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# On the Generalized Sampling Theorem

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Summary

Being based on the generalized sampling theorem along the line of consideration of the reciprocity of integral transforms, presented previously by the authors[1], some of the series-expansions of functions are given as its special examples.

The generalization of the sampling theorem and the reconstruction of a band-limited function from its sampled values and derivatives were made by Kohlenberg[2], Fogel[3], Jagerman and Fogel[4], Bond and Cahn[5], and Linden and Abramson[6]. The sampling theorem was also generalized by Balakrishnan[7] to the case of a continuous-parameter stochastic process. On the other hand, it was pointed out that the sampling intervals need not be uniformly distributed[8].

In the previous paper[1], the authors presented a generalized sampling theorem along the line of consideration of the reciprocity of integral transforms and gave some of its examples, which include Someya-Shannon's sampling theorem[9] [10] [11] as a special case. There the sampling intervals were not uniformly distributed. Recently the authors were informed that the sampling theorem was also generalized by Isomichi[12] in connection with the generalized frequency domain. Some of his results agreed with those given by us[1]. Here in the present paper, the authors wish to discuss the series-expansion of functions in an orthogonal set of functions, and also to give some of the new sampling formulae.

Let $f(t)$ belong to $L_p$, and let the following reciprocity relations hold :

$$F(s) = \mathcal{L}_s \cdot f \equiv \int_A K(s,t)\, f(t)\, dt\,, \qquad \text{for} \quad s \in B \tag{1}$$

and

$$f(t) = \mathcal{L}_t^{-1} \cdot F \equiv \int_B \widehat{K}(t,s)\, F(s)\, ds\,. \qquad \text{for} \quad t \in A \tag{2}$$

From (1) and (2), we obtain at once :

$$\int_A K(s,t)\, \widetilde{K}(t,\sigma)\, dt = \delta(s-\sigma)\,, \tag{3}$$

and

$$\int_B \widetilde{K}(t,s)\, K(s,\tau)\, ds = \delta(t-\tau)\,, \tag{4}$$

with delta function $\delta(t)$.

Further we assume that

$$F(s) = 0\,. \qquad \text{for} \quad s \notin D \subseteq B \tag{5}$$

Accordingly, we obtain from (2) and (5)

$$f(t) = \mathcal{L}_t^{-1} \cdot F = \int_D \widetilde{K}(t,s)\, F(s)\, ds\,. \tag{6}$$

Let $F(s)$ be expanded in a complete orthogonal set of functions :

$$\left\{ \phi_n(s)\,;\ \int_D \phi_m(s)\, \phi_n(s)\, ds = \gamma_m \cdot \delta_{m,n} \quad (m, n = \text{integers}) \right\}, \tag{7}$$

in the domain $D \subseteq B$, i.e.

$$F(s) = \sum_n a_n\, \phi_n(s)\,. \qquad \text{for} \quad s \in D \subseteq B \tag{8}$$

The expression (8), being multiplied by $\phi_m(s)$ and integrated over $D$, gives :

$$\int_D F(s)\, \phi_m(s)\, ds = \sum_n a_n \cdot \int_D \phi_m(s)\, \phi_n(s)\, ds = \sum_n a_n \cdot \gamma_m \cdot \delta_{m,n} = \gamma_m \cdot a_m\,. \tag{9}$$

From (6), (7), (8), and (9), we obtain :

$$f(t) = \mathcal{L}_t^{-1} \cdot F = \sum_n a_n \cdot \mathcal{L}_t^{-1} \cdot \phi_n$$

$$= \sum_n \left( \frac{1}{\gamma_n} \int_D F(s)\, \phi_n(s)\, ds \right) \cdot \int_D \tilde{K}(t,s)\, \phi_n(s)\, ds . \qquad (10)$$

If the functions $f(t)$, $\{\phi_n(s)\}$, and the integral kernels $K(s,t)$ and $\tilde{K}(t,s)$, are given, then we can construct a series-expansion of $f(t)$ by means of (10).

If we can take

$$\phi_n(s) = \tilde{K}(\lambda_n, s), \qquad \text{for} \quad s \in D \qquad (11)$$

with constants $\lambda_n$, i.e. if the kernel $\tilde{K}(\lambda_n, s)$ can be put equal to $\phi_n(s)$, then the expression (10) is simplified into :

$$f(t) = \sum_n \left( \frac{1}{\gamma_n} \int_D F(s)\, \tilde{K}(\lambda_n, s)\, ds \right) \cdot \int_D \tilde{K}(t,s)\, \tilde{K}(\lambda_n, s)\, ds$$

$$= \sum_n \frac{1}{\gamma_n} f(\lambda_n) \cdot \int_D \tilde{K}(t,s)\, \tilde{K}(\lambda_n, s)\, ds , \qquad (12)$$

by means of (6). The expression (12) gives a generalized sampling theorem[1]. The points at the variable $t$ :

$$t = \lambda_n , \qquad (n = \text{integers}) \qquad (13)$$

are called sampling points, and the function $g_n(t)$, defined by

$$g_n(t) \equiv \mathcal{L}_t^{-1} \cdot \phi_n = \int_D \tilde{K}(t,s)\, \phi_n(s)\, ds = \int_D \tilde{K}(t,s)\, \tilde{K}(\lambda_n, s)\, ds, \quad (n = \text{integers}) \qquad (14)$$

is the sampling function.

If the set of functions $\{ \tilde{K}(\lambda_n, s) \; ; n = \text{integers} \}$ is not necessarily orthogonal, but complete and linearly independent, we can find normalized biorthogonal set $\{ \psi_m(s) ; m = \text{integers} \}$, in such a way that

$$\int_D \tilde{K}(\lambda_m, s)\, \psi_n(s)\, ds = \delta_{m,n} , \qquad (15)$$

is to be satisfied. From the completeness of the set $\{ \tilde{K}(\lambda_m, s) ; m = \text{integers} \}$, we have :

$$\sum_n \tilde{K}(\lambda_n, s)\, \psi_n(\sigma) = \delta(s - \sigma) . \qquad (16)$$

Isomichi[12] expanded $F(s)$ in $\{\psi_n(s);\ n = \text{integers}\}$, and obtained :

$$F(s) = \sum_n b_n \psi_n(s), \qquad \text{for} \quad s \in D \subseteq B \tag{17}$$

with

$$b_n = \int_D \widetilde{K}(\lambda_n, s)\, F(s)\, ds. \tag{18}$$

From (6) and (18), we obtain at once :

$$b_n = f(\lambda_n) = \int_D \widetilde{K}(\lambda_n, s)\, F(s)\, ds. \tag{19}$$

Operating $\mathcal{L}_t^{-1}$ on (17), we obtain, by means of (6) :

$$f(t) = \mathcal{L}_t^{-1} \cdot F = \sum_n b_n \cdot \int_D \widetilde{K}(t, s)\, \psi_n(s)\, ds$$

$$= \sum_n f(\lambda_n) \cdot \int_D \widetilde{K}(t, s)\, \psi_n(s)\, ds. \tag{20}$$

The expression (18) or (19) becomes to :

$$f(\lambda_n) = \int_D ds \cdot \widetilde{K}(\lambda_n, s) \cdot \int_A dt \cdot K(s,t)\, f(t)$$

$$= \int_A dt \cdot f(t) \cdot \int_D ds \cdot \widetilde{K}(\lambda_n, s)\, K(s,t), \tag{21}$$

by means of (1). The expression (20) is the generalized sampling theorem obtained by Isomichi[12], and the expression

$$g_n(t) = \int_D \widetilde{K}(t, s)\, \psi_n(s)\, ds, \qquad (n = \text{integers}) \tag{22}$$

is the sampling function, with sampling points $t = \lambda_n$ ($n$ = integers) and with

$$g_n(\lambda_m) = \int_D \widetilde{K}(\lambda_m, s)\, \psi_n(s)\, ds = \delta_{m,n}.$$

The idea of the generalized sampling theorem was also given by one of the present authors, Takizawa, in his book[13] of information theory.

If we take $\{\widetilde{K}(\lambda_n, s)\}$ to be orthogonal, then we have at once

$$\{\psi_n(s)\} = \{\phi_n(s)\} = \{\widetilde{K}(\lambda_n, s)\}, \tag{23}$$

as was given in (11), the expression (20) reduces to the sampling theorem (12), the dual vector $\{\Psi_n(s)\}$ for $\{\widetilde{K}(\lambda_n, s)\}$ reducing to $\{\widetilde{K}(\lambda_n, s)\}$ itself.

The corresponding expressions of $F(t)$ for (20) and (21) are as follows :

$$F(t) = \sum_n F(\mu_n) \cdot \int_E K(t, s)\, \Psi_n(s)\, ds, \tag{24}$$

and

$$F(\mu_n) = \int_B dt \cdot F(t) \cdot \int_E ds \cdot K(\mu_n, s)\, \widetilde{K}(s, t), \tag{25}$$

with the sampling function :

$$p_n(t) = \int_E K(t, s)\, \widetilde{\Psi}_n(s)\, ds, \tag{26}$$

and the sampling points $t = \mu_n$ ($n$ = integers), under the condition that for the complete set $\{K(\mu_n, t);\ n = \text{integers}\}$ we have a set of biorthogonal functions $\{\widetilde{\Psi}_n(s);\ n = \text{integers}\}$ in the domain $E \subseteq A$, i.e.

$$\int_E K(\mu_m, t)\, \widetilde{\Psi}_n(t)\, dt = \delta_{m,n} \cdot \quad \text{for} \quad t \in E \subseteq A \tag{27}$$

From the completeness of the set $\{K(\mu_m, s);\ m = \text{integers}\}$, it follows that the expression :

$$\sum_n K(\mu_n, s)\, \widetilde{\Psi}_n(\sigma) = \delta(s - \sigma), \tag{28}$$

holds.

Now, let us take sampling functions $\{g_n(t);\ n = \text{integers}\}$, $\{h_n(t);\ n = \text{integers}\}$, $\{p_n(t);\ n = \text{integers}\}$, and $\{q_n(t);\ n = \text{integers}\}$, defined as follows :

$$g_n(t) \equiv \int_D \widetilde{K}(t, s)\, \Psi_n(s)\, ds, \tag{29}$$

$$h_n(t) \equiv \int_D K(s, t)\, \phi_n(s)\, ds = \int_D K(s, t)\, \widetilde{K}(\lambda_n, s)\, ds, \tag{30}$$

$$p_n(t) \equiv \int_E K(t, s)\, \widetilde{\Psi}_n(s)\, ds, \tag{31}$$

and

$$g_n(t) \equiv \int_E \widetilde{K}(s,t)\, K(\mu_n, s)\, ds . \qquad (32)$$

From (3) and (4), we have

$$\int_A K(\sigma,t)\, g_n(t)\,dt = \int_D \psi_n(s)\cdot\delta(\sigma-s)\,ds = \begin{cases} \psi_n(\sigma), & \text{for } \sigma\in D \\ 0, & \text{for } \sigma\notin D \end{cases} \qquad (33)$$

$$\int_A \widetilde{K}(t,\sigma)\, h_n(t)\,dt = \int_D \phi_n(s)\cdot\delta(\sigma-s)\,ds = \begin{cases} \widetilde{K}(\lambda_n,\sigma), & \text{for } \sigma\in D \\ 0, & \text{for } \sigma\notin D \end{cases} \qquad (34)$$

$$\int_B \widetilde{K}(\sigma,t)\, p_n(t)\,dt = \int_E \widetilde{\psi}_n(s)\cdot\delta(\sigma-s)\,ds = \begin{cases} \widetilde{\psi}_n(\sigma), & \text{for } \sigma\in E \\ 0, & \text{for } \sigma\notin E \end{cases} \qquad (35)$$

and

$$\int_B K(t,\sigma)\, q_n(t)\,dt = \int_E K(\mu_n,s)\cdot\delta(\sigma-s)\,ds = \begin{cases} K(\mu_n,\sigma), & \text{for } \sigma\in E \\ 0. & \text{for } \sigma\notin E \end{cases} \qquad (36)$$

By means of (3) and (15), we obtain

$$\int_A g_m(t)\, h_n(t)\,dt = \int_D ds\cdot\int_D d\sigma\cdot \psi_m(s)\,\phi_n(\sigma)\cdot\delta(\sigma-s) =$$

$$= \int_D \psi_m(s)\,\phi_n(s)\,ds = \int_D \widetilde{K}(\lambda_n,s)\,\psi_m(s)\,ds = \delta_{m,n} , \qquad (37)$$

for any integers $m$ and $n$. This expression (37) shows that any function of $\{g_n(t);\ n = \text{integers}\}$ and any function of $\{h_n(t);\ n = \text{integers}\}$ are mutually orthogonal in the domain $D$. Similarly, we obtain, from (4) and (27) :

$$\int_B p_m(t)\, q_n(t)\,dt = \int_E \widetilde{\psi}_m(s)\, K(\mu_n, s)\,ds = \delta_{m,n} . \qquad (38)$$

We shall return to the expressions (20), (21), (24), and (25). They are written as follows :

$$f(t) = \sum_n f(\lambda_n)\, g_n(t) , \qquad (39)$$

$$f(\lambda_n) = \int_A f(t)\, h_n(t)\,dt , \qquad (40)$$

$$F(t) = \sum_n F(\mu_n)\, p_n(t)\,, \tag{41}$$

and

$$F(\mu_n) = \int_B F(t)\, f_n(t)\, dt\,, \tag{42}$$

in terms of (29)~(32).

Let $\xi(t)$ be any function which can be expanded in $\{g_n(t);\, n = \text{integers}\}$, i.e.

$$\xi(t) = \sum_n c_n\, g_n(t)\,, \tag{43}$$

with constants $c_n$, then the integral operator :

$$\mathcal{J} \equiv \int_A d\tau \cdot \sum_n g_n(t)\, h_n(\tau) = \int_A d\tau \cdot \delta(t-\tau)\,, \tag{44}$$

applied to (43), is the identical operator, in the sense that

$$\mathcal{J} \cdot \xi(\tau) = \xi(t)\,. \tag{45}$$

The proof of (45) is quite obvious, if we put (43) into (45) and refer (37).

Considering (4) and (44), we shall take

$$F^*(s) = \int_A \tilde{K}(t, s)\, f(t)\, dt\,, \tag{46}$$

with

$$f(t) = \int_B K(s, t)\, F^*(s)\, ds\,, \tag{47}$$

and

$$f^*(\lambda_n) = \int_A g_n(t)\, f(t)\, dt\,, \tag{48}$$

with

$$f(t) = \sum_n f^*(\lambda_n)\, h_n(t)\,. \tag{49}$$

Then we obtain, from (2), (3), (47), (39), (49), and (37), the following expressions :

$$\|f\|^2 = \int_A dt \cdot \int_B ds \cdot K(s,t)\, F^*(s) \cdot \int_B d\sigma \cdot \tilde{K}(t,\sigma)\, F(\sigma)$$

$$= \int_B ds \cdot F^*(s) \cdot \int_B d\sigma \cdot F(\sigma) \cdot \delta(s-\sigma) = \int_B F^*(s) F(s)\, ds, \qquad (50)$$

and

$$\|f\|^2 = \int_A dt \cdot \sum_m \sum_n f^*(\lambda_m)\, h_m(t) \cdot f(\lambda_n)\, g_n(t)$$

$$= \sum_m \sum_n f^*(\lambda_m) f(\lambda_n) \cdot \delta_{m,n} = \sum_n f^*(\lambda_n) f(\lambda_n), \qquad (51)$$

with the norm $\|f\|$ of function $f(t)$ :

$$\|f\|^2 = \int_A [f(t)]^2 dt . \qquad (52)$$

The expressions (37), (45), and the generalized Parseval equalities (50) and (51) are given by Isomichi[12].

Now we shall differentiate (39) with regard to $t$ , then we have

$$f'(t) = \sum_n f(\lambda_n)\, g_n'(t) . \qquad (53)$$

On the other hand, we apply the generalized sampling theorem (39) to $f'(t)$, then we obtain

$$f'(t) = \sum_n f'(\lambda_n)\, g_n(t) , \qquad (54)$$

with

$$f'(\lambda_n) = \left[ \frac{df(t)}{dt} \right]_{t=\lambda_n} .$$

From (53) and (54), we have

$$f'(t) = \sum_n f'(\lambda_n)\, g_n(t) = \sum_n f(\lambda_n)\, g_n'(t). \qquad (55)$$

Accordingly, from the $m$-th derivative of $f(t)$ , we obtain :

$$f^{(m)}(t) = \sum_n f^{(m-r)}(\lambda_n)\, g_n^{(r)}(t) = \sum_n f^{(m-k)}(\lambda_n)\, g_n^{(k)}(t), \qquad (56)$$

for any integers $r$ and $k$ $(0 \leq r \leq m,\ 0 \leq k \leq m)$, and with

$$f^{(m)}(z) = \frac{d^m}{dz^m} f(z), \quad \text{and} \quad g_n^{(m)}(z) = \frac{d^m}{dz^m} g_n(z). \quad (m = \text{integers})$$

Here, we shall mention an expansion-formula, which contains both Taylor's and sampling formulae. If a function $f(t)$ is expressed in (39), i.e.

$$f(t) = \sum_n f(\lambda_n)\, g_n(t), \tag{57}$$

and if we can expand $f(t+\xi)$ in a Taylor series, as

$$f(t+\xi) = \sum_{m=0}^{+\infty} \frac{1}{m!}\, \xi^m \cdot f^{(m)}(t), \tag{58}$$

then we have an expansion-formula :

$$f(t+\xi) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!}\, \xi^m \cdot f^{(m)}(\lambda_n)\, g_n(t) \tag{59}$$

$$= \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!}\, \xi^m \cdot f(\lambda_n)\, g_n^{(m)}(t) . \tag{60}$$

For the special case of $\xi = 0$, the expressions (59) and (60) reduce to (57). While, for the special case of $t = \lambda_k$ (with $k$ fixed) in (59) and (60), we have

$$f(\lambda_k+\xi) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!}\, \xi^m \cdot f^{(m)}(\lambda_n)\, g_n(\lambda_k) = \sum_m \sum_n \frac{1}{m!}\, \xi^m \cdot f^{(m)}(\lambda_n) \cdot \delta_{n,k}$$

$$= \sum_{m=0}^{+\infty} \frac{1}{m!}\, \xi^m \cdot f^{(m)}(\lambda_k), \tag{61}$$

and

$$f(\lambda_k+\xi) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!}\, \xi^m \cdot f(\lambda_n)\, g_n^{(m)}(\lambda_k), \tag{62}$$

i.e.

$$f(t) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} (t-\lambda_k)^m \; f(\lambda_n)\, g_n^{(m)}(\lambda_k). \tag{62'}$$

The expression (61) is nothing but the usual Taylor expansion of $f(\lambda_k + \xi)$, while the expression (62) gives $f(\lambda_k + \xi)$ in terms of the derivatives of sampling functions $g_n$ at the sampling point $\lambda_k$.

Example 1

We shall take the Fourier transforms in (1) and (2) :

$$K(s,t) = \frac{1}{2\pi} \exp[ist], \qquad A = (-\infty, +\infty),$$

$$\tilde{K}(t,s) = \exp[-its], \qquad B = (-\infty, +\infty),$$

$$\phi_n(s) = \tilde{K}(\lambda_n, s) = \exp[-i\lambda_n s], \qquad D = (-\beta, \beta),$$

$\lambda_n = \frac{n-k}{\beta}\pi$, $n$ = integers, $k$ = real number given arbitrarily,

$$\int_{-\beta}^{\beta} \exp[-i\lambda_m s] \cdot \exp[+i\lambda_n s]\, ds = \gamma_m \cdot \delta_{m,n},$$

$$\gamma_n = 2\beta,$$

then the sampling theorem (12) reads as follows :

$$f(t) = \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} f\left(-\frac{n-k}{\beta}\pi\right) \cdot \frac{2\sin(\beta t + (n-k)\pi)}{t + \frac{n-k}{\beta}\pi}$$

$$= \sum_{n=-\infty}^{+\infty} f\left(\frac{n+k}{\beta}\pi\right) \cdot \frac{\sin(\beta t - (n+k)\pi)}{\beta t - (n+k)\pi}. \tag{63}$$

The expression is nothing but Someya's sampling theorem[9] given in 1949. The expression (63) gives $f(t)$ in terms of the sampling function $\sin(\beta t - (n+k)\pi)/(\beta t - (n+k)\pi)$, with values of $f((n\pi + k\pi)/\beta)$ at sampling points $t = (n+k)\pi/\beta$ ($n$ = integers). If we put $k = 0$ in (63), we have Shannon's sampling theorem[10) 11)].

Example 2

We shall take the Fourier cosine transforms for (1) and (2) :

$$K(s,t) = \frac{2}{\pi} \cos(st), \qquad A = (0, +\infty),$$

$$\tilde{K}(t,s) = \cos(ts), \qquad B = (0, +\infty),$$

$$\phi_n(s) = \widetilde{K}(\lambda_n, s) = \cos(\lambda_n s), \qquad D = (0, \beta),$$

$$\gamma_n = \frac{\beta}{2}\left\{ 1 + \frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta} \right\},$$

where $\lambda_n$'s are the roots (arranged in ascending order of magnitude) of the equation :

$$\lambda_n \tan(\lambda_n\beta) = M, \tag{64}$$

with a constant $M$. Then the expression (12) takes form :

$$f(t) = \frac{2}{\beta}\sum_n \frac{1}{1 + \frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}} \left(\int_0^\beta F(s)\cos(\lambda_n s)\,ds\right)\cdot\int_0^\beta \cos(ts)\cos(\lambda_n s)\,ds$$

$$= \frac{2}{\beta}\sum_n f(\lambda_n)\cdot\frac{\cos(\lambda_n\beta)}{1 + \frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t) - M}{t^2 - \lambda_n^2}\cos(\beta t), \tag{65}$$

with sampling points $t = \lambda_n$ ($n$ - integers), and the sampling function:

$$g_n(t) = \frac{2}{\beta}\cdot\frac{\cos(\lambda_n\beta)}{1 + \frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t) - M}{t^2 - \lambda_n^2}\cos(\beta t). \tag{66}$$

Example 3

We shall take the Fourier sine-transforms for (1) and (2) :

$$K(s, t) = \frac{2}{\pi}\sin(st), \qquad A = (0, +\infty),$$

$$\widetilde{K}(t, s) = \sin(ts), \qquad B = (0, +\infty),$$

$$\phi_n(s) = \widetilde{K}(\lambda_n, s) = \sin(\lambda_n s), \qquad D = (0, \beta),$$

$$\gamma_n = \frac{\beta}{2}\left\{ 1 - \frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta} \right\},$$

where $\lambda_n$'s are the roots (arranged in ascending order of magnitude) of the equation :

$$\lambda_n \cot(\lambda_n\beta) = N, \tag{67}$$

with a constant $N$. Then the expression (12) becomes to :

$$f(t)=\frac{2}{\beta}\sum_n \frac{1}{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\sin(\lambda_n s)\,ds\right)\cdot\int_0^\beta \sin(ts)\sin(\lambda_n s)\,ds$$

$$=-\frac{2}{\beta}\sum_n f(\lambda_n)\cdot\frac{\sin(\lambda_n\beta)}{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t), \qquad (68)$$

with sampling points $t=\lambda_n$ ($n$ = integers), and the sampling function :

$$g_n(t)=-\frac{2}{\beta}\cdot\frac{\sin(\lambda_n\beta)}{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t). \qquad (69)$$

The formulae (65) and (68) were obtained by Kroll[14] in connection with the solution of an integral equation.

Example 4

Let us take the Hankel transforms of order $\nu \geq -1/2$ for (1) and (2), and take the Fourier-Bessel series[15] for $F(s)$ :

$$\sqrt{t}\, f(t) \in L_1(0,+\infty),$$

$$K(s,t)=t\,J_\nu(st), \qquad A=(0,+\infty),$$

$$\widehat{K}(t,s)=s\,J_\nu(ts), \qquad B=(0,+\infty),$$

$$\phi_n(s)=J_\nu(j_n s), \qquad D=(0,\beta),$$

with the orthogonality relation :

$$\int_0^\beta s\,J_\nu(j_m s)\,J_\nu(j_n s)\,ds=\gamma_m\cdot\delta_{m,n},$$

and

$$\gamma_n=\frac{\beta^2}{2}J_{\nu+1}^2(j_n\beta),$$

where $j_n\beta$'s ($n$ = 1,2,3, ...) are the positive zeros of $J_\nu(z)$, being arranged in ascending order of magnitude, i.e.

$$J_\nu(j_n\beta)=0. \qquad (70)$$

The expression (12) takes form :

$$f(t) = \frac{2}{\beta^2}\sum_{n=1}^{+\infty} \frac{1}{J_{\nu+1}^2(j_n\beta)}\left(\int_0^\beta s F(s) J_\nu(j_n s)\, ds\right)\cdot \int_0^\beta s J_\nu(ts) J_\nu(j_n s)\, ds$$

$$= -\frac{2}{\beta}\sum_{n=1}^{+\infty} f(j_n)\cdot\frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot\frac{J_\nu(\beta t)}{t^2 - j_n^2}\ , \qquad (71)$$

with sampling points $t = j_n$ ( $n$ = positive integers), and the sampling function :

$$g_n(t) = -\frac{2}{\beta}\cdot\frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot\frac{J_\nu(\beta t)}{t^2 - j_n^2}\ . \qquad (72)$$

The formula (71) was given by the present authors[1].

Example 5

We shall take the Hankel transforms for (1) and (2) just as in Example 4. The function $F(s)$ is assumed to be expanded in the Deni expansion[16] in the domain $D = (0, \beta)$. We shall take

$$K(s, t) = t\, J_\nu(st), \qquad A = (0, +\infty),$$

$$\widetilde{K}(t, s) = s\, J_\nu(ts), \qquad B = (0, +\infty),$$

$$\phi_m(s) = J_\nu(\lambda_n s), \qquad D = (0, \beta),$$

with the orthogonality relation :

$$\int_0^\beta s\, J_\nu(\lambda_m s)\, J_\nu(\lambda_n s)\, ds = \gamma_m \cdot \delta_{m,n}\ ,$$

and

$$\gamma_n = \frac{1}{2\lambda_n^2}\left\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\right\} J_\nu^2(\lambda_n\beta),$$

where $\lambda_n$'s ( $n$ = 1,2,3, ...) are the positive roots (arranged in ascending order of magnitude) of the following equation :

$$\lambda_n\, J_\nu'(\lambda_n\beta) + h\, J_\nu(\lambda_n\beta) = 0, \qquad (73)$$

with a constant $h$.

Then the expression (12) becomes :

$$f(t) = 2\sum_{n=1}^{+\infty} \frac{\lambda_n^2 \cdot}{\{(\lambda_n^2+h^2)\beta^2-\nu^2\} J_\nu^2(\lambda_n\beta)} \left(\int_0^\beta s F(s) J_\nu(\lambda_n s)\, ds\right) \cdot \int_0^\beta s J_\nu(ts) J_\nu(\lambda_n s)\, ds$$

$$= -2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\nu^2\} J_\nu(\lambda_n\beta)} \cdot \frac{t J_\nu'(\beta t) + h J_\nu(\beta t)}{t^2-\lambda_n^2}, \quad (74)$$

with sampling points $t = \lambda_n$ ($n$ = positive integers), and the sampling function :

$$g_n(t) = -2\beta \frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\nu^2\} J_\nu(\lambda_n\beta)} \cdot \frac{t J_\nu'(\beta t) + h J_\nu(\beta t)}{t^2-\lambda_n^2}. \quad (75)$$

The formula (74) was given by Kroll[14].

Letting $h$ tend to infinity in the expressions (73) and (74), we see that they reduce to the following expressions, respectively :

$$J_\nu(\lambda_n\beta) = 0, \quad (76)$$

and

$$f(t) = -\frac{2}{\beta}\sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n}{J_{\nu+1}(\lambda_n\beta)} \cdot \frac{J_\nu(\beta t)}{t^2-\lambda_n^2}. \quad (77)$$

The expressions (76) and (77) are nothing but the expressions (70) and (71), respectively. Accordingly, the expression (71) is a limiting case of (74).

If we put $\nu = 1/2$ in the expression (74), then we obtain

$$f(t) = -2\beta\sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2\sqrt{\lambda_n}}{\{(\lambda_n^2+h^2)\beta^2 - \frac{1}{4}\}\sin(\lambda_n\beta)} \cdot \frac{t\cos(\beta t) + (h-\frac{1}{2\beta})\sin(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}, \quad (78)$$

where $\lambda_n$'s are positive roots of the following equations, being arranged in ascending order of magnitude :

$$\lambda_n \cos(\lambda_n\beta) + (h - \frac{1}{2\beta})\sin(\lambda_n\beta) = 0. \quad (79)$$

In the expression (78), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain :

$$f(t) = -2\beta\sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2 - \frac{1}{4}\}\sin(\lambda_n\beta)} \cdot \frac{t\cot(\beta t) + (h-\frac{1}{2\beta})}{t^2-\lambda_n^2}\sin(\beta t). \quad (80)$$

Further, if we put

$$h - \frac{1}{2\beta} = -N, \tag{81}$$

in the expressions (79) and (80), then the expressions (79) and (80) reduce to (67) and (68), respectively.

If we put

$$h = \frac{1}{2\beta}, \quad N = 0, \tag{82}$$

in the expressions (80) and (79), then we obtain :

$$f(t) = -\frac{2}{\beta}\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{1}{\sin(\lambda_n\beta)}\cdot\frac{t\cos(\beta t)}{t^2-\lambda_n^2}, \tag{83}$$

with

$$\cos(\lambda_n\beta) = 0, \tag{84}$$

i.e.

$$\lambda_n = \frac{2n-1}{2\beta}\pi, \quad (n = 1, 2, 3, \cdots) \tag{85}$$

and

$$\sin(\lambda_n\beta) = (-1)^{n+1}. \quad (n = 1, 2, 3, \cdots) \tag{86}$$

Then the expression (83) reduces to :

$$f(t) = 2\sum_{n=1}^{+\infty} f\left(\frac{2n-1}{2\beta}\pi\right)\cdot(-1)^n\cdot\frac{\beta t\cos(\beta t)}{\beta^2t^2-\frac{(2n-1)^2\pi^2}{4}}. \tag{87}$$

The expression (87) correponds to Someya's sampling theorem (63) with $k = \pm 1/2$ for an odd function $f(t)$.

If we put $\nu = -1/2$ in (74), we obtain :

$$f(t) = 2\beta\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2\sqrt{\lambda_n}}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\cos(\lambda_n\beta)}\cdot\frac{t\sin(\beta t)-(h-\frac{1}{2\beta})\cos(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}, \tag{88}$$

with

$$\lambda_n\sin(\lambda_n\beta) - \left(h-\frac{1}{2\beta}\right)\cos(\lambda_n\beta) = 0. \tag{89}$$

In the expression (88), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain

$$f(t) = 2\beta\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\cos(\lambda_n\beta)}\cdot\frac{t\tan(\beta t)-(h-\frac{1}{2\beta})}{t^2-\lambda_n^2}\cos(\beta t). \tag{90}$$

Further, if we put

$$h - \frac{1}{2\beta} = M, \tag{91}$$

in the expressions (89) and (90), then the expressions (89) and (90) reduce to (64) and (65), respectively.

If we put

$$h = \frac{1}{2\beta}, \qquad M = 0, \tag{92}$$

in the expressions (90) and (89), then we obtain the following expressions :

$$f(t) = \frac{2}{\beta} \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{1}{\cos(\lambda_n \beta)} \cdot \frac{t \sin(\beta t)}{t^2 - \lambda_n^2}, \tag{93}$$

with

$$\sin(\lambda_n \beta) = 0, \tag{94}$$

i.e.

$$\lambda_n = \frac{n\pi}{\beta}, \qquad (n = 1, 2, 3, \dots) \tag{95}$$

and

$$\cos(\lambda_n \beta) = (-1)^n. \qquad (n = 1, 2, 3, \dots) \tag{96}$$

Then the expression (93) reduces to :

$$f(t) = 2 \sum_{n=1}^{+\infty} f\left(\frac{n\pi}{\beta}\right) \cdot (-1)^n \cdot \frac{\beta t \sin(\beta t)}{\beta^2 t^2 - n^2 \pi^2}. \tag{97}$$

The expression (97) corresponds to Shannon's sampling theorem for an even function $f(t)$.

Example 6

We shall take other Hankel transforms due to Weber[17)] for (1) and (2), namely

$$t f(t) \in L_1(p, +\infty), \qquad \alpha \geq p > 0,$$

$$F(s) = \int_A K(s,t) f(t)\, dt, \qquad \text{for } \nu \geq -\frac{1}{2} \tag{98}$$

and

$$G_\nu(\alpha t) f(t) = \int_B \widetilde{K}(t,s) F(s)\, ds, \qquad \text{for } t \in A \tag{99}$$

with

$$K(s,t) = t\, T_\nu(t\alpha, st), \qquad A = (p, +\infty),$$

$$\widetilde{K}(t,s) = s\, T_\nu(t\alpha, ts), \qquad B = (\alpha, +\infty),$$

$$T_\mu(x, z) = Y_\nu(x) J_\mu(z) - J_\nu(x) Y_\mu(z), \tag{100}$$

and

$$G_\nu(z) = J_\nu^2(z) + Y_\nu^2(z), \qquad (101)$$

under the condition that the integral :

$$\int_p^{+\infty} t\, f(t)\, dt < +\infty ,$$

is absolutely convergent.

We shall take another Fourier-Bessel expansion[17] for $F(s)$ of the form :

$$F(s) = \sum_{n=1}^{+\infty} a_n \phi_n(s), \qquad \text{for} \quad s \in D \subseteq B$$

with

$$\phi_n(s) = T_\nu(\lambda_n \alpha, \lambda_n s), \qquad D = (\alpha, \beta),$$

$$\int_\alpha^\beta s\, T_\nu(\lambda_m \alpha, \lambda_m s)\, T_\nu(\lambda_n \alpha, \lambda_n s)\, ds = \gamma_m \cdot \delta_{m,n}, \qquad (102)$$

and

$$\gamma_n = \frac{\beta^2}{2} T_{\nu+1}^2(\lambda_n \alpha, \lambda_n \beta) - \frac{\alpha^2}{2} T_{\nu+1}^2(\lambda_n \alpha, \lambda_n \alpha) = \frac{2}{\pi^2 \lambda_n^2}\left[\frac{J_\nu^2(\lambda_n \alpha)}{J_\nu^2(\lambda_n \beta)} - 1\right], \qquad (103)$$

where $\lambda_n$'s are the positive roots of the equation :

$$T_\nu(\lambda_n \alpha, \lambda_n \beta) = 0, \qquad (104)$$

i.e.

$$Y_\nu(\lambda_n \alpha)\, J_\nu(\lambda_n \beta) = J_\nu(\lambda_n \alpha)\, Y_\nu(\lambda_n \beta).$$

From (99), the expression (12) becomes to :

$$f(t) = \frac{\pi^2}{2} \sum_{n=1}^{+\infty} \frac{\lambda_n^2 J_\nu^2(\lambda_n \beta)}{J_\nu^2(\lambda_n \alpha) - J_\nu^2(\lambda_n \beta)} \cdot G_\nu(\lambda_n \alpha) \cdot f(\lambda_n) \cdot \beta \frac{d}{d\beta} T_\nu(\lambda_n \alpha, \lambda_n \beta) \frac{T_\nu(\alpha t, \beta t)}{(t^2 - \lambda_n^2)\, G_\nu(\alpha t)}$$

$$= -\pi \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 G_\nu(\lambda_n \alpha)\, J_\nu(\lambda_n \alpha)\, J_\nu(\lambda_n \beta)}{J_\nu^2(\lambda_n \alpha) - J_\nu^2(\lambda_n \beta)} \cdot \frac{T_\nu(\alpha t, \beta t)}{(t^2 - \lambda_n^2)\, G_\nu(\alpha t)},$$

$$(105)$$

with sampling points $t = \lambda_n$ $(n = 1,2,3,\ldots)$, and the sampling function :

$$g_n(t) = -\pi \frac{\lambda_n^2 G_\nu(\lambda_n\alpha)\, J_\nu(\lambda_n\alpha)\, J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha) - J_\nu^2(\lambda_n\beta)} \cdot \frac{T_\nu(\alpha t, \beta t)}{(t^2 - \lambda_n^2)\, G_\nu(\alpha t)} . \quad (106)$$

In the expression (105), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/G_\nu(\alpha t)$ and $f(\lambda_n)/G_\nu(\alpha\lambda_n)$ respectively, then we obtain

$$f(t) = -\pi \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2\, J_\nu(\lambda_n\alpha)\, J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha) - J_\nu^2(\lambda_n\beta)} \cdot \frac{T_\nu(\alpha t, \beta t)}{t^2 - \lambda_n^2} , \quad (107)$$

with sampling points $t = \lambda_n$ and the sampling function :

$$g_n(t) = -\pi \frac{\lambda_n^2\, J_\nu(\lambda_n\alpha)\, J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha) - J_\nu^2(\lambda_n\beta)} \cdot \frac{T_\nu(\alpha t, \beta t)}{t^2 - \lambda_n^2} . \quad (108)$$

The expressions (63),(65), (68), (71), (74),(78), (80), (87), (88), (90), (97), (105), and (107), are examples of the sampling formulae derived from our generalized sampling theorem (12).

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