数理解析研究所講究録
第 90 巻 1970 年 1-12

I　最適制御

A MINIMUM COST CONTROL PROBLEM IN BANACH SPACE

N. MINAMIDE AND K. NAKAMURA

1. INTRODUCTION.

In recent years, Porter and Williamd [1] considered the following abstract version of the minimum effort control problem by function space methods. Let S be a bounded linear transformation between Banach spaces X and Z, respectively. With S onto Z and $\eta\varepsilon Z$ arbitrary, find (if one exists) a preimage of η with minimum norm. It was shown that for this problem to have a unique solution it is necessary and sufficient that X be both reflexive and rotund. Furthermore, the solution was completely characterized in terms of a hyperplane. In [2], [3], several extensions and generalizations of the initial problem were also considered.

In the present paper the following related problem is considered. Let X,Y and Z be Banach spaces, T a bounded linear transformation from X into Y, and S a bounded linear transformation from X onto Z. Let Ω X be a closed convex body containing the origin in its interior. Also, let $J(\cdot,\cdot)$ be a continuous convex functional defined on $X\times Y$ such that

(1.1) $\qquad J(x,y)\geq 0, \qquad$ *for all* $(x,y)\varepsilon X\times Y$,

(1.2) $\qquad J(0,0)=0,$

(1.3) $\qquad J(x,y)\longrightarrow+\infty, \qquad$ *as* $\|x\|+\|y\|\longrightarrow+\infty$[1].

Problem (P). With $\xi\varepsilon Y$ and $\eta\varepsilon int(S(\Omega))$, the interior of the image of Ω under S, arbitrary, find an element (if one exists) $u\varepsilon\Omega$ satisfying $Su=\eta$ which minimizes $J(u,\xi-Tu)$.

Such an element will be called an optimal solution. Interesting and important cases may arise when the functional J and the constraint set Ω are described in terms of norms, among which are the following:

Problem (P$_1$). $\min\limits_{\|u\|\leq\rho}\|\xi-Tu\|$ *subject to* $Su=\eta$, $(0<\rho<+\infty)$,

Problem (P$_2$). $\min\limits_{\|u\|\leq\rho}\{\|u\|^p+\|\xi-Tu\|^p\}$, *subject to* $Su=\eta$, $(1<p<+\infty, 0<\rho\leq\infty)$

Our main objectives are to discuss existence and uniqueness of the solution and to characterize it in terms of a hyperplane. Problem (P$_1$) has been studied by Porter [5] and Kirillova [6] when $S=\eta=0$, while

[1] This assumption may be removed if Ω is bounded.

Problem (P_2) was considered by Porter and Williams [2] when p=2 and $\rho=+\infty$.

In the articles [1]-[3], a key role was played by the Minkowski functional associated with $S(U_X)$ the image of the unit ball under S, and by the Hahn-Banach produced hyperplane of support to a convex body at each of its boundary points. In this study we shall define the extended version of the Minkowski functional. This extended version and the supporting hyperplane are our principal tools for characterization of the solution.

2. Some preliminaries.

Throghout the paper we shall restrict attention, without loss of generality, to real spaces. Let B be a real Banach space and B' the conjugate of B. The unit ball and unit sphere of B will be denoted by U_B and ∂U_B, respectively. Let K B be a convex set. For every $\phi\varepsilon B'$ let the number $\langle K,\phi\rangle$ be defined by

$$\langle K,\phi\rangle = \sup_{x\in K}\langle x,\phi\rangle,$$

and suppose that ϕ attains its supremum $\langle K,\phi\rangle$ on K at the vector $x_0\varepsilon K$. We shall denote by $[\phi:K]$ the set of all such vectors and shall refer to it as an extremal of ϕ with respect to K. Especially, $[\phi:U_B]$ will be denoted by $\bar{\phi}$, usually called an extremal of ϕ (See [1] or [4]). For convenience we shall identify a suitable element $x\varepsilon[\phi:K]$ with the set $[\phi:K]$ itself. It may be obvious from the context whether $[\phi:K]$ indicates a member or a set. Let $B_1\times B_2$ be a product Banach space equipped with the usual product topology. Let $K\subset B_1\times B_2$ be a convex set. Motivated by the above identification, we shall loosely set

$$([\phi_1:K],[\phi_2:K]) \triangleq [(\phi_1,\phi_2):K], \quad (\phi_1,\phi_2)\in(B_1\times B_2)'=B_1'\times B_2'.$$

In order to discuss uniqueness of the solution, the following concept is needed. A convex body K in B is called rotund (or strictly convex) if K contains no straight-line segments in its boundary. A Banach space B is called rotund if its unit ball is rotund. For each $\phi\varepsilon B'$ and a convex body $K\subset B$, $[\phi:K]$ has at most one element if and only if K is rotund. Moreover, since rotundity of B implies weak compactness of U_B (Milman's Theorem, see [7]), $\bar{\phi}$ has exactly one element if and only if B is rotund.

3. THE SOLUTION TO PROBLEM (P).

To motivate what follows, let us first suppose that $_0$ is an optimal solution with $J(u_0, \xi - Tu_0) > 0$. Then,

(3.1) $\quad \alpha_0 \triangleq J(u_0, \xi - Tu_0) \leqq J(u, \xi - Tu), \qquad$ for all $u \in \Omega \cap S^{-1}(\eta)$,

(3.2) $\quad Su_0 = \eta$.

where $S^{-1}(\eta)$ denotes the set of all preimages of η for further study. Let us define the set $J(\alpha)$ by

$$J(\alpha) = \{(x,y) \mid J(x,y) \leqq \alpha, \ (x,y) \in X \times Y\},$$

and denote by $\partial J(\alpha)$ the boundary of $J(\alpha)$. Clearly, for $\alpha > 0$, $J(\alpha)$ is a closed convex body and $\partial J(\alpha) = \{(x,y) \mid J(x,y) = \alpha, \ (x,y) \varepsilon X \times Y\}$. We consider the mapping $\bar{T}$ of $X \times Y$ to $Y \times Z$ defined by

$$\bar{T} : (u, y) \longrightarrow (Tu + y, Su), \qquad (u, y) \in X \times Y.$$

It then intuitively clear, from (3.1) and (3.2), that

$$(\xi, \eta) = (Tu_0 + (\xi - Tu_0), Su_0) = \bar{T}(u_0, \xi - Tu_0) \in \partial \bar{T}(J(\alpha_0) \cap (\Omega \times Y)) \cap \bar{T}(J(\alpha_0) \cap (\Omega \times Y)),$$

where $A \times B$ denotes the rectangular set, i.e., $A \times B = \{(a,b) \mid a \varepsilon A, \ b \varepsilon B\}$. We shall show this inference in Lemma 3.3 below.

Lemma 3.1. If $(\xi, \eta) \varepsilon \partial \bar{T}(J(\alpha) \cap (\Omega \times Y))$, *with* $\eta \varepsilon int(S))$, *then*

$$J(u, \xi - Tu) \geqq \alpha, \qquad \textit{for all } u \varepsilon \Omega \cap S^{-1}(\eta).$$

Proof. Suppose that for some $u_0 \varepsilon \Omega \cap S^{-1}(\eta)$, $J(u_0, \xi - Tu_0) < \alpha (\neq 0)$. By the assumption that $\eta \varepsilon int(S(\Omega))$, there exists an element $u \varepsilon int(\Omega)$ which satisfies $S\bar{u} = \eta$. Set $u_\lambda = \lambda \bar{u} + (1-\lambda) u_0$. It then follows easily that for sufficiently small $\lambda > 0$, $u_\lambda \varepsilon int(\Omega)$ satisfies $Su_\lambda = \eta$ and $J(u_\lambda, \xi - Tu_\lambda) < \alpha$. Hence by appealing again to the continuity of J, a neighborhood $U \times V$ of the origin in $X \times Y$ exists such that

(3.3) $\quad (u_\lambda, \xi - Tu_\lambda) + U \times V \subset J(\alpha) \cap (\Omega \times Y)$.

Operating on (3.3) with $\bar{T}$ and noting that $\bar{T}$ is open-mapping, we have $(\xi, \eta) \varepsilon int\{\bar{T}(J(\alpha) \cap (\Omega \times Y))$, which contradicts our hypothesis.

Corollary. Problem (P) has a solution for each $(\xi, \eta) \in \partial \bar{T}(J(\alpha) \cap (\Omega \times Y))$ *if and only if* $\bar{T}(J(\alpha) \cap (\Omega \times Y))$ *is closed in* $Y \times Z$.

Henceforth, we shall assume that X is a reflexive Banach space[1]. The following lemma may justify this point.

Lemma 3.2. In order for Problem (P) to have a solution for every bounded linear transformation, convex continuous functional J and closed convex body Ω, *it is necessary and sufficient that X be a reflexive Banach space.*

Proof. The necessity is shown in [4, §4.3]. Hence we shall show

[1] This assumption can be modified as in [4, §4.3].

the sufficiency. By the corollary to Lemma 3.1, it suffices to show that $\bar{T}(J(\alpha)\cap(\Omega\times Y))$ is closed in $Y\times Z$. Let $\{(u_n,y_n)\}\subset J(\alpha)\cap(\Omega\times Y)$ be a sequence such that $(\xi_n,\eta_n)=\bar{T}(u_n,y_n)$ converges to (ξ_0,η_0). Since, in view of (1.3), $\{u_n\}$ is bounded, and since in a reflexive Banach space every bounded sequence contains a weakly convergent subsequence, we may suppose that $\{u_n\}$ itself converges weakly to $u_0\in X$: $u\xrightarrow{w}u_0$. That Ω is closed, convex implies $u_0\in\Omega$. Furthermore, by the weak continuity of T and S we have

(3.4) $$y_n=\xi_n-Tu_n\xrightarrow{w}\xi_0-Tu_0=y_0\in Y,$$

(3.5) $$\eta_n=Su_n\xrightarrow{w}Su_0=\eta_0.$$

Since $J(\alpha)$ is weakly closed, we get $(u_n,y_n)\xrightarrow{w}(u_0,y_0)\in J(\alpha)$, which, combined with (3.4),(3.5), implies $(\xi_0,\eta_0)\in\bar{T}(J(\alpha)\cap(\Omega\times Y))$.

For each $(\xi,\eta)\in Y\times int(S(\Omega))$, we consider the set $C(\xi,\eta)$ defined by

$$C(\xi,\eta)=\{\alpha\mid(\xi,\eta)\in\bar{T}(J(\alpha)\cap(\Omega\times Y))\}.$$

Obviously $C(\xi,\eta)$ is non-void. We then define

$$p((\xi,\eta);\Omega)=\inf\{\alpha\mid\alpha\in C(\xi,\eta)\}.$$

Lemma 3.3. With α_0 *defined by* $\alpha_0=p((\xi,\eta);\Omega)$ *and* $\alpha_0>0$, *then*

$$(\xi,\eta)\in\partial T(J(\alpha_0)\cap(\Omega\times Y))\cap T(J(\alpha_0)\cap(\Omega\times Y)).$$

Proof. Let $\{(u_n,y_n)\}$ and a monotone decreasing sequence $\{\alpha_n\}$, with $\alpha\downarrow\alpha_0$, be such that

$$(\xi,\eta)=\bar{T}(u_n,y_n),\qquad (u_n,y_n)\in J(\alpha_n)\cap(\Omega\times Y),\quad (n=1,2,\cdots)$$

Then, arguing just as in the proof of the previous lemma, we see that there exists a subsequence $\{(u_k,y_k)\}$ which converges weakly to $(u_0,y_0)\in\Omega\times Y$, so that $(\xi,\eta)=\bar{T}(u_0,y_0)$. Since a convex continuous functional J is weakly lower semi-continuous, we have

$$J(u_0,y_0)\leq\liminf_{k\to\infty}J(u_k,y_k)=\alpha_0$$

Thus, $(\xi,\eta)\in\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$. To see $(\xi,\eta)\in\partial\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$, suppose that $(\xi,\eta)\notin\partial\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$. $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ being a closed convex body, there exists a neighborhood W of the origin in $Y\times Z$ such that $\{(\xi,\eta)+W\}\subset\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$. Since for sufficiently small $\lambda>0$, it follows that

$$(\xi,\eta)\in\frac{1}{1+\lambda}\bar{T}(J(\alpha_0)\cap(\Omega\times Y))\subset\bar{T}(J(\alpha_0)/(1+\lambda))\cap(\Omega\times Y)),$$

which contradicts definition of α_0.

We now state the main result in this section.

Proposition 3.1. Let X be a reflexive Banach space. Then there exists a solution of (P). If $p((\xi,\eta);\Omega)>0$, *then the necessary condition* *for* u_0 *to be optimal is that* u_0 *takes the following form*

(3.6) $$u_0=[T'\phi_1+S'\phi_2 : J(\alpha_0)\cap(\Omega\times Y)]$$

where $\alpha_0(=p((\xi,\eta);\Omega))$ *and* (ϕ_1,ϕ_2) *of norm 1 satisfy any of the following equivalent conditions:*

(1) $$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle \geqq \langle \bar{T}(J(\alpha_0)\cap(\Omega\times Y)),(\phi_1,\phi_2)\rangle,$$
$$(\xi,\eta)\in\bar{T}(J(\alpha_0)\cap(\Omega\times Y)),$$

(2) $$\xi-T[T'\phi_1+S'\phi_2 : J(\alpha_0)\cap(\Omega\times Y)]=[\phi_1 : J(\alpha_0)\cap(\Omega\times Y)],$$
$$S[T'\phi_1+S'\phi_2 : J(\alpha_0)\cap(\Omega\times Y)]=\eta,$$

(3) $$\max\{\langle\xi,\psi_1\rangle+\langle\eta,\psi_2\rangle+\min_{u\in\Omega}\min_{y\in Y}(J(u,y)-\langle u,T'\psi_1+S'\psi_2\rangle-\langle y,\psi_1\rangle)\}$$
$$=\langle\xi,\phi_1\rangle+\langle\eta,\phi_2\rangle+\min_{u\in\Omega}\min_{y\in Y}(J(u,y)-\langle u,T'\phi_1+S'\phi_2\rangle-\langle y,\phi_1\rangle)=\alpha_0$$

Conversely, suppose that $\{\alpha_0,(\phi_1,\phi_2)\}$ *satisfies any of conditions (1)-(3). Then the suitable version* $u_0=[T'\phi_1+S'\phi_2:J(\alpha_0)\ (\Omega\times Y)]$ *is optimal. Moreover, if* $J(\alpha_0)\cap(\Omega\times Y)$ *is rotund, the solution is unique.*

Proof. We first show the necessity. Let u_0 be an optimal solution. Then, by the preceding lemma we have $J(u_0,\xi-Tu_0)=p((\xi,\eta);\Omega)=\alpha_0$, $Su_0=\eta$ and $(\xi,\eta)\varepsilon\partial\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$. Since $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ is a convex body, there exists a hyperplane $(\phi_1,\phi_2)\neq 0$ which supports $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ at (ξ,η):

(3.7) $$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\langle\bar{T}(J(\alpha_0)\cap(\Omega\times Y)),(\phi_1,\phi_2)\rangle=\langle J(\alpha_0)\cap(\Omega\times Y),(T'\phi_1+S'\phi_2,\phi_1)\rangle$$

On the other hand, we have

(3.8) $$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle=\langle(Tu_0+(\xi-Tu_0),Su_0),(\phi_1,\phi_2)\rangle$$
$$=\langle(u_0,\xi-Tu_0),(T'\phi_1+S'\phi_2,\phi_1)\rangle\leqq\langle J(\alpha_0)\cap(\Omega\times Y),(T'\phi_1+S'\phi_2,\phi_1)\rangle$$

Hence combining (3.7) with (3.8) yields

(3.9) $$(u_0,\xi-Tu_0)=[(T'\phi_1+S'\phi_2,\phi_1) : J(\alpha_0)\cap(\Omega\times Y)]$$
$$\triangleq([T'\phi_1+S'\phi_2 : J(\alpha_0)\cap(\Omega\times Y)],[\phi_1 : J(\alpha_0)\cap(\Omega\times Y)]),$$

from which Eq.(3.6) and (2) follow. To see (3), note that $(\xi,\eta)\varepsilon\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ implies and is implied by

(3.10) $$\langle(\xi,\eta),(\psi_1,\psi_2)\rangle\leqq\langle J(\alpha_0)\cap(\Omega\times Y),(T'\psi_1+S'\psi_2,\psi_1)\rangle,$$
$$\text{for all } (\psi_1,\psi_2)\in Y'\times Z',$$

where equality holds if and only if $(\psi_1,\psi_2)\neq 0$ supports $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ at (ξ,η). Now, by the Kuhn-Tucker Theorem in a locally convex linear topological space [8] we have

(3.11) $$\langle J(\alpha_0)\cap(\Omega\times Y),(T'\psi_1+S'\psi_2,\psi_1)\rangle=\sup_{u\in\Omega,\ J(u,y)\leqq\alpha_0}\langle(u,y),(T'\psi_1+S'\psi_2,\psi_1)\rangle$$
$$=\sup_{u\in\Omega}\sup_{y\in Y}\{\langle(u,y),(T'\psi_1+S'\psi_2,\psi_1)\rangle-\lambda(J(u,y)-\alpha_0)\}$$

for some $\lambda>0$. Substituting (3.11) into (3.10), deviding by λ both sides of the resulting equation and setting $\|(\psi_1,\psi_2)/\lambda\|=1$ for normalization, we obtain (3).

To show the converse part, note that conditions (1)-(3) are equivalent to one another, as will be seen by following the above argument in reverse order. Hence, if $\{\alpha_0,(\phi_1,\phi_2)\}$ satisfies any of conditions (1)-(3), we see that in any case, $(\xi,\eta)\in\partial\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ and (ϕ_1,ϕ_2) supports $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ at (ξ,η). Let $(u_0,y_0)\in J(\alpha_0)\cap(\Omega\times Y)$ be any preimage of (ξ,η). It then follows from (3.9) and Lemma 3.1 that $u_0\in[T'\phi_1+S'\phi_2:J(\alpha_0)\cap(\Omega\times Y)]$ is an optimal solution. Finally, it remains to prove the last assertion. This is easily done as follows. Let $u_1,u_2\in\Omega$ be two solutions and $(\phi_1,\phi_2)\neq 0$ a hyperplane of support to $\bar{T}(J(\alpha_0)\cap(\Omega\times Y))$ at (ξ,η). In view of (3.7), we have

(3.12) $\langle(u_i,\xi-Tu_i),(T'\phi_1+S'\phi_2,\phi_1)\rangle\geqq\langle J(\alpha_0)\cap(\Omega\times Y),(T'\phi_1+S'\phi_2,\phi_1)\rangle,\quad i=1,2.$

Eq.(3.12) tells us that $(T'\phi_1+S'\phi_2,\phi_1)\neq 0$ supports $J(\alpha_0)\cap(\Omega\times Y)$ at $(u_1,\xi-Tu_1)$ and at $(u_2,\xi-Tu_2)$ as well. By rotundity of $J(\alpha_0)\cap(\Omega\times Y)$, this implies $(u_1,\xi-Tu_1)=(u_2,\xi-Tu_2)$. Consequently, the solution is unique.

Remark. The simplest problem in the calculus of variations is that of finding, in a class of arcs

$$x(t),\qquad (t_0\leqq t\leqq t_1)$$

joining two fixed points $x(t_0)=x_0$ and $x(t_1)=x_1$, one which minimizes an integral of the form $J(\dot{x},x)=\int_{t_0}^{t_1}f(\dot{x}(t),x(t),t)\,dt$, $(\dot{x}(t)=dx(t)/dt)$

Problem (P) may be interpreted as the function space version of this problem, if we set

$$u(t)=\dot{x}(t),\qquad \xi(t)=x_0,\qquad \eta=x_1-x_0,\qquad Su=\int_{t_0}^{t_1}u(s)\,ds,$$
$$(Tu)(t)=\int_{t_0}^{t}u(s)\,ds,\quad (t_0\leqq t\leqq t_1).$$

Suppose that the solution u_0 lies in the interior of Ω and $J(u,y)$ is Gateaux differentiable. Then, it turns out from Proposition 3.1 that $J(u,y)$ is Fréchet differentiable at $(u_0,\xi-Tu_0)$, and we have

(3.13) $\phi_1=\nabla_y J(u_0,\xi-Tu_0),$

(3.14) $T'\nabla_y J(u_0,\xi-Tu_0)+S'\phi_2=\nabla_u J(u_0,\xi-Tu_0),$

(3.15) $J(u,\xi-Tu_0)-J(u_0,\xi-Tu_0)-\langle u-u_0,\nabla_u J(u_0,\xi-Tu_0)\rangle\geqq 0,$ for all $u\in\Omega$.

Eq.(3.14) and Eq.(3.15) are the versions of Euler equation and of Weierstrass condtion, respectively.

4. MINIMIZATION PROBLEMS WITH NORM CRITERIA.

In the previous section, a function space minimum cost control problem with convex functional criteria was considered. If the functional J and the constraint set Ω are specified in terms of norms, more explicit characterization of the solution is possible. We shall now treat these cases below.

4.1. The solution to Problem (P_1).

As an important special case of Problem (P), we set

$$J(u, \xi - Tu) = \|\xi - Tu\|, \quad \Omega = \{u \mid \|u\| \leq \rho,\ u \in X\}. \quad (0 < \rho < +\infty)$$

We shall make the following definition.

Definition. We shall say that the pair (ξ,η), with $\eta \in int(S(\Omega))$, is regular if

$$\min_{\substack{\|u\| \leq \rho \\ Su = \eta}} \|\xi - Tu\| > \inf_{Su = \eta} \|\xi - Tu\|$$

holds.

Note that if the mapping $\bar{S}: u \to (Tu, Su)$, has dense range and if $p((\xi,\eta);\Omega)$ is positive, then the pair (ξ,η), with $\eta \in int(S(\Omega))$, is regular.

Theorem 4.1. Let X be a reflexive Banach space, T an into-mapping and S an onto-mapping. Then Problem (P_1) has a solution. Suppose that the pair (ξ,η) is regular. In this case, $u_0 \in \partial \rho U_X$ is optimal if and only if

$$u_0 = \rho \overline{T'\phi_1 + S'\phi_2}, \tag{4.1}$$

where (ϕ_1,ϕ_2) of norm 1 may be determined by either of the following

(1)

$$\xi - \rho T(\overline{T'\phi_1 + S'\phi_2}) = \frac{(\langle \xi, \phi_1 \rangle + \langle \eta, \phi_2 \rangle - \rho \|T'\phi_1 + S'\phi_2\|)}{\|\phi_1\|} \bar{\phi}_1 \tag{4.2}$$

$$\rho S(\overline{T'\phi_1 + S'\phi_2}) = \eta \tag{4.3}$$

(2)

$$\max_{(\phi_1 \neq 0,\, \phi_2)} \frac{\langle \xi, \phi_1 \rangle + \langle \eta, \phi_2 \rangle - \rho \|T'\phi_1 + S'\phi_2\|}{\|\phi_1\|} \tag{4.4}$$

Moreover, if X is rotund, the solution is unique.

Proof. Let us first note that a hyperplane (ϕ_1,ϕ_2) with $\phi_1 \neq 0$ supports $\bar{T}(\rho U_X \times \alpha_0 U_Y)$ at (ξ,η):

$$\langle (\xi,\eta),(\phi_1,\phi_2) \rangle \geq \langle \rho U_X, T'\phi_1 + S'\phi_2 \rangle + \langle \alpha_0 U_Y, \phi_1 \rangle = \rho \|T'\phi_1 + S'\phi_2\| + \alpha_0 \|\phi_1\|, \tag{4.5}$$

where $\alpha_0 = p((\xi,\eta);\Omega)$. In fact, suppose contrary that $\phi_1 = 0$. Then $\phi_2 \neq 0$ and by (4.5) we have

$$\langle \eta, \phi_2 \rangle \geq \rho \|S'\phi_2\|$$

which contradicts the assumption $\eta \in int(S(\Omega))$. We shall next show that if (ξ,η) is a regular pair, then $T'\phi_1 + S'\phi_2 \neq 0$, whence $u_0 \in \partial \rho U_X$ If $T'\phi_1 + S'\phi_2 = 0$ is true, then by (4.5) we have, for all $u \in S^{-1}(\eta)$,

$$\langle \xi, \phi_1 \rangle + \langle \eta, \phi_2 \rangle = \langle \xi - Tu + Tu, \phi_1 \rangle + \langle Su, \phi_2 \rangle = \langle \xi - Tu, \phi_1 \rangle \geq \alpha_0 \|\phi_1\|$$

Hence

$$\|\xi - Tu\| \|\phi_1\| \geq \langle \xi - Tu, \phi_1 \rangle \geq \alpha_0 \|\phi_1\|; \quad \text{for all } u \in S^{-1}(\eta),$$

which, by regularity of the pair (ξ,η), is impossible. Now, by virtue of Proposition 3.1 the proof of this theorem may be completed if it is shown that the solution is unique under the hypothesis in the theorem. Let u_0 be any solution of (P_1). Then, in view of (4.5) we have

$$\langle u_0, T'\phi_1+S'\phi_2\rangle=\rho\|T'\phi_1+S'\phi_2\|=\langle\rho\bar{v}_X, T'\phi_1+S'\phi_2\rangle, \tag{4.6}$$

$$\langle\xi-Tu_0,\phi_1\rangle=\alpha_0\|\phi_1\|=\langle\alpha_0\bar{v}_Y,\phi_1\rangle \tag{4.7}$$

Eq.(4.6) shows that if (ϕ_1,ϕ_2) supports $\bar{T}(\rho U_X\times\alpha_0U_Y)$ at (ξ,η), then $T'\phi_1+S'\phi_2(\neq 0)$ necessarily defines the supporting hyperplane of ρU_X at u_0. Hence if X is rotund, then $u_0=\rho\overline{T'\phi_1+S'\phi_2}$ is unique.

Remark 1. It is interesting to observe that Eq. (4.3) coincides with the formal differential of the dual problem (4.4).

Remark 2. Eq.(4.2) can be replaced by

$$\|\xi-\rho T(\overline{T'\phi_1+S'\phi_2})\|=\frac{\langle\xi,\phi_1\rangle+\langle\eta,\phi_2\rangle-\rho\|T'\phi_1+S'\phi_2\|}{\|\phi_1\|}. \tag{4.8}$$

This follows from the fact that

$$\|\xi-\rho T(\overline{T'\phi_1+S'\phi_2})\|\geqq p((\xi,\eta);\Omega)\geqq(\langle\xi,\phi_1\rangle+\langle\eta,\phi_2\rangle-\rho\|T'\phi_1+S'\phi_2\|)/\|\phi_1\|$$

holds for all (ϕ_1,ϕ_2) satisfying $\rho T(\overline{T'\phi_1+S'\phi_2})=\eta$.

Corollary. Suppose that (ξ,η) is a regular pair. Then the unique solution of the Hilbert space version of Problem (P_1) is given by

$$u_0=(\lambda I+T^*T)^{-1}T^*\xi-(\lambda I+T^*T)^{-1}S^*\{S(\lambda I+T^*T)^{-1}S^*\}^{-1}\{S(\lambda I+T^*T)^{-1}T^*\xi-\eta\},$$

where λ is a constant uniquely determined by $\|u_0\|=\rho$.

Proof. In a Hilbert space the extremal of x takes the form $\bar{x}=x/\|x\|$. Hence by (4.1), (4.2) and (4.3) we have

$$u_0=\rho(T^*\phi_1'+S^*\phi_2'), \tag{4.1'}$$

$$\rho T(T^*\phi_1'+S^*\phi_2')+\alpha\phi_1'=\xi, \tag{4.2'}$$

$$\rho S(T^*\phi_1'+S^*\phi_2')=\eta. \tag{4.3'}$$

Here, we put $(\phi_1',\phi_2')=(\phi_1,\phi_2)/\|T^*\phi_1+S^*\phi_2\|$ and $\alpha=\|T^*\phi_1+S^*\phi_2\|p((\xi,\eta);\Omega)$. Operate on (4.2') with T^* and solve for $T^*\phi_1'$ to obtain

$$T^*\phi_1'=(\alpha I+T^*T)^{-1}(T^*\xi-\rho T^*TS^*\phi_2'). \tag{4.9}$$

By substituting (4.9) into (4.3') and making use of the relation that

$$SS^*-\rho S(\alpha I+T^*T)^{-1}T^*TS^*=S\{\alpha(\alpha I+\rho T^*T)^{-1}\}S^*$$

ϕ_2' is found

$$\phi_2'=(\alpha\rho)^{-1}\{S(\alpha I+T^*T)^{-1}S^*\}^{-1}\{\eta-S(\alpha I+\rho T^*T)^{-1}T^*\xi\}, \tag{4.10}$$

where $\{S(\alpha I+\rho T^*T)^{-1}S^*\}^{-1}$ exists and defines a continuous linear oprator as will be seen easily. This corollary follows from (4.1'), (4.9) and (4.10).

4.2. The solution for Problem (P_2).

Another interesting case follows if we specify

$$J(u, \xi - Tu) = \|u\|^p + \|\xi - Tu\|^p \quad (1<p<+\infty), \quad \Omega = \{u \mid \|u\| \leqq \rho,\ u \in X\} \quad (0<\rho \leqq +\infty),$$

where we have made the obvious convention for $\rho=+\infty$. Consider a product Banach space $X \times Y$ equipped with the norm

$$\|(u, y)\| = (\|u\|^p + \|y\|^p)^{1/p}, \quad u \in X, \quad y \in Y \quad (1<p<+\infty)$$

We then observe that for each $(u, y) \in X \times Y$ and $(\varphi_1, \varphi_2) \in X' \times Y'$,

$$|\langle (u, y), (\varphi_1, \varphi_2) \rangle| = |\langle u, \varphi_1 \rangle + \langle y, \varphi_2 \rangle| \leqq (\|u\|^p + \|y\|^p)^{1/p} (\|\varphi_1\|^q + \|\varphi_2\|^q)^{1/q},$$

from which the extremal of $(\varphi_1, \varphi_2) \in X' \times Y'$, if one exists, takes the form

(4.11) $$(\varphi_1, \varphi_2) = \left(\{(\|\varphi_1\|^q + \|\varphi_2\|^q)^{-1/q} \|\varphi_1\|\}^{q-1} \bar{\varphi}_1,\ \{(\|\varphi_1\|^q + \|\varphi_2\|^q)^{-1/q} \|\varphi_2\|\}^{q-1} \bar{\varphi}_2 \right),$$

where $1/p + 1/q = 1$ (See [1] or [4]).

Theorem 4.2. Let X be a reflexive Banach space, T an into-mapping, and S an onto-mapping. Then there exists an element $u \in \rho U_X$ satisfying $Su=\eta$ which minimizes $\|u\|^p + \|\xi - Tu\|^p$ $(1<p<+\infty)$. In order for u_0 to be optimal, it is necessary and sufficient that u_0 is of the form:

(4.12), (4.13)
$$u_0 = \begin{cases} \dfrac{(\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle) \|T'\varphi_1 + S'\varphi_2\|^{q-1}}{\|T'\varphi_1 + S'\varphi_2\|^q + \|\varphi_1\|^q} \overline{T'\varphi_1 + S'\varphi_2}, & \text{if } \rho_0 \leqq \rho, \\ \rho \overline{T'\varphi_1 + S'\varphi_2}, & \text{if } \rho_0 > \rho. \end{cases}$$

The functional (ϕ_1, ϕ_2) of norm 1 may be computed be either of the following

(I) (4.14), (4.15)
$$\begin{cases} \xi - \mu_1 T(\overline{T'\varphi_1 + S'\varphi_2}) = \mu_2 \bar{\varphi}_1, \\ \mu_1 S(\overline{T'\varphi_1 + S'\varphi_2}) = \eta \end{cases} \quad \text{if } \rho_0 \leqq \rho,$$
$$\begin{cases} \xi - \rho T(\overline{T'\varphi_1 + S'\varphi_2}) = \mu_3 \bar{\varphi}_1, \\ \rho S(\overline{T'\varphi_1 + S'\varphi_2}) = \eta \end{cases} \quad \text{if } \rho_0 > \rho,$$

(4.16) $$\max_{(\varphi_1, \varphi_2)} \frac{\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle}{(\|T'\varphi_1 + S'\varphi_2\|^q + \|\varphi_1\|^q)^{1/q}} \quad \text{if } \rho_0 \leqq \rho,$$

(4.17) $$\max_{(\varphi_1 \neq 0, \varphi_2)} \frac{\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle - \rho \|T'\varphi_1 + S'\varphi_2\|}{\|\varphi_1\|} \quad \text{if } \rho_0 > \rho.$$

where

$$\mu_1 = (\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle) \|T'\varphi_1 + S'\varphi_2\|^{q-1} / (\|T'\varphi_1 + S'\varphi_2\|^q + \|\varphi_1\|^q),$$

$$\mu_2 = (\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle) \|\varphi_1\|^{q-1} / (\|T'\varphi_1 + S'\varphi_2\|^q + \|\varphi_1\|^q),$$

$$\mu_3 = (\langle \xi, \varphi_1 \rangle + \langle \eta, \varphi_2 \rangle - \rho \|T'\varphi_1 + S'\varphi_2\|) / \|\varphi_1\|,$$

and ρ_0 is the norm of the solution for $\rho=+\infty$. Moreover, if X is rotund, the solution is unique.

Proof. Let us note that for u_1, $u_2 \in X$ and $0<\lambda<1$, where equality holds if and only if $\|u_1\|=\|u_2\|$ and $(1-\lambda)\|u_1\|+\|\lambda u_2\|=\|(1-\lambda)u_1+\lambda u_2\|$. Thus, if u_1 and u_2 are distinct solutions for $\rho=+\infty$, then $\|u_1\|=\|u_2\|$, and hence $\rho_0=\|u_1\|=\|u_2\|$ is uniquely determined. Furthermore, we notice from (4.18) that rotundity of X guarantees uniqueness of the solution for Problem (P_2) (See [4, §4.2] concerning the equivalent properties for rotundity). This proves the last statement. To complete the proof it suffices to show that the solution of (P_2) is given by (4.12) of (4.13) in accordance with $\rho_0\leq\rho$ or $\rho_0>\rho$ Now, if $\rho_0\leq\rho$, the result follows easily from Proposition 3.1 and Eq.(4.11). Hence suppose that $\rho_0>\rho$. We observe, in this case, that $(\xi,\eta)\in\overline{T}(p((\xi,\eta);X)U_{X\times Y}\cap(\rho U_X\times Y))$, whence $p((\xi,\eta);\rho U_X)>p((\xi,\eta);X)$, and so $(\xi,\eta)\in int\{\overline{T}(p((\xi,\eta);\rho U_X)U_{X\times Y})\}=\overline{T}(int\{p((\xi,\eta);\rho U_X)U_{X\times Y}\})$. But obviously we have

$$(\xi,\eta)\in\partial\overline{T}(p((\xi,\eta);\rho U_X)U_{X\times Y}\cap(\rho U_X\times Y))\subset\overline{T}(\partial\{p((\xi,\eta);\rho U_X)U_{X\times Y}\cap(\rho U_X\times Y)\})$$
$$\subset\overline{T}(\partial\{p((\xi,\eta);\rho U_X)U_{X\times Y}\}\cup\partial\{\rho U_X\times Y\})$$

We thus see that $(\xi,\eta)\in\overline{T}(\partial\rho U_X\times\partial\{(p((\xi,\eta);\rho U_X)-\rho^p)^{1/p}U_Y\})$. The rest of the argument may be carried out just as in the proof of the previous theorem.

Remark. If $\rho_0>\rho$ holds, then (ξ,η) is necessarily a regular pair. To see this, let $\bar{u}$ be a solution of (P_2) for $\rho=+\infty$. It then follows that

$$\min_{\substack{\|u\|\leq\rho\\ Su=\eta}}\{\|u\|^p+\|\xi-Tu\|^p\}=\rho^p+\min_{\substack{\|u\|\leq\rho\\ Su=\eta}}\|\xi-Tu\|^p\geq\min_{u\in S^{-1}(\eta)}\{\|u\|^p+\|\xi-Tu\|^p\}=\|\bar{u}\|^p+\|\xi-T\bar{u}\|^p.$$

Hence

$$\min_{\substack{\|u\|=\rho\\ Su=\eta}}\{\|\xi-Tu\|^p\}-\|\xi-T\bar{u}\|^p\geq\rho_0^p-\rho^p>0$$

Corollary. The unique solution of the Hilbert space version of Problem (P_2) is given by

$$u_0=\begin{cases}(I+T^*T)^{-1}T^*\xi-(I+T^*T)^{-1}S^*\{S(I+T^*T)^{-1}S^*\}^{-1}\{S(I+T^*T)^{-1}T^*\xi-\eta\} & \text{if } \rho_0\leq\rho,\\ (\lambda I+T^*T)^{-1}T^*\xi-(\lambda I+T^*T)^{-1}S^*\{S(\lambda I+T^*T)^{-1}S^*\}^{-1}\{S(\lambda I+T^*T)^{-1}T^*\xi-\eta\} & \text{if } \rho_0>\rho.\end{cases}$$

where ρ_0 is the norm of the solution for $\rho=+\infty$, and λ is a constant uniquely determined by $\|u_0\|=\rho$.

Acknowledgement. The authors are greatly indebted to Mr. T. Matsuo, a member of uor laboratory, for his helpful and suggestive discussions on the present paper.

REFERENCES

1. W.A.PORTER AND J.P.WILLIAMS. A note on the minimum effort control problem. J. Math. Anal. Appl. 13(1966), 251-264.

2. W.A.PORTER AND J.P.WILLIAMS. Extensions of the minimum effort control problem. J. Math. Anal. Appl. 13(1966), 539-549.

3. W.A.PORTER. On the optimal control of distributive systems. SIAM J. On Control 4. NO.3(1966), 466-472.

4. W.A.PORTER. "Modern Foundations of Systems Engineering." Macmillan, New York, 1966.

5. W.A.PORTER. A minimization problem and its applications to optimal control and system sensitivity. SIAM J. On Control 6, No.2(1968), 303-311.

6. E.M.KIRILLOVA. Applications of functional analysis to the theory of optimal control. SIAM J. On Control 5 No.1(1967), 25-30.

7. KOSAKU YOSIDA. "Functional Analysis." Springer-verlag, 1966.

8. K.ARROW, L.HURWICZ AND H.UZAWA. "Studies in Linear and Nonlinear Programming." Stanford Univ. Press, Stanford, 1958.