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On the Principles of Invariance in the Non-Stationary Radiation Field

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1. Introduction.

In the theory of radiative transfer, the principles of invariance play an important role. Chandrasekhar's principles of invariance can be applied to the stationary radiation field in a homogeneous atmosphere [1]. Ueno [2,3] extented the principles for the stationary radiation field in an inhomogeneous atmosphere and the non-stationary radiation field in a finite homogeneous atmosphere.

In the present paper, we shall extend the principles for the non-stationary and three-dimensional radiation field in an inhomogeneous atmosphere. The principles can be applied to the problem of diffuse reflection and transmission of a pulsed searchlight beam by a finite atmosphere. To simplify the discussion, we shall assume the scattering is isotropic. In the theory of non-stationary radiative transfer, we should take account of the following two time scales: the duration of temporal capture and the mean free time. Starting with the equation of transfer involving these two time-scales, we shall derive the principles of invariance.

2. The Equation of Transfer

In an isotropically scattering medium, the intensity of radiation at time t at point $\underline{R}$ in direction $\underline{\Omega}$ satisfies the equation of transfer in the form

$$[\, c^{-1}D_t + \underline{\Omega}\cdot\nabla + \ell(\underline{R})\,]\, I(t,\underline{R},\underline{\Omega}) = \zeta(t,\underline{R}), \qquad (1)$$

where $D_u \equiv \frac{\partial}{\partial u}$. In the above expression, c, ℓ, $\underline{\Omega}$ and $\underline{R}$

are, respectively, the velocity of light, the volume attenuation coefficient, the directional unit vector and the position vector. ζ is the source function given by

$$\zeta(t,\underline{R}) = \sigma(\underline{R}) \int_0^t J(t',\underline{R})\exp[-(t-t')/t_1]dt'/t_1 \ , \qquad (2)$$

where σ is the volume scattering coefficient, t_1 is the duration of temporal capture and J is the mean intensity such that

$$J(t,\underline{R}) = (1/4\pi) \int_{4\pi} I(t,\underline{R},\underline{\Omega})d\underline{\Omega}. \qquad (3)$$

The integral with subscript 4π is taken over the all directions.

In a system of Cartesian or cylindrical coordinates, the z-axis is chosen to normal to a finite plane-parallel atmosphere whose optical properties vary with z . In the system of coordinates, the position vector and the directional unit vector can be expressed in the form

$$\underline{R} = (\underline{r},z), \quad \text{and} \quad \underline{\Omega} = (\underline{\omega},\mu),$$

where the two-dimensional vectors $\underline{r}$ and $\underline{\omega}$ are, respectively, the orthogonal projections of the vectors $\underline{R}$ and $\underline{\Omega}$ on the xy-plane, $\cos^{-1}\mu$ is the co-latitude of the unit vector $\underline{\Omega}$ $(-1 \leq \mu \leq 1)$ and $|\underline{\omega}| = \surd(1-\mu^2)$. Since the optical properties vary along the z-axis, we can write $\ell(\underline{R}) = \ell(z)$, $\sigma(R) = \sigma(z)$. For ℓ and σ , we require the following conditions:

$$0 < \ell_1 \leq \ell(z), \quad 0 \leq \sigma(z) \leq \sigma_1 \text{ and } 0 \leq \sigma_1/\ell_1 \leq 1. \quad (4)$$

We shall use $I^-(t,\underline{R},\underline{\Omega})$ to denote the intensity directed towards the upper surface $z = \alpha$ and $I^+(t,\underline{R},\underline{\Omega})$ for that directed towards the lower surface $z = \beta$, where $\alpha \leq z \leq \beta$ and $0 < \mu \leq 1$. Then Eq. (1) can be written

$$[\ c^{-1}D_t + \underline{\omega}\cdot\nabla_2 \pm \mu D_z\ ]\ I^{\pm}(t,\underline{R},\underline{\Omega}) = \zeta(t,\underline{R}), \qquad (5)$$

where ∇_2 is the two-dimensional nabla operator.

Let radiation of intensities $I_0(t,r,\underline{\Omega})$ and $I_1(t,r,\underline{\Omega})$ ($0 \leq$

t) ($0 < \mu \leq 1$) fall on the upper and lower surfaces, respectively. Then the initial and boundary conditions for Eq. (5) are

$$\left.\begin{aligned} I(0,\underline{R},\underline{\Omega}) &= 0\ , \\ I(t,\underline{R},\underline{\Omega}) &\to 0 \quad \text{as} \quad |\underline{r}| \to \infty, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} I^{+}(t,\underline{r},\alpha,\underline{\Omega}) &= I_0(t,\underline{r},\underline{\Omega}), \\ I^{-}(t,\underline{r},\beta,\underline{\Omega}) &= I_1(t,\underline{r},\underline{\Omega}). \end{aligned}\right\} \tag{7}$$

Now, we consider the following transform:

$$\overline{f}(s,\underline{p},z,\underline{\Omega}) = \int_0^{\infty} dt \int f(t,\underline{r},z,\underline{\Omega}) \exp(-st+i\underline{p}\cdot\underline{r}) d\underline{r}, \tag{8}$$

where the r-integral is taken over the entire xy-plane. We assume that $I(t,\underline{R},\underline{\Omega})$ is bounded with respect to t and satisfies

$$\int |I(t,\underline{r},z,\underline{\Omega})| d\underline{r} \qquad .$$

Then the transform of $I^{\pm}$ converges uniformly for $\mathrm{Re}\ s > 0$ and $0 \leq |\underline{p}| < \infty$. With the aid of Eq. (6), the transform of Eq. (5) gives

$$[\ \pm\mu D_z + \{\ell(z) + c/s - i\underline{\omega}\cdot\underline{p}\}]\overline{I}^{\pm}(s,\underline{p},z,\underline{\Omega}) = \gamma(s,z)\overline{J}(s,\underline{p},z), \tag{9}$$

where

$$\gamma(s,z) = \sigma(z)(1 + st_1)^{-1}, \tag{10}$$

and

$$\overline{J}(s,\underline{p},z) = (1/4\pi) \int_{4\pi} \overline{I}(s,\underline{p},z,\underline{\Omega}) d\underline{\Omega}. \tag{11}$$

Eq. (7) is transformed into

$$\left.\begin{aligned} \overline{I}^{+}(s,\underline{p},\alpha,\underline{\Omega}) &= \overline{I}_0(s,\underline{p},\underline{\Omega}), \\ \overline{I}^{-}(s,\underline{p},\beta,\underline{\Omega}) &= \overline{I}_1(s,\underline{p},\underline{\Omega}). \end{aligned}\right\} \tag{12}$$

The formal solution of Eq. (9) subject to (12) is

$$\overline{I}^{+}(s,\underline{p},z,\underline{\Omega}) = \overline{I}_0(s,\underline{p},\underline{\Omega})\eta(\alpha,z,\underline{\Omega}) + L^{+}(\alpha,z,\underline{\Omega})\{\overline{J}(s,\underline{p},z')\}/\mu \;, \tag{13}$$

$$\overline{I}^{-}(s,\underline{p},z,\underline{\Omega}) = \overline{I}_1(s,\underline{p},\underline{\Omega})\eta(z,\beta,\underline{\Omega}) + L^{-}(z,\beta,\underline{\Omega})\{\overline{J}(s,\underline{p},z')\}/\mu \;, \tag{14}$$

where

$$\eta(\alpha,\beta,\underline{\Omega}) = \exp[-\{(\tau_\beta-\tau_\alpha) + (s/c - i\underline{\omega}\cdot\underline{p})(\beta-\alpha)\}/\mu] = 1/\eta(\beta,\alpha,\underline{\Omega}) \;, \tag{15}$$

and

$$\tau_z = \int_0^z \ell(z')dz' \;. \tag{16}$$

In Eqs. (13) and (14), L^{+} and L^{-} are

$$L^{+}(\alpha,\beta,\underline{\Omega})\{f(z')\} = \int_\alpha^\beta \gamma(s,z')\eta(z',\beta,\underline{\Omega})f(z')dz' \;, \tag{17}$$

and

$$L^{-}(\alpha,\beta,\underline{\Omega})\{f(z')\} = \int_\alpha^\beta \gamma(s,z')\eta(\alpha,z',\underline{\Omega})f(z')dz' \;. \tag{18}$$

From Eqs. (15) (17) and (18), for $\alpha \leqq u \leqq \beta$, we have

$$\eta(\alpha,\beta,\underline{\Omega}) = \eta(\alpha,u,\underline{\Omega})\eta(u,\beta,\underline{\Omega}) \;, \tag{19}$$

$$L^{-}(\alpha,\beta,\underline{\Omega}) = L^{-}(\alpha,u,\underline{\Omega}) + \eta(\alpha,u,\underline{\Omega})L^{-}(u,\beta,\underline{\Omega}) \;, \tag{20}$$

and

$$L^{+}(\alpha,\beta,\underline{\Omega}) = L^{+}(u,\beta,\underline{\Omega}) + \eta(u,\beta,\underline{\Omega})L^{+}(\alpha,u,\underline{\Omega}) \;. \tag{21}$$

Substituting Eqs. (13) and (14) into Eq. (11), we obtain the integral equation for $\overline{J}$ as follows:

$$(1 - L)(\alpha,\beta,z)\{J(s,\underline{p},z')\} = B(s,\underline{p},z), \tag{22}$$

where the linear operator L is defined by

$$L(\alpha,\beta,z)\{f(s,\underline{p},z')\} = (1/4\pi)\int_\alpha^\beta \gamma(s,z')K(z,z')f(s,\underline{p},z')dz', \tag{23}$$

with

$$K(z,z') = \int\eta(z,z',\underline{\Omega})d\underline{\Omega}/\mu \text{ if } z' > z,$$
$$= \int\eta(z',z,\underline{\Omega})d\underline{\Omega}/\mu \text{ if } z > z', \tag{24}$$

1 is the identity operator and B is

$$B(s,\underline{p},z) = (1/4\pi)\int_\alpha^\beta I_0(s,\underline{p},\underline{\Omega})\eta(\alpha,z,\underline{\Omega})d\underline{\Omega}$$
$$+ (1/4\pi)\int_\alpha^\beta I_I(s,\underline{p},\underline{\Omega})\eta(z,\beta,\underline{\Omega})d\underline{\Omega}. \tag{25}$$

THEOREM 1. If B is bounded on the following set D:

$D = \{(s,\underline{p},z) |$ Re $s > 0,\ 0 \leqq |\underline{p}| < \infty,\ \alpha \leqq z \leqq \beta$, .

then Eq. (22) has one and only one solution.

PROOF. Let $\Phi(D)$ be a set of functions bounded on D. Then $\Phi(D)$ is a Banach space with the norm

$$||f|| = \sup |f(s,\underline{p},z)| \quad (s,p,z) \in D.$$

Hence L is regarded as a linear operator on $\Phi(D)$ into $\Phi(D)$. From Eq. (23) it follows that

$$||Lf|| \leqq ||f||(1/4\pi)\int_\alpha^\beta|\gamma(s,z')||K(z,z')|dz'. \tag{26}$$

By Eqs. (10) and (15), we have

$$|\gamma(s,z)| \leqq \sigma(z)\cdot|1+st_1|^{-1} \leqq \sigma(z) \leqq \sigma_1, \tag{27}$$

and

$$|\eta(z,z',\underline{\Omega})| \leqq \exp[-(\ell+ \text{Re } s/c)|z-z'|/\mu]. \tag{28}$$

Hence from (24) we have

$$|K(z,z')| \leq 2\pi E_1[\ (\ell_1 + \text{Re } s/c)|z-z'|\ ], \tag{29}$$

where

$$E_n(u) = \int_1^\infty t^{-n}\exp(-tu)dt. \tag{30}$$

From (27) and (29) it follows that

$$(1/4\pi)\int_\alpha^\beta |\gamma(s,z')| \cdot |K(z',z)|dz'$$

$$\leq \sigma_1(\ell_1 + \text{Re } s/c)^{-1}[2 - E_2\{(\ell_1 + \text{Re } s/c)(z-\alpha)\}$$

$$- E_2\{(\ell_1 + \text{Re } s/c)(\beta-z)\}]/2$$

$$\leq (\sigma_1/\ell_1)[1 - E_2\{\ell_1(\beta-\alpha)/2\}]$$

$$\leq 1 - E_2[\ell_1(\beta-\alpha)/2] \equiv \lambda < 1, \tag{31}$$

where (4) is used. Therefore from (26) we obtain

$$||Lf|| \leq \lambda||f||, \text{ and hence } ||L|| < 1 .$$

Then Eq. (22) has one and only one solution for $B \in \Phi(D)$. This completes the proof.

Now we consider decomposition formulae for operator L. For an arbitrary u in the interval $[\alpha,\beta]$, we have

$$L(\alpha,\beta,z) = L(u,\beta,z)$$

$$+ (1/4\pi)\int\eta(u,z,\underline{\Omega})L^+(\alpha,u,\underline{\Omega})d\underline{\Omega}/\mu \quad (\alpha \leq u \leq z \leq \beta), \tag{32}$$

$$L(\alpha,\beta,z) = L(\alpha,u,z)$$

$$+ (1/4\pi)\int\eta(z,u,\underline{\Omega})L^-(u,\beta,\underline{\Omega})d\underline{\Omega}/\mu \quad (\alpha \leq z \leq u \leq \beta). \tag{33}$$

These equations can be obtained by Eqs. (20), (21   nd (23).

3. The S- and T-Functions

In the theory of radiative transfer, it is convenient to distinguish between the diffuse radiation field which arises in consequence of one or more scattering processes, and the reduced incident radiation field which penetrates to the level z without any collision process. We shall put a subscript d to all quantities referring to the diffuse radiation field. Then we have

$$I^{+}(t,\underline{R},\underline{\Omega}) = I_d^{+}(t,\underline{R},\underline{\Omega})$$

$$+ I_0[t-(z-\alpha)/c\mu,\underline{r}-(z-\alpha)\underline{\omega}/\mu,\underline{\Omega}]\exp[-(\tau_z-\tau_\alpha)/\mu], \quad (34)$$

$$I^{-}(t,\underline{R},\underline{\Omega}) = I_d^{-}(t,\underline{R},\underline{\Omega})$$

$$+ I_1[t-(\beta-z)/c\mu,\underline{r}-(\beta-z)\underline{\omega}/\mu,\underline{\Omega}]\exp[-(\tau_\beta-\tau_z)/\mu]. \quad (35)$$

From Eqs. (13) and (14) it follows that

$$\overline{I}_d^{+}(s,\underline{p},z,\underline{\Omega}) = L^{+}(\alpha,z,\underline{\Omega})\{\overline{J}(s,\underline{p},z')\}/\mu, \quad (36)$$

and

$$\overline{I}_d^{-}(s,\underline{p},z,\underline{\Omega}) = L^{-}(z,\beta,\underline{\Omega})\{\overline{J}(s,\underline{p},z')\}/\mu. \quad (37)$$

In what follows, we consider the diffuse radiation field alone. We omitt the subscript d .

Now we consider the diffuse reflection of a pencile of radiation of net flux 4π incident at time 0 at point $\underline{r} = \underline{0}$ on the upper surface in direction $\underline{\Omega}_0$. In this case I_0 and I_1 are

$$I_0(t,\underline{r},\underline{\Omega}) = 4\pi\delta(t)\delta(\underline{r})\delta(\underline{\Omega}-\underline{\Omega}_0), \quad I_1(t,\underline{r},\underline{\Omega}) = 0. \quad (38)$$

We denote the J-function by $J(t,\underline{r},z,\alpha,\beta,\underline{\Omega}_0)$. From Eqs.(22),(25) and (38) we find that $\overline{J}$ satisfies the following equation:

$$(1-L)(\alpha,\beta,z)\{\overline{J}(z',\alpha,\beta,\underline{\Omega}_0)\} = \eta(\alpha,z,\underline{\Omega}_0), \quad (39)$$

where s and $\underline{p}$ are supressed. Now, we define the S- and T-functions as follows.

$$S(t,\underline{r},\alpha,\beta,\underline{\Omega},\underline{\Omega}_0) = I^-(t,\underline{r},\alpha,\underline{\Omega}), \qquad (40)$$

$$T(t,\underline{r},\alpha,\beta,\underline{\Omega},\underline{\Omega}_0) = I^+(t,\underline{r},\beta,\underline{\Omega}). \qquad (41)$$

Then, by Eqs. (36) and (37),

$$\overline{S}(\alpha,\beta,\underline{\Omega},\underline{\Omega}_0) = L^-(\alpha,\beta,\underline{\Omega})\{\overline{J}(z',\alpha,\beta,\underline{\Omega}_0)\}, \qquad (42)$$

and

$$\overline{T}(\alpha,\beta,\underline{\Omega},\underline{\Omega}_0) = L^+(\alpha,\beta,\underline{\Omega})\{\overline{J}(z',\alpha,\beta,\underline{\Omega}_0)\}. \qquad (44)$$

THEOREM 2. Let $\overline{J}$ be the solution of Eq. (39). Then we have the following functional equation:

$$\overline{J}(z,\alpha,\beta,\underline{\Omega}_0) = \eta(\alpha,u,\underline{\Omega}_0)\overline{J}(z,u,\beta,\underline{\Omega}_0)$$

$$+ (1/4\pi)\int \overline{J}(z,u,\beta,\underline{\Omega}')\overline{I}^+(u,\underline{\Omega}')d\underline{\Omega}', \qquad (44)$$

where $\alpha \leq u \leq z \leq \beta$.

PROOF. Using Eqs. (32) in Eq. (39), from Eq. (19) we have

$$(1-L)(u,\beta,z)\{\overline{J}(z',\alpha,\beta,\underline{\Omega}_0)\} = \eta(\alpha,u,\underline{\Omega}_0)\eta(u,z,\underline{\Omega}_0)$$

$$+ (1/4\pi)\int \eta(u,z,\underline{\Omega}')L^+(\alpha,u,\underline{\Omega}')\{\overline{J}(z',\alpha,\beta,\underline{\Omega}_0)\}d\underline{\Omega}'. \qquad (45)$$

It is easy to show that the right hand side of Eq.(45) satisfies the assumption of THEOREM 1, whence Eq. (45) has only one solution. Operating $L(u,\beta,z)$ on the right hand side of Eq.(44), we obtain the right hand side of Eq. (45). Therefore the right hand side of Eq. (44) is the solution of Eq. (45). Hence, Eq. (44) follows. This completes the proof.

4. The Principles of Invariance

Operating $\mu^{-1}L^{-}(u,\beta,\underline{\Omega})$ on the both sides of Eq. (44), and making use of Eqs. (36), (38) and (40), we obtain

$$\overline{I}^{-}(u,\underline{\Omega}) = \mu^{-1}\overline{S}(u,\beta,\underline{\Omega},\underline{\Omega}_0)\eta(\alpha,u,\underline{\Omega}_0)$$

$$+ (1/4\pi)\int\overline{S}(u,\beta,\underline{\Omega},\underline{\Omega}')\overline{I}^{+}(u,\underline{\Omega}')d\underline{\Omega}'. \qquad (46)$$

Taking the inverse transform and replacing u by z, we find

$$I^{-}(t,\underline{r},z,\underline{\Omega}) = \mu^{-1}S[t-(z-\alpha)/c\mu_0,\underline{r}-(z-\alpha)\underline{\omega}_0/\mu_0,z,\beta,\underline{\Omega},\underline{\Omega}_0]$$

$$\times \exp[-(\tau_z-\tau_\alpha)/\mu_0]$$

$$+ (1/4\pi\mu)\int_0^t dt'\int d\underline{r}'\int S(t-t',\underline{r}-\underline{r}',z,\beta,\underline{\Omega},\underline{\Omega}')$$

$$\times I^{+}(t',\underline{r}',z,\underline{\Omega}')d\underline{\Omega}'. \qquad (47)$$

Eq. (47) is a mathematical expression of one of the principles of invariance in the non-stationary radiation field. In order to obtain the complete set of the principles, we must consider the illumination of the lower surface simultaneously.

The physical interpretation of Eq. (47) is stated as follows.

The intensity $I^{-}(t,\underline{r},z,\underline{\Omega})$ in the uppersard direction $\underline{\Omega}$ at time t at point P = $(\underline{r},z)$ results from the diffuse reflection of the reduced incident radiation $4\pi\exp[-(\tau_z-\tau_\alpha)/\mu_0]$ at point Q = $[(z-\alpha)\underline{\omega}_0/\mu_0,z]$ at time $(z-\alpha)/c\mu_0$ and the diffuse reflection of diffuse radiation field $I^{+}(t',\underline{r}',z,\underline{\Omega}')$ incident on the surface z at time t' at point R = $(\underline{r}',z)$ by the atmosphere below z (See Fig. 1).

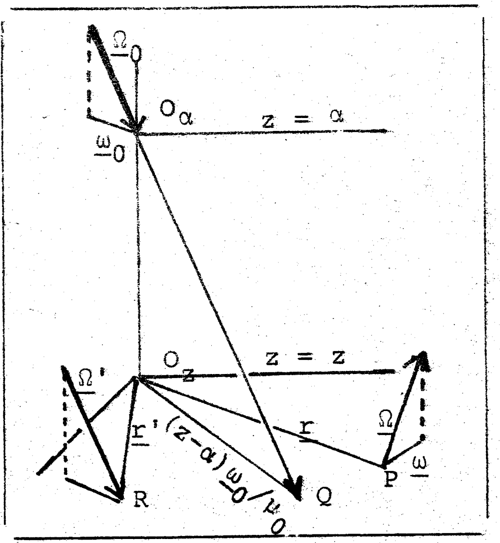

Fig. 1

References

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