数理解析研究所講究録
第 33 巻 1967 年 73-91

ON LINEAR DIFFERENTIAL GAMES

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1. Statement of the Problems

This paper treats a certain class of linear differential games, the state of which is specified by a state vector in an n-dimensional Euclidean space R^n. It is assumed that the state vector $z(t)$ in R^n can be expressed as

$$z(t)=z_1(t)+\int_0^t F(t,s)u(s)ds+\int_0^t W(t,s)v(s)ds, \tag{1}$$

where $u(t)$ and $v(t)$ are r-vectors called strategies or controls of the first and the second players, respectively, $F(t,s)$ and $W(t,s)$ are $n\times r$ matrix functions which are assumed to be continuous, and $z_1(t)$ is a known n-dimensional vector function. It is assumed that the strategies $u(t)$ and $v(t)$ of the first and second players are measurable and constrained as

$$u(t)\in U, \quad v(t)\in V, \tag{2}$$

where U and V are certain sets in an r-dimensional Euclidean space. Such strategies $u(t)$ and $v(t)$ which satisfy above mentioned conditions are called admissible strategies. In this paper, we treats the case where the sets U and V are unit cube, respectively, i.e.,

$$U=V=\Omega=\{u:\ |u_i|\leq 1 \quad (i=1,\cdots,r)\}. \tag{3}$$

We consider two kinds of differential game problems stated as follows:

Problem 1. Determine a saddle point for

$$J(u,v)=(z(T),z(T))+\int_0^T\{(u(t),\ Cu(t))-(v(t),\ Dv(t))\}dt, \qquad (4)$$

subject to the constraints (1) and (2), where $(\cdot,\cdot)$ denotes the inner product in some finite-dimensional Euclidean space, C and D are $r\times r$ positive semi-definite diagonal matrices with nonnegative constant elements, and T is a fixed time. $J(u, v)$ is called a payoff. A saddle point is defined as the pair $u^o(t)$, $v^o(t)$ satisfying the relation

$$J(u^o,v)\le J(u^o,\ v^o)\le J(u,\ v^o) \qquad (5)$$

for arbitrary admissible strategies u and v. Namely, the first player is to select a strategy that minimizes the payoff and the second player is to select a strategy that maximizes the payoff. If (5) can be realized, u^o and v^o are called the optimal strategies and $J(u^o, v^o)$ is called the value of the game.

Problem 2. Let $T_{u,v}$ be a time corresponding to the strategies $u(t)\in U$, $v(t)\in V$ and satisfying $z(T_{u,v})=0$. Determine a time T_{u^o,v^o} and a pair of admissible strategies $u^o(t)$, $v^o(t)$ such that

$$z(T_{u^o,v^o})=0, \qquad T_{u^o,v^o}=\min_{u\in U}\max_{v\in V}T_{u,v}. \qquad (6)$$

The second equation of (6) is equivalent to

$$T_{u^o,v}\le T_{u^o,v^o}\le T_{u,v^o}.$$

Such differential game problems have been treated by several authors [1]-[7]. In this paper, we consider the problems as an infinite-dimensional nonlinear programming problem [8]. By using the generalized Kuhn-Tucker theorem [9] in nonlinear programming, we derive a system of transcendental equations, the solution of which directly yields the optimal strategies.

We can mention two kinds of games describable in the form (1). One is a pursuit-evasion game governed by

$$\left.\begin{aligned} dx_p/dt &= A_p(t)x_p + B_p(t)u(t) + h_p(t), \quad x_p(0)= x_{po}, \\ dx_e/dt &= A_e(t)x_e + B_e(t)v(t) + h_e(t), \quad x_e(0)= x_{eo}, \end{aligned}\right\} \qquad (7)$$

where x_p is an n-vector representing the state of the pursuer, $u(t)\in U$ is an r-vector representing the control of the pursuer, $A_p(t)$, $B_p(t)$, and $h_p(t)$ are n×n, n×r, and n×1 known matrices, respectively, continuous in t, and identical statements apply to the evader and x_e, $v(t)\in V$, $A_e(t)$, $B_e(t)$, and $h_e(t)$. x_{po} and x_{eo} are initial values. The state of the game is defined by

$$z(t)= Q\left\{x_p(t)- x_e(t)\right\}, \qquad (8)$$

where Q is an n×n positive semidefinite constant matrix. Then, the $z_1(t)$, $F(t,s)$, and $W(t,s)$ in (1) are given by

$$\left.\begin{aligned} z_1(t) &= Q\left\{ X_p(t)\left(x_{po}+ \int_0^t X_p^{-1}(s)h_p(s)ds \right) - X_e(t)\left(x_{eo}+ \int_0^t X_e^{-1}(s)h_e(s)ds \right)\right\}, \\ F(t, s) &= QX_p(t)X_p^{-1}(s)B_p(s), \\ W(t, s) &= -QX_e(t)X_e^{-1}(s)B_e(s), \end{aligned}\right\} \qquad (9)$$

where $X_p(t)$ and $X_e(t)$ are n×n matrix functions satisfying

$$\left.\begin{aligned} dX_p(t)/dt &= A_p(t)X_p(t), \quad X_p(0)= I \text{ (the identity)}, \\ dX_e(t)/dt &= A_e(t)X_e(t), \quad X_e(0)= I. \end{aligned}\right\}$$

The other game describable in the form (1) is a control system subject to unpredictable disturbances. The state of the control system is governed by

$$dz/dt = A(t)z+ B_1(t)u(t)+ B_2(t)v(t)+ h(t), \quad z(0)= z_o, \qquad (10)$$

where z(t) is an n-vector describing the state of the system, $u(t)\in U$ is an r-vector representing the control, $v(t)$ is an r-vector representing unpredictable disturbance functions, $A(t)$, $B_1(t)$, $B_2(t)$, and $h(t)$ are $n\times n$, $n\times r$, $n\times r$, and $n\times 1$ known matrices, respectively, which are continuous in t. The known information concerning $v(t)$ is only the fact that $v(t)\in V$. In this case, the $z_1(t)$, $F(t, s)$, and $W(t, s)$ in (1) are given by

$$\left.\begin{aligned} z_1(t) &= X(t)\left(z_0 + \int_0^t X^{-1}(s)h(s)ds\right), \\ F(t, s) &= X(t)X^{-1}(s)B_1(s), \\ W(t, s) &= X(t)X^{-1}(s)B_2(s), \end{aligned}\right\} \qquad (11)$$

where z_0 is the initial value of $z(t)$, and $X(t)$ is an $n\times n$ matrix function satisfying

$$dX(t)/dt = A(t)X(t), \quad X(0) = I.$$

2. Fundamental Theorem and its Application to the Problem

Since the solution of the Problem 2 is obtained from the solution of the Problem 1, we consider the Problem 1 first. Let H be a real Hilbert space of r-dimensional functions square integrable over $[0, T]$. Then, the strategies $u(t)$ and $v(t)$ $(0\le t\le T)$ can be taken in H. Define the inner product of two vectors u^1 and u^2 in the Hilbert space H by

$$[u^1, u^2] = \int_0^T u^{1*}(t)u^2(t)dt = \int_0^T (u^1(t), u^2(t))dt,$$

where * denotes the transpose of a vector or a matrix. We define linear operators P and R, respectively, by

$$\left.\begin{aligned} Pu &= \int_0^T F(T, s)u(s)ds, \\ Rv &= \int_0^T W(T, s)v(s)ds, \end{aligned}\right\} \qquad (12)$$

which map the Hilbert space H into the n-dimensional Euclidean space R^n. Then, from (1) it follows that

$$z_T = z_1 + Pu + Rv, \tag{13}$$

where $z_T = z(T)$ and $z_1 = z_1(T)$. The payoff (4) is rewritten as

$$J(u, v) = (z_T, z_T) + [u, Cu] - [v, Dv]. \tag{14}$$

By using (13), it follows that

$$\begin{aligned}(z_T, z_T) &= (Pu + Rv + z_1, Pu + Rv + z_1) \\ &= (Pu, Pu) + 2(Pu, Rv + z_1) + (Rv + z_1, Rv + z_1).\end{aligned} \tag{15}$$

Let P^* now be the adjoint operator of P, then P^* maps R^n into H and satisfies the relation

$$(x, Pu) = [P^*x, u],$$

where $x \in R^n$ and $u \in H$. From (12), it can be easily seen that [10]

$$P^*x = F^*(T, t)x, \tag{16}$$

where F^* denotes the transposed matrix of F. Then, (15) can be rewritten as

$$(z_T, z_T) = [u, P^*Pu] + 2[u, P^*(Rv + z_1)] + (Rv + z_1, Rv + z_1).$$

Therefore, the payoff (14) can be written as

$$\begin{aligned}J(u, v) &= [u, (P^*P + C)u] + 2[u, P^*(Rv + z_1)] + (Rv + z_1, Rv + z_1) - [v, Dv] \\ &= [v, (R^*R - D)v] + 2[v, R^*(Pu + z_1)] + (Pu + z_1, Pu + z_1) + [u, Cu].\end{aligned} \tag{17}$$

Since $[u, (P^*P + C)u] = (Pu, Pu) + [u, Cu] \geq 0$, it can be seen that $J(u, v)$ is convex with respect to $u \in H$. But the convexity or concavity of $J(u, v)$ with respect to $v \in H$ cannot be asserted. Define a mapping g, which maps H_1 ($H_1 \subset H$) into H, by

$$g(u) = \begin{pmatrix} 1 - u_1^2(t) \\ 1 - u_2^2(t) \\ \vdots \\ 1 - u_r^2(t) \end{pmatrix}. \tag{18}$$

Then, the constraints $u(t)$, $v(t) \in \Omega$ can be expressed as

$$g(u) \geq 0, \quad g(v) \geq 0 \ . \tag{19}$$

Define a closed bounded convex subset X of H by

$$X = \{u \in H : g(u) \geq 0\} \ .$$

Since $J(u, v)$ is continuous and convex with respect to u on X, from [11, Theorem 2.1] there exists an element u^o in X such that

$$\inf_{u \in X} J(u, v) = J(u^o, v) \ .$$

Concerning an element v^o in X which maximizes $J(u, v)$, the existence cannot be asserted. Therefore, we assume the existence of v^o. In the following, necessary conditions for the optimal strategies will be derived.

First, we show a theorem which corresponds to a special case of the Theorem V.3.3.2 in [9] and is available for our problem.

Theorem 1. : Let f be a real-valued differentiable functional on the Hilbert space H. Let $x^o \in X$ maximize $f(x)$ subject to the constraint $g(x) \geq 0$. Then, there exists a $\lambda^o \in H$ such that

$$\lambda^o \geq 0 \ , \tag{20}$$

and that the Lagrangian expression

$$\Phi(x, \lambda) = f(x) + [\lambda, g(x)] \tag{21}$$

satisfies the following relations:

$$\delta_x \Phi((x^o, \lambda^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{22}$$

$$[\lambda^o, g(x^o)] = 0 \ , \tag{23}$$

where $\delta_x \Phi((x^o, \lambda^o); \xi)$ represents the partial Fréchet differential of Φ with respect to x at (x^o, λ^o) with increment ξ, which is defined by

$$\delta_x \Phi((x^o, \lambda^o); \xi) = \lim_{\varepsilon \to 0} \frac{\Phi(x^o + \varepsilon\xi, \lambda^o) - \Phi(x^o, \lambda^o)}{\varepsilon} . \tag{24}$$

<u>Proof</u>. Since the function g defined by (18) is regular [9] at every point of X, it follows from [9, Theorem V.3.3.1] that the relation

$$\left.\begin{aligned} &\delta g(x^o; \xi) + g(x^o) \geq 0 \\ \text{implies} \quad & \\ &-\delta f(x^o; \xi) \geq 0 . \end{aligned}\right\} \tag{25}$$

Define a linear continuous transformation $A(\xi)$ on H into H by

$$A(\xi) = \delta g(x^o; \xi) , \tag{26}$$

and a linear functional $\phi(\xi)$ on H by

$$\phi(\xi) = -\delta f(x^o; \xi) . \tag{27}$$

Since A is bounded, it follows that a subset Y_A in H defined by

$$Y_A = \left\{ A^*(\lambda) \in H : \lambda \in H, \ \lambda \geq 0 \right\} \tag{28}$$

is regularly convex [9], where A^* is the adjoint transformation of A. Further,

$$A(\xi) + g(x^o) \geq 0 \quad \text{implies} \quad \phi(\xi) \geq 0 .$$

Therefore, from [9, Corollary IV. 3], there exists a $\lambda^o \geq 0$ ($\lambda^o \in H$) such that

$$[\lambda^o, A(\xi)] = \phi(\xi) \quad \text{for all} \quad \xi \in H .$$

Furthermore,

$$[\lambda^o, g(x^o)] = 0 .$$

Thus, the Theorem 1 is proved.

Now, we apply Theorem 1 to our problem. Let $u^o \in X$ minimize J(u, v) subject to the constraint $g(u) \geq 0$, then there exists a $\lambda^o \in H$ such that

$$\lambda^o \geq 0 , \tag{29}$$

and that the Lagrangian expression

$$K(u, v, \lambda) = J(u, v) - [\lambda, g(u)] \tag{30}$$

satisfies the following relations:

$$\delta_u K((u^o, v, \lambda^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{31}$$

$$[\lambda^o, g(u^o)] = 0 . \tag{32}$$

Analogously, let $v^o \in X$ maximize $J(u, v)$ subject to the constraint $g(v) \geq 0$, then there exists a $\mu^o \in H$ such that

$$\mu^o \geq 0 , \tag{33}$$

and that the Lagrangian expression

$$L(u, v, \mu) = J(u, v) + [\mu, g(v)] \tag{34}$$

satisfies the following relations:

$$\delta_v L((u, v^o, \mu^o); \xi) = 0 \quad \text{for all} \quad \xi \in H, \tag{35}$$

$$[\mu^o, g(v^o)] = 0 . \tag{36}$$

Henceforth, u^o, v^o, λ^o, and μ^o which satisfy these relations are simply written as u, v, λ, and μ, since no confusion may occur. In view of (17) and (24), the partial Fréchet differential of $K(u, v, \lambda)$ with respect to u at (u, v, λ) with increment ξ can be evaluated as

$$\delta_u K((u, v, \lambda); \xi) = 2\left[(P^*P + C)u, \xi\right] + 2\left[P^*(Rv + z_1), \xi\right] - \left[\frac{\partial g(u)}{\partial u}\lambda, \xi\right] ,$$

where $\partial g(u)/\partial u$ denotes an $r \times r$ matrix defined by

$$\frac{\partial g(u)}{\partial u} = \left[\frac{\partial g_i(u)}{\partial u_j}\right] = -2 \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & u_r \end{bmatrix} .$$

Hence, from (31) it follows that

$$(P^*P+C)u+P^*Rv+P^*z_1-\frac{1}{2}\frac{\partial g(u)}{\partial u}\lambda=0 . \tag{37}$$

Define a new vector x in H by

$$x=Cu-\frac{1}{2}\frac{\partial g(u)}{\partial u}\lambda=\begin{pmatrix} c_1u_1+\lambda_1u_1 \\ \vdots \\ c_ru_r+\lambda_ru_r \end{pmatrix} , \tag{38}$$

where c_i $(i=1, \cdots ,r)$ are the elements of the diagonal matrix C and non-negative. Then, (37) is rewritten as

$$P^*(Pu+Rv+z_1)+x=0 . \tag{39}$$

In the same way, since

$$\delta_v L((u, v, \mu); \xi)=2\left[(R^*R-D)v, \xi\right]+2\left[R^*(Pu+z_1), \xi\right]+\left[\frac{\partial g(v)}{\partial v}\mu, \xi\right],$$

it follows from (35) that

$$(R^*R-D)v+R^*Pu+R^*z_1+\frac{1}{2}\frac{\partial g(v)}{\partial v}\mu=0 . \tag{40}$$

Define a new vector y in H by

$$y=Dv-\frac{1}{2}\frac{\partial g(v)}{\partial v}\mu=\begin{pmatrix} d_1v_1+\mu_1v_1 \\ \vdots \\ d_rv_r+\mu_rv_r \end{pmatrix} , \tag{41}$$

where d_i $(i=1,\cdots,r)$ are the elements of the diagonal matrix D and nonnegative. Then, (40) is rewritten as

$$R^*(Rv+Pu+z_1)-y=0 . \tag{42}$$

Since $\lambda(t)\geq 0$ and $g(u(t))\geq 0$ on the interval $[0, T]$, it follows from (32) that the equation

$$(\lambda(t),\ g(u(t))) = \lambda^*(t)g(u(t)) = 0$$

holds for almost all $t \in [0, T]$. Therefore,

$$\left.\begin{array}{lll} \lambda_i(t) = 0 & \text{if} & -1 < u_i(t) < 1\ , \\ \lambda_i(t) \geq 0 & \text{if} & |u_i(t)| = 1\ . \end{array}\right\}$$

Thus, the relation between $u_i(t)$ and $\lambda_i(t)$ can be shown as Fig. 1 (a). The relation between $u_i(t)$ and $\lambda_i(t)u_i(t)$ and then the relation between $u_i(t)$ and $c_i(t)u_i(t) + \lambda_i(t)u_i(t)$ can also be obtained successively from Fig. 1 (a) as shown in Fig. 1 (b) and (c), respectively. Hence, the relation between $x \in H$ and $u \in X$, which is defined by (38), can be expressed as

$$u(t) = \varphi(x(t)),\quad \text{or} \quad u_i(t) = \varphi_i(x_i(t)) \quad (i = 1, \cdots, r), \tag{43}$$

where the nonlinear function φ_i is shown in Fig. 2 (a), which can be obtained from Fig. 1 (c) directly.

Analogously, the relation between $y \in H$ and $v \in X$, which is defined by (41), can be expressed as

$$v(t) = \psi(y(t)),\quad \text{or} \quad u_i(t) = \psi_i(y_i(t)) \quad (i = 1, \cdots, r), \tag{44}$$

where the nonlinear function ψ_i is shown in Fig. 2 (b). If we use a notation such that

$$\left.\begin{array}{lll} \text{sat}\,\alpha = \alpha & \text{if} & |\alpha| \leq 1, \\ \text{sat}\,\alpha = \text{sgn}\,\alpha & \text{if} & |\alpha| \geq 1, \end{array}\right\}$$

then,

$$\varphi(x) = \begin{bmatrix} \text{sat}(x_1/\ c_1) \\ \vdots \\ \text{sat}(x_r/\ c_r) \end{bmatrix}, \qquad \psi(y) = \begin{bmatrix} \text{sat}(y_1/\ d_1) \\ \vdots \\ \text{sat}(y_r/\ d_r) \end{bmatrix}.$$

Substituting (43) and (44) into (39) and (42) yields

$$\left.\begin{aligned} &P^*(P\varphi(x)+R\psi(y)+z_1)+x=0, \\ &R^*(P\varphi(x)+R\psi(y)+z_1)-y=0. \end{aligned}\right\} \tag{45}$$

Equation (45) is a system of nonlinear integral equations from which $x\in H$ and $y\in H$ can be determined. This system of nonlinear integral equations can be reduced to a system of transcendental equations as follows.

By setting

$$P\varphi(x)+R\psi(y)+z_1=\alpha\ , \tag{46}$$

we obtain

$$x=-P^*\alpha\quad,\qquad y=R^*\alpha\quad, \tag{47}$$

where α is a vector in R^n. In view of (13), it is clear that the α defined by (46) represents $z_T=z(T)$. Substituting (47) into (46) yields a transcendental equation:

$$-P\varphi(P^*\alpha)+R\psi(R^*\alpha)+z_1=\alpha\ . \tag{48}$$

Let $f_i(T,\ t)$ and $w_i(T,\ t)$ $(i=1,\cdots,r)$ be n-dimensional column vectors of the matrices $F(T,t)$ and $W(T,\ t)$, respectively, i.e.,

$$\left.\begin{aligned} F(T,\ t)&=\begin{bmatrix} f_1(T,\ t) & f_2(T,\ t) & \cdots & f_r(T,\ t)\end{bmatrix}, \\ W(T,\ t)&=\begin{bmatrix} w_1(T,\ t) & w_2(T,\ t) & \cdots & w_r(T,\ t)\end{bmatrix}. \end{aligned}\right\}$$

Then, since

$$P^*\alpha=F^*(T,\ t)\alpha=\begin{bmatrix} f_1{}^*(T,\ t)\alpha \\ \vdots \\ f_r{}^*(T,\ t)\alpha\end{bmatrix},\quad R^*\alpha=W^*(T,\ t)\alpha=\begin{bmatrix} w_1{}^*(T,\ t)\alpha \\ \vdots \\ w_r{}^*(T,\ t)\alpha\end{bmatrix},$$

(48) can be rewritten as

$$\alpha = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \psi_i\big(w_i^*(T,t)\alpha\big) - f_i(T,t)\, \varphi_i\big(f_i^*(T,t)\alpha\big) \Big] dt + z_1 . \tag{49}$$

3. Solution of the Transcendental Equation

Let us define a mapping A by

$$A\alpha = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \psi_i\big(w_i^*(T,t)\alpha\big) - f_i(T,t)\, \varphi_i\big(f_i^*(T,t)\alpha\big) \Big] dt + z_1 , \tag{50}$$

which maps R^n into R^n. Let $(A\alpha)_j$, f_{ij}, w_{ij}, and z_{1j} denote the jth component of n-vectors $A\alpha$, f_i, w_i, and z_1, respectively. Then,

$$(A\alpha)_j = \sum_{i=1}^{r} \int_0^T \Big[w_{ij}(T,t)\, \psi_i\big(w_i^*(T,t)\alpha\big) - f_{ij}(T,t)\, \varphi_i\big(f_i^*(T,t)\alpha\big) \Big] dt + z_{1j} .$$

Since $|\varphi_i| \le 1$, $|\psi_i| \le 1$ $(i = 1, \cdots, r)$, it follows that

$$\left. \begin{aligned} |(A\alpha)_j| \le \sum_{i=1}^{r} \int_0^T \Big\{ |w_{ij}(T,t)| + |f_{ij}(T,t)| \Big\} dt + |z_{1j}| \\ (j = 1, \cdots, n) . \end{aligned} \right\} \tag{51}$$

Since the functions f_{ij}, w_{ij} $(i = 1, \cdots, r;\ j = 1, \cdots, n)$ are assumed to be continuous on the closed interval $0 \le t \le T$, (51) shows that the mapping A maps a closed bounded convex subset of R^n into itself. Furthermore, the mapping is continuous. Therefore, by Brouwer's fixed-point theorem [12], we can conclude that there exists a point α such that $A\alpha = \alpha$. Namely, Theorem 2. Let the vector functions f_i and w_i $(i=1,\cdots,r)$ be continuous on the closed interval $[0, T]$. Then, there exists a solution of (49).

The solution of (49) may be computed by the method of successive approximations:

$$\alpha_k = A\alpha_{k-1} \qquad (k=1,2,\cdots). \tag{52}$$

As to the convergence of the successive approximations (52), we can propose:

Theorem 3. Let the nonnegative constants c_i, d_i $(i=1,\cdots,r)$ be all positive. Further, let us assume that

$$\sum_{i=1}^{r}\int_0^T\left[\frac{1}{c}\|f_i(T,t)\|^2+\frac{1}{d}\|w_i(T,t)\|^2\right]dt<1, \tag{53}$$

where $c=\min(c_1, c_2,\cdots, c_r)$, $d=\min(d_1, d_2,\cdots, d_r)$, and $\| \ \|$ denotes the Euclidean norm in R^n. Then, the successive approximations (52), starting with an arbitrary α_0, converge to a unique solution of (49).

Proof. Let α and β be arbitrary points in R^n. By using the Schwarz inequality and the relations:

$$\left.\begin{aligned}
|\varphi_i(f_i^*(T,t)\alpha)-\varphi_i(f_i^*(T,t)\beta)| &\le \frac{1}{c}|f_i^*(T,t)(\alpha-\beta)| \\
&\le \frac{1}{c}\|f_i(T,t)\|\,\|\alpha-\beta\|, \\
|\psi_i(w_i^*(T,t)\alpha)-\psi_i(w_i^*(T,t)\beta)| &\le \frac{1}{d}\|w_i(T,t)\|\,\|\alpha-\beta\|, \\
(i=1,\cdots,r)&
\end{aligned}\right\} \tag{54}$$

it follows that

$$|(A\alpha)_j-(A\beta)_j| \le \|\alpha-\beta\|\sum_{i=1}^{r}\int_0^T\Big[\frac{1}{c}|f_{ij}(T,t)|\,\|f_i(T,t)\| + \frac{1}{d}|w_{ij}(T,t)|\,\|w_i(T,t)\|\Big]dt$$

$$\leq \|\alpha - \beta\| \left[\frac{1}{c} \left\{ \int_0^T \sum_{i=1}^r f_{ij}^2 (T, t) dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^r \| f_i (T,t)\|^2 dt \right\}^{\frac{1}{2}} + \frac{1}{d} \left\{ \int_0^T \sum_{i=1}^r w_{ij}^2 (T, t) dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^r \| w_i (T, t)\|^2 dt \right\}^{\frac{1}{2}}.$$

Hence,

$$\|A\alpha - A\beta\| \leq \left[\frac{1}{c} \int_0^T \sum_{i=1}^r \| f_i (T, t)\|^2 dt + \frac{1}{d} \int_0^T \sum_{i=1}^r \| w_i (T, t)\|^2 dt \right] \|\alpha - \beta\| . \tag{55}$$

Inequality (55) shows that the mapping A defined by (50) is a contraction mapping under the condition (53) [13]. Thus, the theorem is proved.

4. Solution to the Problem 2

If $C = D = 0$ in (4), the payoff becomes

$$\hat{J}(u, v) = (z(T), z(T)) . \tag{56}$$

In this case, equation (49) which determines the vector α becomes

$$\alpha = \sum_{i=1}^r \int_0^T \left[w_i (T, t) \, sgn \left(w_i^* (T, t) \alpha \right) - f_i (T, t) sgn \left(f_i^* (T, t) \alpha \right) \right] dt + z_1 . \tag{57}$$

Let us introduce a real number ε and a vector β in R^n such that

$$\alpha = \varepsilon \beta , \quad \varepsilon > 0 , \quad \|\beta\| = 1 .$$

Then, (57) can be rewritten as

$$\varepsilon \beta = \sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \mathrm{sgn}\big(w_i^*(T,t)\beta\big) - f_i(T,t)\, \mathrm{sgn}\big(f_i^*(T,t)\beta\big)\Big] dt + z_1 . \qquad (58)$$

As mentioned before, the vector α defined by (46) represents $z_T = z(T)$. Hence, if there exists a time T such that the value $\hat{J}(u^o, v^o)$ of the game vanishes, then there exists a solution of the Problem 2. Letting $\varepsilon \to 0$ in (58) yields a equation

$$\sum_{i=1}^{r} \int_0^T \Big[w_i(T,t)\, \mathrm{sgn}\big(w_i^*(T,t)\beta\big) - f_i(T,t)\, \mathrm{sgn}\big(f_i^*(T,t)\beta\big)\Big] dt + z_1 = 0 . \qquad (59)$$

The time T and the n-vector β $(\|\beta\| = 1)$ which satisfy (59) give a solution to the Problem 2, i.e.,

$$T_{u^o, v^o} = T . \qquad (60)$$

Furthermore, from (43), (44), and (47) it follows that

$$u^o(t) = -\begin{bmatrix} \mathrm{sgn}\,(f_1^*(T,t)\beta) \\ \vdots \\ \mathrm{sgn}\,(f_r^*(T,t)\beta) \end{bmatrix}, \quad v^o(t) = \begin{bmatrix} \mathrm{sgn}\,(w_1^*(T,t)\beta) \\ \vdots \\ \mathrm{sgn}\,(w_r^*(T,t)\beta) \end{bmatrix}. \qquad (61)$$

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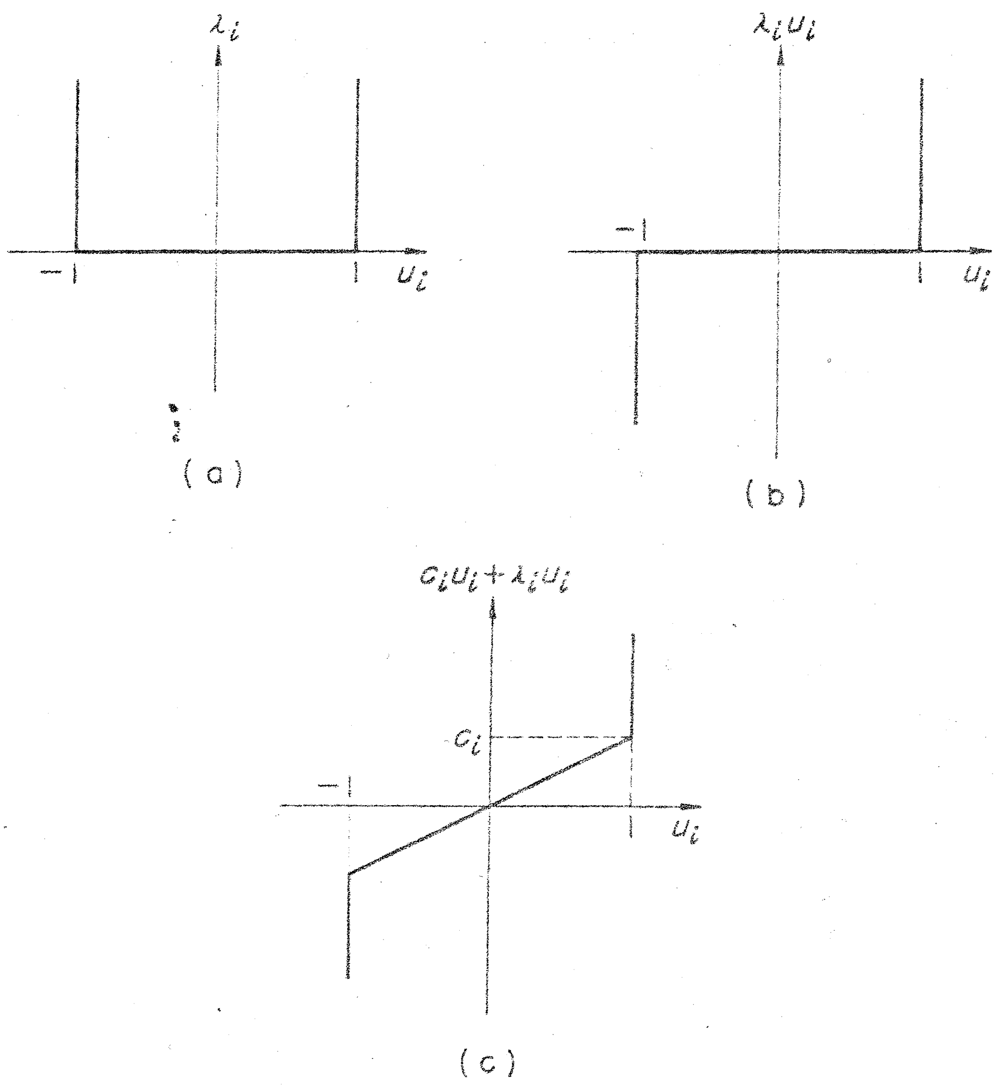

Fig. 1 Relations between the variables.

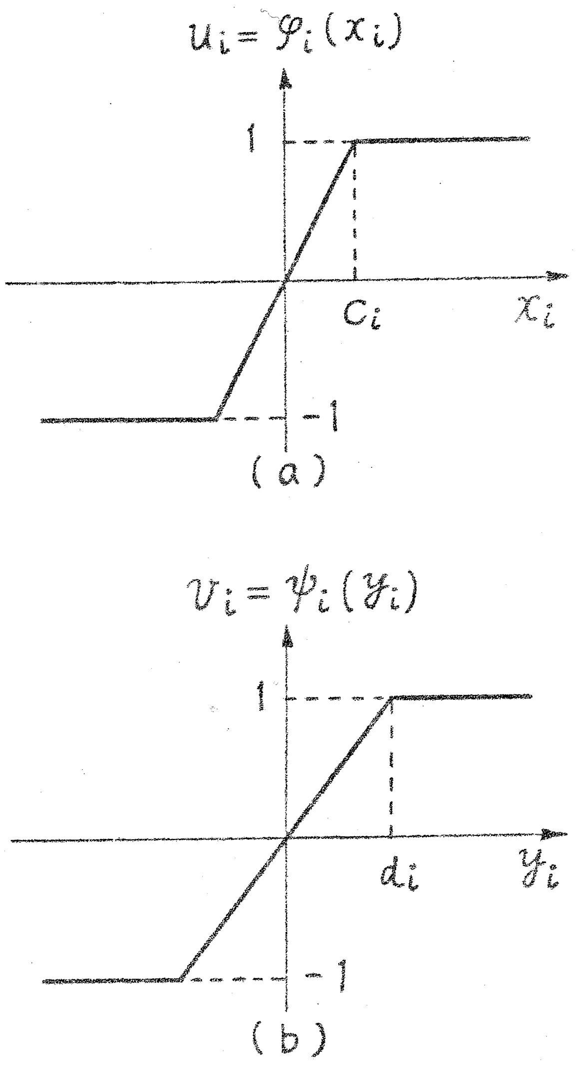

Fig. 2 Nonlinear characteristics.