definition of $x \leqslant y$.

7 $\quad x * x = 0$, i. e. $x \leqslant x$.

By axiom 1, we have $(x * x) * x = 0$ and $(x * (x * x)) * x = 0$. By axiom 2, $(x * x) * ((x * x) * x) \leqslant (x * (x * x)) * x$. This shows $(x * x) * 0 \leqslant 0$ by axioms 4 and 5.

8 $\quad$ If $x * y = 0$, $y * z = 0$, then $x * z = 0$, i. e. $x \leqslant y$, $y \leqslant z$ imply $x \leqslant z$.

By axiom 2 and proposition 6, we have $(x * z) * 0 \leqslant 0 * z = 0$. Hence $x * z = 0$, i. e. $x \leqslant z$.

9 $\quad$ If $x * y \leqslant z$, then $x * z \leqslant y$, i. e. $(x * y) * z = 0$ implies $(x * z) * y = 0$.

Axiom 2 and $(x * y) * z = 0$ imply $(x * z) * (y * z) = 0$. By $y * z \leqslant y$, we have $x * z \leqslant y$, i. e. $(x * z) * y = 0$.

10 $\quad$ If $x \leqslant y$, then $z * y \leqslant z * x$ and $x * z \leqslant y * z$.

Axioms 1 and 2 mean $(z * x) * y \leqslant z * x$ and $(z * y) * (z * y) \leqslant (z * x) * y$ respectively. By $x * y = 0$ and the inequalities above, we have $(z * y) * 0 \leqslant (z * x) * y \leqslant z * x$. Hence by proposition 9, $(z * y) * (z * x) = 0$, i. e. $z * y \leqslant Z * x$. Next axiom 2 and proposition 6, we $(x * z) * (y * z) \leqslant (x * y) * z = 0 * z = 0$. Hence by axioms 4, 5 we have $(x * z) * (y * z) = 0$ which proves proposition 10.

11 $\quad y * x = (y * x) * x.$

By axiom 2, we have $(y * x) * (x * x) \leqslant (y * x) * x$. By proposition 7, $(y * x) * 0 \leqslant (y * x) * x$ and $y * x \leqslant (y * x) * x \leqslant y * x$ by proposition 9 and axiom 1.

12 $(z * x) * (y * x) \leqslant z * y.$

If we substitute $(z * x) * (y * x)$ for x, $(z * y) * x$ for x and $z * y$ for z in axiom 2, then we have

$$(((z * x) * (y * x)) * (z * y)) * (((z * y) * x) * (z * y))$$
$$(((z * x) * (y * x)) * ((z * y) * x)) * (z * y)$$
$$= 0 * (z * y) = 0$$

by axiom 2. Further, by axiom 1, we have $((z * y) * x) * (z * y) = 0$, which completes the proof of proposition 12.

13 $(x * y) * z = (x * z) * y.$

By proposition 10, $y * x \quad y$ implies

$$(z * x) * y \leqslant (z * x) * (y * x).$$

In this formula, put $z = (x * y) * z$, $x = (x * z) * y$ and $y = (x * y) * (z * y)$, then we have

$$(((x * y) * z) * ((x * z) * y)) * (((x * y) * (z * y)) * ((x * z) * y))$$
$$\leqslant ((x * y) * z) * ((x * y) * (z * y)).$$

The right side is 0 by the formula above, and the second term of the left side is 0 by axiom 2, hence

$$(x * y) * z \leqslant (x * z) * y.$$

If we use the formula above again, we have the equality 12.

For the proofs of propositions 6-13, we do not use axiom 3. Next we prove some propositions using axiom 3.

14 $x * y \leqslant \sim y, \quad x * (\sim y) \leqslant y.$

By axioms 1 and 3, we have

$$x * y \leqslant (\sim y) * (\sim x) \leqslant \sim y$$

and moreover, by proposition 9, we have $x * (\sim y) \leqslant y$.

15 $x \leqslant \sim(\sim x).$

By axiom 3, we have

$$x * (\sim(\sim x)) \leqslant \sim(\sim(\sim x)) * (\sim x).$$

On the other hand, in the first inequality of proposition 14, we substitute $\sim(\sim(\sim x))$ for x and $\sim x$ for y, then we have

$$\sim(\sim(\sim x)) * (\sim x) \leq \sim(\sim x).$$

Hence by proposition 8, $x * (\sim(\sim x)) \leq \sim(\sim x)$, i. e., $(x * (\sim(\sim x))) * (\sim(\sim x)) = 0$. Applying proposition 11, we have

$$x * (\sim(\sim x)) = (x * (\sim(\sim x))) * (\sim(\sim x)) = 0,$$

which shows $x * (\sim(\sim x)) = 0$. This means $x \leq \sim(\sim x)$.

16 $\qquad \sim(\sim x) \leq x.$

By axiom 3 and $x * (\sim(\sim x)) = 0$, we have

$$(\sim(\sim x)) * x \leq (\sim x) * (\sim(\sim(\sim x))) = 0.$$

Hence $\sim(\sim x) \leq x$.

17 $\qquad \sim(\sim x) = x.$

This formula follows from propositions 15 and 16.

Therefore we have

18 $\qquad$ In the B-algebra, the following equalities hold:

$$(\sim x) * y = (\sim y) * x, \qquad x * (\sim y) = y * (\sim x),$$
$$(\sim x) * (\sim y) = y * x.$$

19 $\qquad x * 0 = x.$

$x * y \leq x * y$ and proposition 9 imply $x * (x * y) \leq y$. If we substitute 0 for y, we have $x * (x * 0) \leq 0$, and hence $x \leq x * 0$. Axiom 1 implies $x * 0 \leq x$, so we have the formula 19.

I stop proofs of other propositions in B-algebra. I think you arise a question whether the concepts of the B-algebra and the Boolean algebra are equivalent. If we define $x \vee y = (\sim y * x)$, $x \wedge y = y * (\sim x)$ and $1 = \sim 0$, then the B-algebra is the Boolean algebra and these concepts are same. But this result is not essential to me. In the above discussion, we may find some other important results. Next I point out these results. First of all, the proofs of formulas are very simplified, and these proofs are transformed in the proofline method of J. Łukasiewicz (10) without difficulty.

As an example, we take up a single axiom of the classical propositional calculus by J. Łukasiewicz:

CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv.

In my formulation, this is written in the form of

$$x * w \leq (x * ((((u * t) * (s * t)) * ((u * s) * r)) * (((\sim t) * s) * (\sim r)))) * ((y * z) * y)).$$

If we use the method of proofline by J. Łukasiewicz to get the formula above, its proof is very complicated(for details, see S. Tanaka (14)). Following my method, its proof is as follows: As already seen, $y * z \leq y$, i. e., $(y * z) * y = 0$. So we shall prove:

$$(((u * t) * (s * t)) * ((u * s) * r)) * (((\sim t) * s) * (\sim r)) = 0,$$

the the right side of the formula is $x * (0 * 0) = x$. Hence we have $x * w \leq x$, which completes the proof. Next we shall prove an auxiliar formula:

$$((u * t) * (s * t)) * ((u * s) * r) \leq ((u * s) * t) * ((u * s) * r)$$
$$= (\sim t * \sim(u * s)) * (\sim r) * \sim(u * s))$$
$$(\sim t * \sim r) * \sim(u * s) = (r * t) * \sim(u * s)$$
$$= (r * \sim(u * s)) * t = (\sim r * (u * s)) * t.$$

Consider $((r * t) * \sim(u * s)) * ((\sim t * s) * r)$, then this is equal to

$$((r * t) * \sim(u * s)) * ((\sim t * \sim r) * s)$$
$$= ((r * t) * \sim(u * s)) * ((r * t) * s)$$
$$\leq (s * \sim(u * s)) * (r * t) \leq s * (\sim(u * s)).$$

Then the right side is equal to 0. Hence we have

$$(((u * t) * (s * t)) * ((u * s) * r)) * ((\sim t * s) * \sim r) = 0.$$

We had a non-associative, non-commutative ordered algebra with respect to the operation *, and its operation does not preserved by the order(see proposition 10). This property is also found in (11). As an algebraic formulation of the negationless logic, I considered an algebra $\langle X, 0, \vee, * \rangle$ with two operations $\vee$, $*$ on a set X satisfying the following axioms:

1) $x \vee x \leqslant x$,

2) $x \vee y \leqslant y \vee x$,

3) $(x \vee y) * (x \vee z) \leqslant y * z$,

4) $x * y \leqslant (x * z) \vee (z * y)$,

5) $x \leqslant y \vee x$,

6) $0 \leqslant x$,

7) if $x \leqslant y$ and $y \leqslant x$, then $x = y$,

where $x \leqslant y$ is defined by $x * y = 0$.

With some additional conditions, I defined Griss algebra, and developed the ideal theory of Griss algebra. My plan is to get the representation theorem of Griss algebra, but I have not yet completed.

By the way, I give some axiom systems of the B-algebra. In my seminar, we gave some new proofs of equivalences of several axiom systems of the classical propositional calculus using my technique. These are found in Axioms systems of B-algebra, I-VI, Proc. Japan Acad., 41-2(1965-6). On the other hand, by my method, Roumanian mathematician C. Sicoe found a new axiom system:

a) $(x * y) * z \leqslant x$,

b) $(x * (y * z)) * (x * z) \leqslant x * (\sim z)$,

c) $x * y \leqslant \sim y$,

and axioms 4 and 5. The proof is given in (12).

Next I explain on proposition 19. When we begun our discussion of my algebraic formulation, $x * 0 = x$ was quite useful for proofs of many formulas. Later we found that $x * 0 = x$ corresponds with a general detachment rule by R. B. Angell (1): if α and $\varphi(C\alpha\beta)$ are theses in a propositional calculus, then $\varphi(\beta)$ is a thsis in it. If we use $x * 0 = 0$ as an axiom, many axiom systems are quite simplified(see K. Iseki (7)).

Axiom 1 and proposition 9 imply $0 * y = 0$ for every y, and we have $0 \leqslant y$ for every y. On the other hand, there are many propositional calculi that the axiom $x * y \leqslant x$ does not hold. As an example, I take up the BCI propositional calculus defined by C. A. Meredith:

a) CCpqCCqrCpr,

b) CpCCpqq,

c) Cpp.

In the BCI system, CpCqp is not provable. For such a propositional calculus, I replaced axiom 4 by another axiom, and succeeded its algebraic formulation.

Let $\langle X, 0, * \rangle$ be an algebra with a zero element 0 and a binary operation $*$. If the algebra satisfies the following axioms, it is called <u>BCI-algebra</u>.

1 $\quad (x * y) * (x * z) \leqslant z * y$,

2 $\quad x * (x * y) \leqslant y$,

3 $\quad x \leqslant x$,

4 $\quad x \leqslant 0$ implies $x = 0$,

5 $\quad x \leqslant y$ and $y \leqslant x$ imply $x = y$.

In this algebra, we can prove the following formula:

6 $\quad (x * y) * z \leqslant (x * z) * y$.

As easily seen, we have

7 $\quad$ If $x \leqslant y$, then $z * y \leqslant z * x$.

By axiom 1, we have

$$u * (z * y) \leqslant u * ((x * y) * (x * z)).$$

If we substitute $x * u$ for x, $x * z$ for z and $((x * u) * y) * (z * u)$ for u in the formula above, then

$$(((x * u) * y) * (z * u)) * ((x * z) * y))$$
$$\leqslant (((x * u) * y) * (z * u)) * (((x * u) * y) * ((x * u) * (x * z)))$$

Applying the same formula, the right side is equal to 0, we have

$$(((x * u) * y) * (z * u)) \leqslant ((x * z) * y).$$

Put $u =$ [illegible], $z = x * y$ in this formula, then

$$((x * z) * y) * ((x * y) * z) \leqslant (x * (x * y)) * y = 0.$$

Hence by axiom 4, we have proposition 6.

By the way, if we replace axiom by $0 \leqslant x$, then we have

$$(x * y) * x \leqslant (x * x) * y = 0,$$

therefore $x * y \leqslant x$.

As an another important example, we know a three valued propositional calculus by B. Sobocinski. This contains a negation functor N, so the axiom system of it is written in the form of

1' $\quad (x * y) * (x * z) \leqslant z * y,$

2' $\quad x * (x * y) \leqslant y,$

3' $\quad x * y \leqslant (x * y) * y,$

4' $\quad (x * (\sim y)) * y \leqslant x,$

5' $\quad x * y \leqslant (\sim y) * (\sim x),$

6' $\quad x \leqslant 0$ implies $x = 0$.

By a truth-table, we can verify that $x * y \quad x$ does not hold.

From these algebraic formulations, we conclude that there are two different kinds of propositional calculi. As in the B-algebra, the element 0 is the least element in it. On the other hand, as in the BCI-algebra, the zero element is a minimal element in it. The classification corresponds to a question: whether any expression is a truth function or not.

In the BCI-algebra, $x * 0 = x$ holds. As already proved, we

$$(x * y) * z = (x * z) * y,$$

then put $y = 0$, $z = x$ in this formula, we have

$$(x * 0) * x = (x * x) * 0 = 0.$$

In axiom 2, put $y = o$, then $x * (x * 0) \leqslant 0$. By axiom 4, we have $x \leqslant x * 0$. Therefore we have $x * 0 = x$.

From the result mentioned above, we have a characterization of the BCI-algebra as follows: An algebra is the BCI-algebra if and only if it satisfies

1'' $\quad (x * y) * (x * z) \leqslant z * y,$

2'' $\quad x * 0 = x,$

3'' $\quad x \leqslant 0$ implies $x = 0$.

The proof is very simple. As already seen, the three conditions hold in the BCI-algebra. To prove the converse, in the first formula, put $y = 0$, then we have $(x * 0) * (x * z) \leqslant z * 0$. By the second formula, we can write this as $x * (x * y) \leqslant y$, which is axiom 2. In the formula obtained, put $y = 0$, then $x * (x * 0) \leqslant 0$. This means $x * x \leqslant 0$. The third condition implies $x * x = 0$. Hence we complete the proof.

If axiom 4 in the BCI-algebra is replaced by " $0 \leqslant x$ for every x ", the algebra is called BCK-algebra. The BCK-algebra corresponds to the BCK-propositional calculus introduced by C. A. Meredith.

In my seminar, we have studied on an algebraic formulation of propositional calculi with variable functors, and members of my seminar have obtained some interesting results. On the other hand, by using my formulation, C. Sicoe has studied on an algebraic formulation of the Lukasiewicz three valued logic. His result is contained in (13), and it shows that my method is useful.

Finally I give some remarks on my methods.

We hope to have an algebraic formulation of NK-systems. One part of its formulation is contained in our papers (8), (15). But the results do not give us satisfaction.

A question naturally aries how to get the substitution formulas in the proof mentioned above. We have not any algorithm to get these formulas. We only follow some methodology in mathematics. I asked in my seminar: Are there some useful methods to introduce new mathematical concepts and to get mathematical proofs? One of the answers is as follows. From Euclid to the present mathematics(non-standard analysis as a realization of the Democritos idea), almost all mathematical concepts have been introduced by making the product of some mathematical concepts already defined or its equivalence classes. The analoguous method have been used in the proofs of many theorems. The same considerations are very important for my formulation. The operation *. has played an important role as the product, and propositions 9 and 10 are quite useful for making the product. First we make a formula with many terms applying proposition 10 several times. Then we substitute some formulas proved for the terms, and if we have a formula such that its right side is equal to 0, we have omitted the unnecessary terms. Repeating this technique, we approach to the formula to be proved.

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