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INTRODUCTION TO A POTENTIAL THEORY ON A DIFFERENTIABLE MANIFOLD

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Consider a uniformly elliptic differential equation

$$(1) \qquad \sum_i \frac{\partial}{\partial x_i}\left(\sum_j a_{ij}\frac{\partial u}{\partial x_j}\right) = 0$$

with not necessarily continuous coefficients a_{ij}. Properties of solutions of such an equation were studied by many authors; cf. [11], [12], [13], [8] and [14]. Using them, R.-M. Hervé [6], [7] developed a potential theory with respect to this type of equation on a bounded domain in the Euclidean space. It was shown that almost all basic results in the classical potential theory are generalized to this case.

In this paper, we show how to extend Hervé's results to a potential theory on a differentiable manifold with respect to a differential equation which is locally of the form (1). Since the classical approach is no longer valid in this case, we employ a different approach, which is essentially due to Hervé (and to G. Stampacchia; cf. [8], [14]). In the last part (sections 10 and 11) we also give fundamental results which are necessary in the discussion on boundary value problems with respect to an ideal boundary (cf. [9], [10]).

§1

1. Metric tensor.

We consider a connected non-compact C^1-manifold Ω of dimension $d \geqq 2$ and a symmetric covariant tensor (g_{ij}) on Ω which satisfies

(G): On each relatively compact coordinate neighborhood U, each g_{ij} is a bounded measurable function on U and there exists $\lambda > 0$ such that

$$\lambda \sum \xi_i{}^2 \leqq \sum g_{ij}(x)\xi_i\xi_j$$

for all $x \in U$ and real numbers $\xi_1,\ldots,\xi_d$.

Let G be the determinant of (g_{ij}). $dV = \sqrt{G}\, dx_1\ldots dx_d$ defines a measure on U. Thus dV is defined to be a positive measure on Ω. Let $L^2(dV)$ (resp. $L^2_{loc}(dV)$) be the space of square summable (resp. locally square summable) functions on Ω with respect to dV.

The space Ω with such a metric tensor (g_{ij}) is a locally compact metrizable space, and hence it is countable at infinity.

We denote by $C^1(\Omega)$ the space of continuously differentiable functions on Ω and by $C^1_o(\Omega)$ the subspace consisting of functions with compact support.

2. The spaces $\mathcal{L}(\Omega)$ and $\mathcal{L}_o(\Omega)$.

Given $f \in C^1(\Omega)$,

$$D[f] = \int_\Omega \sum g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}\, dV$$

is well-defined, where (g^{ij}) is the inverse matrix of (g_{ij}). Let $C_D^1(\Omega) = \{f \in C^1(\Omega);\ D[f] < \infty\}$. Obviously $C_0^1(\Omega) \subset C_D^1(\Omega)$.

Let U_0 be a fixed relatively compact coordinate neighborhood in Ω and let

$$\|f\|_0{}^2 = \int_{U_0} f^2\, dV$$

for $f \in L^2_{loc}(dV)$. For $f \in C_D^1(\Omega)$, we define a norm

$$\|f\| = D[f]^{1/2} + \|f\|_0.$$

LEMMA 1. (cf. [4]) If $f_n \in C_D^1(\Omega)$ and $D[f_n] \to 0$, then there exist constants c_n such that $\|f_n + c_n\|_0 \to 0$.

LEMMA 2. If U_1 is another relatively compact coordinate neighborhood, then $\|f\|$ and $D[f]^{1/2} + (\int_{U_1} f^2 dV)^{1/2}$ are equivalent norms on $C_D^1(\Omega)$.

DEFINITION.

$\mathcal{D}(\Omega)$ = the completion of $C_D^1(\Omega)$ with respect to $\|f\|$.

$\mathcal{D}_0(\Omega)$ = the closure of $C_0^1(\Omega)$ in $\mathcal{D}(\Omega)$.

For any $f \in \mathcal{D}(\Omega)$, $D[f]$ and $\|f\|$ are well-defined; in fact we see:

(i) f is identified with a function in $L^2_{loc}(dV)$.

(ii) There corresponds a covariant vector grad f $= \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right)$ such that $|\text{grad } f| = \left(\sum g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}\right)^{1/2}$ is a function in $L^2(dV)$ and $D[f] = \int_\Omega |\text{grad } f|^2 dV$.

By (i), we may regard as $\mathcal{D}(\Omega) \subset L^2_{loc}(dV)$. Thus $\mathcal{D}(\Omega)$

consists of elements f in $L^2_{loc}(dV)$ for each of which there exist $\phi_n \in C^1_D(\Omega)$ such that $\phi_n \to f$ in $L^2_{loc}(dV)$ and $D[\phi_n - \phi_m] \to 0$ $(n, m \to \infty)$.

Also for any $f, g \in \mathcal{D}(\Omega)$,

$$D[f, g] = \int_\Omega \sum g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} dV$$

is well-defined. $\mathcal{D}(\Omega)$ is a Hilbert space with respect to the inner product $D[f, g] + \int_{U_o} fg\, dV$.

By Lemma 1, we have

LEMMA 3. If $1 \notin \mathcal{D}_o(\Omega)$, then $\|f\|$ is equivalent to $D[f]^{1/2}$ on $\mathcal{D}_o(\Omega)$; in fact there exists $M_o > 0$ such that $\int_{U_o} f^2 dv \leqq M_o D[f]$ for all $f \in \mathcal{D}_o(\Omega)$.

3. The space $\mathcal{D}_{loc}(\Omega)$.

If ω is a subdomain of Ω, then it is a C^1-manifold and we may restrict (g_{ij}) to ω. Then we have the spaces $\mathcal{D}(\omega)$ and $\mathcal{D}_o(\omega)$ relative to ω.

DEFINITION.

$$\mathcal{D}_{loc}(\Omega) = \left\{ f \in L^2_{loc}(dV);\ \begin{array}{l} \text{for any relatively compact subdomain} \\ \omega, \text{ there exists } f_\omega \in \mathcal{D}(\omega) \text{ such that} \\ f_\omega = f \text{ (a.e.) on } \omega \end{array} \right\}.$$

It is easy to see that $\mathcal{D}(\Omega) \subset \mathcal{D}_{loc}(\Omega)$; for any $f \in \mathcal{D}(\Omega)$, its restriction f_ω to ω is well-defined and $f_\omega \in \mathcal{D}(\omega)$.

The following results are proved as in the classical case:

LEMMA 4. If $f \in \mathcal{D}_{loc}(\Omega)$ and $\phi \in C^1_o(\Omega)$, then $\phi f \in \mathcal{D}_o(\Omega)$.

COROLLARY. If $f \in \mathcal{D}_{loc}(\Omega)$ and $f = 0$ (a.e.) outside a compact set in Ω, then $f \in \mathcal{D}_0(\Omega)$.

LEMMA 5. If $f \in \mathcal{D}_{loc}(\Omega)$ and $|\text{grad } f| \in L^2(\Omega)$, then $f \in \mathcal{D}(\Omega)$.

4. The lattice structure of $\mathcal{D}(\Omega)$ and $\mathcal{D}_0(\Omega)$.

PROPOSITION 1. (cf. [2], [4]) If $f, g \in \mathcal{D}(\Omega)$, then $\max(f, g), \min(f, g) \in \mathcal{D}(\Omega)$ and $D[\max(f, g)] + D[\min(f, g)] = D[f] + D[g]$.

PROPOSITION 2. For $f \in \mathcal{D}(\Omega)$, let $f_n = \max(\min(f, n), -n)$. Then $\|f - f_n\| \to 0$ $(n \to \infty)$.

LEMMA 6. (cf. [6]) If $f \in \mathcal{D}(\Omega)$ and $f \geqq 0$ (a.e.) on Ω and if $g \in \mathcal{D}_0(\Omega)$, then $\min(f, g) \in \mathcal{D}_0(\Omega)$.

Sketch of the proof: Choose $\phi_n \in C_0^1(\Omega)$ such that $\|\phi_n - g\| \to 0$ $(n \to \infty)$. Then $\min(f, \phi_n) \in \mathcal{D}_0(\Omega)$ by the corollary to Lemma 4. We can show that $\|\min(f, \phi_n) - \min(f, g)\| \to 0$ (cf. [6], [2]). Hence $\min(f, g) \in \mathcal{D}_0(\Omega)$.

COROLLARY 1. For $f \in \mathcal{D}(\Omega)$, $f \in \mathcal{D}_0(\Omega)$ if and only if $|f| \in \mathcal{D}_0(\Omega)$.

COROLLARY 2. If $f \in \mathcal{D}(\Omega)$ and $f \geqq 0$ (a.e.) on Ω and if $f \leqq g$ (a.e.) outside a compact set in Ω for some $g \in \mathcal{D}_0(\Omega)$, then $f \in \mathcal{D}_0(\Omega)$.

LEMMA 7. If $g \in \mathcal{D}_0(\Omega)$ and $g \geqq 0$ (a.e.) on Ω, then there exist $\phi_n \in C_0^1(\Omega)$ such that $\phi_n \geqq 0$ on Ω and $\|\phi_n - g\| \to 0$ $(n \to \infty)$.

5. Solutions and supersolutions of $\Delta u = 0$.

DEFINITION. (i) $u \in \mathcal{D}_{loc}(\Omega)$ is called a solution of $\Delta u = 0$ on Ω if $D[u, \phi] = 0$ for all $\phi \in C_0^1(\Omega)$.

(ii) $u \in \mathcal{D}_{loc}(\Omega)$ is called a supersolution of $\Delta u = 0$ on Ω if $D[u, \phi] \geqq 0$ for all $\phi \in C_0^1(\Omega)$ such that $\phi \geqq 0$.

By Lemma 7, we see

LEMMA 8. If u is a supersolution on Ω and $u \in \mathcal{D}(\Omega)$, then $D[u, g] \geqq 0$ for all $g \in \mathcal{D}_0(\Omega)$ such that $g \geqq 0$ (a.e.) in Ω.

LEMMA 9. Suppose $1 \notin \mathcal{D}_0(\Omega)$. If $u \in \mathcal{D}(\Omega)$ is a supersolution on Ω and if $u \geqq g$ (a.e.) outside a compact set in Ω for some $g \in \mathcal{D}_0(\Omega)$, then $u \geqq 0$ (a.e.) on Ω.

Proof: Since $u^- \leqq g^-$ outside a compact set in Ω, $u^- \in \mathcal{D}_0(\Omega)$ by Corollary 2 to Lemma 6. Hence, by Lemma 8, $D[u, u^-] \geqq 0$. Therefore $0 \leqq D[u^-, u^-] \leqq D[u^+, u^-] = 0$, which implies $u^- = 0$ (a.e.) on Ω.

DEFINITION. $\mathcal{H}(\Omega) = \{u \in \mathcal{D}(\Omega);\ u$ is a solution of $\Delta u = 0$ on $\Omega\}$.

PROPOSITION 3. (Royden decomposition) (i) For any $u \in \mathcal{H}(\Omega)$ and $g \in \mathcal{D}_0(\Omega)$, $D[u, g] = 0$.

(ii) Any $f \in \mathcal{D}(\Omega)$ has a decomposition $f = u + g$ with $u \in \mathcal{H}(\Omega)$ and $g \in \mathcal{D}_0(\Omega)$. This decomposition is unique if $1 \notin \mathcal{D}_0(\Omega)$.

Hereafter we shall assume that $1 \notin \mathcal{D}_0(\Omega)$. The function $u \in \mathcal{H}(\Omega)$ determined by f in the above proposition is denoted by $h_f \equiv h_f^{\Omega}$.

PROPOSITION 4. The mapping $f \to h_f$ is continuous linear and non-negative on $\mathcal{D}(\Omega)$.

Proof: The linearity is obvious. If $f \geqq 0$, then $h_f \geqq -g$. Hence $h_f \geqq 0$ by Lemma 9. Obvionsly, $D[h_f] \leqq D[f]$ and $D[g] \leqq D[f]$. By Lemma 3, $\|g\|_o^2 \leqq M_o D[g] \leqq M_o D[f]$. Hence $\|h_f\|_o \leqq \sqrt{M_o} D[f] + \|f\|_o$, so that $\|h_f\| \leqq \sqrt{1+M_o}\|f\|$. Thus the mapping is continuous.

Now we shall extend Lemma 9 to supersolutions which do not necessarily belong to $\mathcal{D}(\Omega)$. We first prove the following two lemmas which are due to Hervé [6]:

LEMMA 10. Let $f \in \mathcal{D}(\Omega)$ and let ω be a subdomain of Ω. Then $f_\omega = \begin{cases} h_f^\omega & \text{on } \omega \\ f & \text{on } \Omega - \omega \end{cases}$ belongs to $\mathcal{D}(\Omega)$. Furthermore there exists $M_1 > 0$ independent of ω such that

$$\|f_\omega\| \leqq M_1 \|f\|.$$

Proof: It is easy to see that $f_\omega - f \in \mathcal{D}_o(\Omega)$. Hence $f \in \mathcal{D}(\Omega)$. By Lemma 3, $\|f_\omega - f\|_o^2 \leqq M_o D[f_\omega - f]$. Hence

$$\|f_\omega - f\|^2 \leqq (1+M_o)D[f_\omega - f] = (1+M_o)D_\omega[h_f^\omega - f]$$

$$\leqq (1+M_o)D_\omega[f] \leqq (1+M_o)D[f].$$

Therefore,

$$\|f_\omega\| \leqq (1 + \sqrt{1+M_o})\|f\|.$$

LEMMA 11. Let $f \in \mathcal{D}(\Omega)$ and let $\{\omega_n\}$ be an exhaustion of

Ω. Then f_{ω_n} tends to h_f weakly in $\mathcal{D}(\Omega)$.

Proof: By the above lemma, $\{f_{\omega_n}\}$ is bounded, so that it is weakly relatively compact. Let u be any weak limit of $\{f_{\omega_n}\}$. For any $\phi \varepsilon C_0^1(\Omega)$, $D[u, \phi] = \lim_{n\to\infty} D[f_{\omega_n}, \phi] = 0$. Hence $u \in \mathcal{H}(\Omega)$. Since $f_{\omega_n} - f \varepsilon \mathcal{D}_0(\Omega)$, $u - f \varepsilon \mathcal{D}_0(\Omega)$. Thus, $u = h_f$ by Proposition 3. Hence we have the lemma.

Now we prove

PROPOSITION 5. (cf. [6]) If u is a supersolution on Ω and if there exists $g \varepsilon \mathcal{D}_0(\Omega)$ such that $u \geqq g$ (a.e.) outside a compact set in Ω, then $u \geqq 0$ (a.e.) on Ω.

Proof: Choose an exhaustion $\{\omega_n\}$ of Ω in such a way that $u \geqq g$ (a.e.) outside a compact set in ω_n for each n. Since $u - h_g^{\omega_n} \varepsilon \mathcal{D}(\omega_n)$ and $g - h_g^{\omega_n} \varepsilon \mathcal{D}_0(\omega_n)$, Lemma 9 implies that $u - h_g^{\omega_n} \geqq 0$ on ω_n. By Lemma 11, $g_{\omega_n} \to 0$ weakly in $\mathcal{D}(\Omega)$, since $g \varepsilon \mathcal{D}_0(\Omega)$. Hence, for any $\phi \varepsilon C_0^1(\Omega)$ such that $\phi \geqq 0$, we have

$$0 = \lim_{n\to\infty} \int g_{\omega_n} \phi \, dV \leqq \int u\phi \, dV.$$

Hence $u \geqq 0$ (a.e.) on Ω.

COROLLARY. If u is a supersolution on Ω and if $\underline{\lim}_{x\to id(\Omega)} u(x) \geqq 0$, then $u \geqq 0$ (a.e.) on Ω.

6. The harmonic structure.

PROPOSITION 6. (See [11], [12], [13]) Any solution of $\Delta u = 0$ on a coordinate neighborhood is Hölder continuous.

PROPOSITION 7. (See [13]) If u is a non-negative solution of $\Delta u = 0$ on a coordinate neighborhood U, then for any compact set K in U

$$\sup_K u \leqq c \inf_K u$$

with c depending only on U and K.

From this proposition we obtain

PROPOSITION 8. If u is a non-negative solution of $\Delta u = 0$ on a domain ω, then either $u \equiv 0$ on ω or $u > 0$ everywhere on ω.

DEFINITION. For any domain ω of Ω, let

$$H(\omega) = \{u;\ \text{continuous solution of } \Delta u = 0 \text{ on } \omega\}$$

and for any open set ω with decomposition $\omega = \bigcup \omega_i$ into components, let

$$H(\omega) = \{u;\ u|\omega_i \in H(\omega_i) \quad \text{for all } i\}.$$

THEOREM 1. (cf. [6]) $\{H(\omega)\}_{\omega:\text{open}}$ gives a harmonic structure satisfying the axioms 1, 2 and 3'(= 3) of M. Brelot ([1]).

Axiom 1: $\{H(\omega)\}$ is a sheaf of linear spaces of continuous functions (Definition and Proposition 5).

Axiom 2: Regular domains form a base of open sets; in fact any ball in a coordinate neighborhood is regular (cf. [8], [11],

[14]).

Axiom 3': The statement of Proposition 7 is true and $\{u \in H(\omega);\ u \geqq 0,\ u(x_o) = 1\}$ is equi-continuous at $x = x_o$ for any domain ω and $x_o \in \omega$ (shown by Hervé [6] (using Proposition 6) in case ω is a coordinate neighborhood).

Thus functions in $H(\omega)$ are called harmonic on ω and we have notions of superharmonic functions and potentials with respect to this harmonic structure (see [1]).

7. Superharmonic functions belonging to $\mathcal{D}_{loc}(\Omega)$.

PROPOSITION 9. (Hervé [7]) If u is superharmonic on Ω and if $u \in \mathcal{D}_{loc}(\Omega)$, then u is a supersolution of $\Delta u = 0$ on Ω. Conversely, for any supersolution v on Ω, there exists a unique superharmonic function u on Ω such that $u = v$ a.e. on Ω.

LEMMA 12. If u is superharmonic on Ω and if $u \in \mathcal{D}_o(\Omega)$, then u is a potential.

Proof: By Proposition 9, u is a supersolution. Hence by Lemma 9 $u \geqq 0$ on Ω. Let h be the greatest harmonic minorant of u. Then $0 \leqq h \leqq u$. By Proposition 5, we conclude that $-h \geqq 0$. Hence $h = 0$, so that u is a potential.

Now the following theorem is an immediate consequence of this lemma:

THEOREM 2. If u is superharmonic on Ω and $u \in \mathcal{D}(\Omega)$, then u has a harmonic minorant in Ω and h_u is the greatest harmonic minorant of u.

COROLLARY. If $u_1, u_2 \in \mathcal{H}(\Omega)$, then the least harmonic majorant v of $\max(u_1, u_2)$ and the greatest harmonic minorant w of $\min(u_1, u_2)$ both belong to $\mathcal{H}(\Omega)$ and

$$D[v] + D[w] \leqq D[u_1] + D[u_2].$$

(Cf. Proposition 1)

8. Measures associated with supersolutions.

Let u be a supersolution on Ω. Then there exists a non-negative Radon measure μ on Ω such that

$$D[u, \phi] = \int \phi \, d\mu$$

for all $\phi \in C_0^1(\Omega)$. We call μ the measure associated with u.

LEMMA 13. Given $\psi \in C_0^1(\Omega)$, $\psi \geqq 0$, there exists a continuous potential g_ψ belonging to $\mathcal{D}_0(\Omega)$ such that ψdV is its associated measure.

Proof: Since the linear functional $g \to \int g\psi \, dV$ is continuous on $\mathcal{D}_0(\Omega)$ (Lemma 3), there exists $g_\psi \in \mathcal{D}_0(\Omega)$ such that $D[g_\psi, g] = \int g\psi \, dV$ for all $g \in \mathcal{D}_0(\Omega)$. If $g \geqq 0$, then $D[g_\psi, g] \geqq 0$. Hence g_ψ is a supersolution on Ω. On each coordinate neighborhood, g_ψ is a solution of

$$-\frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x_j} \left(\sum_i \sqrt{G}\, g^{ij} \frac{\partial u}{\partial x_i} \right) = \psi.$$

Therefore, by the results in [11] or in [14], we may assume that g_ψ is continuous. Then g_ψ is superharmonic, and hence it is a potential by Lemma 12.

REMARK. By the above lemma, we see that there exists a positive potential on Ω. Note that this fact essentially relies on the assumption $1 \notin \mathcal{S}_o(\Omega)$.

9. Green functions.

It is shown in [7] that for any coordinate neighborhood U in Ω, there exists the Green function $g_y^U(x)$ on U, uniquely determined by the following two conditions:

(i) For each $y \in U$, g_y^U is a potential on U and is harmonic on $U - \{y\}$;

(ii) If $u \in \mathcal{D}(U)$ and is a supersolution on U and if μ is the associated measure, then

$$u(x) = \int_U g_y^U(x) d\mu(y) + h_u^U(x)$$

a.e. on U.

Also it is shown that if μ is a measure on U such that $\mu(U) < \infty$, then $\int_U g_y^U d\mu(y) \in L^1(U)$.

From this result we obtain

PROPOSITION 10. There exists a uniquely determined function $g_y(x)$ defined for $x \in \Omega$, $y \in \Omega$ such that

(i) For each $y \in \Omega$, g_y is a potential on Ω and is harmonic on $\Omega - \{y\}$;

(ii) For each coordinate neighborhood U, $g_y - g_y^U$ is harmonic on U.

Furthermore, we have

(iii) $(x, y) \to g_y(x)$ is lower semi-continuous on $\Omega \times \Omega$, continuous on $\Omega \times \Omega - \{(x, x);\ x \in \Omega\}$.

Sketch of the proof: First remark that there is a positive potential on Ω (cf. the remark after Lemma 13). The uniqueness of g_y^U for each coordinate neighborhood U implies the proportionality of two potentials supported by $\{y\}$. Hence, by Théorème 18.1 in [5], there exists a function $p_y(x)$ on $\Omega\times\Omega$ such that for each $y \in \Omega$ p_y is a potential supported by $\{y\}$ and the mapping $y \to p_y(x)$ is continuous on $\Omega - \{x\}$ for each $x \in \Omega$. On each coordinate neighborhood U, we can write

$$p_y(x) = \lambda_y g_y^U(x) + h_y(x)$$

for $x, y \in U$, where h_y is harmonic on U. It is easy to see that $y \to \lambda_y$ is continuous on U and λ_y does not depond on U as long as $y \in U$. Then $g_y(x) = p_y(x)/\lambda_y$ satisfies (i) and (ii) of the proposition and $y \to g_y(x)$ is continuous on $\Omega - \{x\}$ for each $x \in \Omega$. The uniqueness of $g_y(x)$ is easy to see. The property (iii) follows from Proposition 18.1 in [5].

PROPOSITION 11. If u is a superharmonic function on Ω having the greatest harmonic minorant h on Ω, then there corresponds a unique measure μ such that

$$u(x) = \int_\Omega g_y(x)d\mu(y) + h(x)$$

for all $x \in \Omega$. If, furthermore, $u \in \mathcal{L}_{loc}(\Omega)$, then μ is the measure associated with u.

The integral representation follows from Théorème 18.2 of

[5]. The last assertion of this proposition is a consequence of condition (ii) for g_y^U and g_y.

COROLLARY. Any superharmonic function on Ω belongs to $L^1_{loc}(dV)$.

PROPOSITION 12. $g_y(x) = g_x(y)$ for any $x, y \in \Omega$.

Proof: Let $\psi_1, \psi_2 \in C_0(\Omega)$, $\psi_1, \psi_2 \geqq 0$. By Lemma 15 and the above proposition we have

$$D[g_{\psi_1}, g_{\psi_2}] = \int g_{\psi_1}\psi_2 \, dV$$
$$= \iint g_y(x)\psi_1(y)\psi_2(x)dV(x)dV(y).$$

Since $D[g_{\psi_1}, g_{\psi_2}] = D[g_{\psi_2}, g_{\psi_1}]$ and ψ_1, ψ_2 are arbitary, we have $g_y(x) = g_x(y)$.

Using this proposition, we obtain the following result as in the classical case:

LEMMA 14. If μ is a non-negative measure on Ω with finite total mass, then $\int_\Omega g_y(x)d\mu(y)$ is a potential.

10. <u>Dirichlet problem with respect to an ideal boundary</u>.

Let Ω^* be a compactification of Ω. For an extended real valued function ϕ on $\Gamma = \Omega^* - \Omega$, let

$$\overline{\mathcal{S}}_\phi = \left\{u; \begin{array}{l} \text{superharmonic on } \Omega, \text{ bounded below} \\ \underline{\lim}_{x\to\xi} u(x) \geqq \phi(\xi) \text{ for all } \xi \in \Gamma \end{array}\right\} \cup \{\infty\}$$

and $\underline{\mathcal{S}}_\phi = \{-u;\ u \in \overline{\mathcal{S}}_{-\phi}\}$. By a general theory (cf. [1], [3]) it is known that each of $\overline{H}_\phi = \inf \overline{\mathcal{S}}_\phi$ and $\underline{H}_\phi = \sup \underline{\mathcal{S}}_\phi$ is either harmonic on Ω, $\equiv +\infty$ or $\equiv -\infty$ and that $\underline{H}_\phi \leqq \overline{H}_\phi$. If $\underline{H}_\phi = \overline{H}_\phi$ and are harmonic, then ϕ is called resolutive and the common function is denoted by H_ϕ. Let $R(\Gamma)$ be the set of all resolutive functions on Γ. We have ([1], [3]; cf. [2])

PROPOSITION 13. (i) $R(\Gamma)$ is a linear space and the mapping $\phi \to H_\phi \in H(\Omega)$ is non-negative linear on $R(\Gamma)$; $H_1 \equiv 1$.

(ii) $H_{\max(\phi_1,\phi_2)}$ = the least harmonic majorant of $\max(H_{\phi_1}, H_{\phi_2})$,

$H_{\min(\phi_1,\phi_2)}$ = the greatest harmonic minorant of $\min(H_{\phi_1}, H_{\phi_2})$.

If every finite continuous function on Γ is resolutive, then we call Ω^* a resolutive compactification (with respect to (g_{ij})). In this case we have the harmonic measure ω_x on Γ for each $x \in \Omega$, which is defined by

$$\int \phi \, d\omega_x = H_\phi(x)$$

for all $\phi \in C(\Gamma)$. Obviously $\omega_x(\Gamma) = 1$.

11. <u>The space</u> $R_D(\Gamma)$.

We assume that Ω^* is a resolutive compactification. Let

$$R_D(\Gamma) = \{\phi \in R(\Gamma);\ H_\phi \in \mathcal{H}(\Omega)\}.$$

This is a linear subspace of $R(\Gamma)$.

PROPOSITION 14. If ϕ_1, $\phi_2 \in R_D(\Gamma)$, then $\max(\phi_1, \phi_2)$, $\min(\phi_1, \phi_2) \in R_D(\Gamma)$ and

$$D[H_{\max(\phi_1,\phi_2)}] + D[H_{\min(\phi_1,\phi_2)}] \leqq D[H_{\phi_1}] + D[H_{\phi_2}];$$

in particular, if $\phi \in R_D(\Gamma)$, then $|\phi| \in R_D(\Gamma)$ and $D[H_{|\phi|}] \leqq D[H_\phi]$. (Cf. the corollary to Theorem 2 and Proposition 13; also cf. [10])

PROPOSITION 15. For $\phi \in R_D(\Gamma)$, let $\phi_n = \max(\min(\phi, n), -n)$, $n = 1,2,\ldots$. Then $D[H_{\phi_n} - H_\phi] \to 0$ $(n \to \infty)$. (Cf. Proposition 2; also cf. [10])

THEOREM 3. (Doob's lemma) For fixed $x_o \in \Omega$, there exists a constant $M > 0$ such that for any $\phi \in R_D(\Gamma)$

$$\int \phi^2 d\omega_{x_o} \leqq M\|H_\phi\|^2.$$

The proof of this theorem is similar to the classical case (see e.g., [9], [10]) once we obtain the following lemma:

LEMMA 15. If $u \in \mathcal{H}(\Omega)$, then the superharmonic function $-u^2$ has a harmonic minorant and

$$-u^2(x) = 2\int_\Omega g_y(x)|\text{grad } u|^2(y)dV(y) + h(x),$$

where h is the greatest harmonic minorant of $-u^2$.

Proof: Since u is continuous, we easily see that $-u^2 \in \mathcal{L}_{loc}(\Omega)$ and $\frac{\partial(-u^2)}{\partial x_i} = -2u\frac{\partial u}{\partial x_i}$. Hence, for any $\phi \in C_o^1(\Omega)$,

$$D[-u^2, \phi] = -2\int \sum g^{ij}\frac{\partial u}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}u\right)dV$$

$$= -2D[u, \phi u] + 2\int|\text{grad } u|^2\phi dV.$$

Since $\phi u \in \mathcal{L}_o(\Omega)$ (Lemma 4), $D[u, \phi u] = 0$. Hence μ

$= 2|\text{grad } u|^2 dV$ is the measure associated with $-u^2$. Since $u \in \mathcal{D}(\Omega)$, $\mu(\Omega) < \infty$. Therefore, by Lemma 14, $p(x) = 2\int_\Omega g_y(x)|\text{grad } u|^2(y)dV(y)$ is a potential on Ω. Obviously, $-u^2 - p$ is harmonic on Ω. Thus we obtain the lemma.

COROLLARY 1. $\{H_\phi;\ \phi \in R_D(\Gamma)\}$ is a closed subspace of $\mathcal{H}(\Omega)$.

COROLLARY 2. If Λ is an ω-measurable subset of Γ such that $\omega(\Gamma - \Lambda) > 0$. Then there exists a constant $M_\Lambda > 0$ such that

$$\int \phi^2 d\omega_{x_0} \leqq M_\Lambda\ D[H_\phi]$$

for any $\phi \in R_D(\Gamma)$ such that $\phi = 0$ ω-a.e. on $\Gamma - \Lambda$. Thus

$$\{H_\phi;\ \phi \in R_D(\Gamma),\ \phi = 0 \quad \omega\text{-a.e. on } \Gamma - \Lambda\}$$

is a closed subspace of $\mathcal{H}(\Omega)$. (Cf. [9], [10]) References

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