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Indices of Function Spaces

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§ 1. Symmetric Banach function spaces.

The theory of symmetric function spaces which are natural generalizations of the familiar Lebesgue spaces L^p $(1 \leq p \leq \infty)$ and of the Orlicz spaces L_M has their roots in the papers [5,6] by G. G. Lorentz and [4] by I. Halperin. Since then, the literature on this subject has been grown, and the importance of such spaces in analysis was shown in several papers, for instance, in [1-3,5-7, 12-15,17-21]. In a symmetric function space several indices, which characterize some intrinsic properties of the space, can be defined. In the present paper we introduce three kind of indices on such spaces, and show their applications together with some new results. In this section we give an introduction to symmetric Banach function spaces. For details, see [9, 10].

Let $(E, \mathfrak{m}, \mu)$ be a σ-finite measure space. For each real measurable function f on E the function $d_f(t) = \mu\{x : f(x) > t\}$, $t \in (-\infty, \infty)$ is called the distribution function of f. Let $(E_i, \mathfrak{m}_i, \mu_i)$, i = 1, 2 be σ-finite measure spaces, and let f_i , i = 1,2 be $\mathfrak{m}_i$-measurable. f_1 and f_2 are called spectrally equivalent and denoted by $f_1 \sim f_2$, if $d_{f_1} = d_{f_2}$ holds. Clearly $f_1 \sim f_2$ is equivalent to the fact that the spectral measures of f_i, $i = 1,2$ coincide. M (M^+) denotes the set of all real (resp. non-negative) measurable functions on E. A mapping ρ from M^+ into $\bar{R}^+$ is called a <u>function norm</u>, if it satisfies

$$
(1.1)\quad \begin{cases}
\text{i)} & \rho(f) = 0 \ \underline{\text{if and only if}}\ f = 0 \ \mu\text{-a.e.;} \\
\text{ii)} & \rho(f+g) \le \rho(f) + \rho(g), \quad f, g \in M^+ ; \\
\text{iii)} & \rho(af) = a\rho(f), \quad a \in R^+, \ f \in M^+ ; \\
\text{iv)} & \rho(f) \le \rho(g), \ \underline{\text{if}}\ f \le g.
\end{cases}
$$

If ρ satisfies the following condition:

(1.2) $\quad 0 \le f_n \uparrow f$ <u>implies</u> $\rho(f) = \sup_{1 \le n} \rho(f_n)$,

ρ is called <u>semi-continuous</u>. For each function norm ρ we denote by $X = X_\rho$ the set of all measurable functions such that $\rho(|f|) < \infty$. Equipped with the norm $\|f\| = \rho(|f|)$, X is then a normed linear space, if μ-almost equal functions are identified in the usual way. It is known that if ρ is semi-continuous $(X, \|\cdot\|)$ is a Banach space, which we shall call a <u>Banach function space</u> determined by ρ in the sequel. For each ρ, $\rho'(g) = \sup_{\rho(f) \le 1} \int f\, g\, d\mu$, $g \in M^+$ is called the <u>conjugate function norm</u> of ρ, which in fact satisfies the condition (1.1) under a certain assumption on ρ. The <u>conjugate space</u> of $X = X_\rho$ is the Banach function space $X' = X_{\rho'}$ determined by the conjugate function norm ρ' of ρ. For function norms the following theorem is essentially of importance [9, 10]:

(1.3) Theorem (Lorentz-Luxemburg). ρ <u>is reflexive</u>, i.e. $\rho = \rho''$ <u>if and only if</u> ρ <u>is semi-continuous</u>.

Hence, $X_\rho = X_{\rho''}$ holds whenever ρ is semi-continuous. A function norm ρ is called <u>symmetric</u> (or <u>rearrangement invariant</u>), if $f_1 \sim f_2$ implies $\rho(f_1) = \rho(f_2)$, and $X = X_\rho$ is then called a <u>symmetric Banach function space</u>.

For each $f \in M^+$ such that $d_f(\alpha) < \infty$ for all $\alpha > 0$, δ_f denotes the <u>decreasing rearrangement</u> of f, which is the right continuous inverse of the function d_f. Then $\delta_f \sim f$ holds. For any $f \in M$, f^* denotes $\delta_{|f|}$, if it has a sense. A measure space $(E, \mathfrak{M}, \mu)$ is called <u>adequate</u>, if for any $f, g \in M^+$

(1.4) $$\sup_{g \sim g'} \int_E f\, g'\, d\mu = \int_0^{\ell} \delta_f \delta_g \, dt,$$

where $\ell = \mu(E)$. Non-atomic measure spaces and discrete measure spaces having the atoms of equal measure are adequate, as is easily seen. For an adequate measure space E and a semi-continuous symmetric function norm ρ over E we have

(1.5) $$\rho(f) = \sup_{\rho(g) \leq 1} \int_0^{\ell} \delta_f \delta_g \, dt, \quad f \in M^+.$$

Let λ be a symmetric function norm on the interval $(0, \ell)$. Then it is clear that the functional $\rho(f) = \lambda(\delta_f)$, $f \in M^+(E, \mathfrak{M}, \mu)$ is a symmetric function norm on M^+. Conversely, on account of (1.3) and (1.5), it is shown that all symmetric function norms are represented in this way [10 ; (12.2)].

(1.6) Theorem. *Let $(E, \mathfrak{M}, \mu)$ be a σ-finite measure space which is adequate. Suppose that ρ is semi-continuous and $\rho(f) < \infty$ implies $d_f(\alpha) < \infty$ for all $\alpha > 0$. Then, ρ is symmetric if and only if there exists a symmetric function norm λ on $M^+(0, \ell)$ such that $\rho(f) = \lambda(\delta_f)$, $f \in M^+(E, \mathfrak{M}, \mu)$.*

If $f, g \in L^1(E, \mu)$, then we shall write $g \prec f$ (the Hardy-Littlewood-Pólya's preorder relation), whenever $\int_0^x g^* \, dt \leq \int_0^x f^* \, dt$ for all $x \in (0, \ell)$. An important property of a symmetric Banach function space is the following:

(1.7) $f \in X$, $g \prec f$ <u>implies</u> $g \in X$ <u>and</u> $\|g\| \leq \|f\|$, if $(E, \mathfrak{M}, \mu)$ <u>is adequate</u>.

This proposition is an immediate consequence of (1.3) and (1.5).

The Lebesgue spaces L^p ($1 \le p \le \infty$) and the Orlicz space L_M are symmetric Banach function spaces. The Lorentz spaces $\Lambda(\varphi)$[5, 6] are also symmetric and play a special role in the class of the symmetric Banach function spaces. It follows from (1.3), (1.6) and (1.7) that, for every symmetric Banach function space X, there exists a set C of positive decreasing functions such that

(1.8) $\quad X = \bigcap_{\varphi \in C} \Lambda(\varphi) \quad$ and $\quad \| f \| = \sup_{\varphi \in C} \| f \|_{\Lambda(\varphi)}, \quad f \in X.$

§ 2. The indices $\bar{\gamma}$ and $\underline{\gamma}$.

In what follows, let X be a symmetric Banach function space determined by a semi-continuous function norm ρ over $I = (0, \ell)$, $\ell < \infty$ or $= \infty$.

The function $\gamma(x) = \gamma_X(x) = \| \chi_{(0,\ x)} \|$, $x \in I$ is called the fundamental function of $(X, \| \cdot \|)$. If $e \subset I$ and $\text{mes}(e) = x$, then $\| \chi_e \| = \gamma(x)$.

We have on account of (1.3) and (1.7)

(2.1) i) $\gamma_X(x)\ \gamma_{X'}(x) = x, \quad x \in I.$

ii) $\gamma(x) / x$ is a continuous decreasing function of x, $x \in I$.

iii) There exists a norm $\| \cdot \|_0$ equivalent to $\| \cdot \|$ such that the fundamental function γ_0 of the space $(X, \| \cdot \|_0)$ is concave and $\gamma(x) \le \gamma_0(x) \le 2\gamma(x), \quad x \in I.$

The proposition iii) above is stated without proof in [11]. Here we give a sketch of a simple proof based on the representation theorem (1.8). The fundamental function γ_φ of $(\Lambda(\varphi), \| \ \|_{\Lambda(\varphi)})$ is concave for every positive decreasing function φ . Hence $\gamma = \gamma_X$ is the supremum of positive concave functions. It is also easy to see that there exists a concave function majorizing all γ_φ , $\varphi \in C$ by virtue of ii) of (2.1). Therefore, the proof of the proposition iii) can be reduced to the following elementary fact :

(2.2) <u>If</u> C <u>is a family of positive increasing concave functions on</u> I <u>with a concave majorant function</u> c_o <u>satisfying the condition</u> ii) <u>of</u> (2.1), <u>there exists the least concave function</u> c_o <u>majorizing all functions of</u> C <u>and</u> $\sup_{c\in C} c(t) \leq c_0(t) \leq 2 \sup_{c\in C} c(t)$ <u>holds for all</u> $t \in I$.

If we take the least concave majorant γ_o for the family $\{\gamma_\varphi ; \varphi \in C\}$ in (1.8), it is then easy to verify that the functional $\|f\|_o = \mathrm{Max}\{\|f\|, \sup_{x>0} (\frac{1}{x}\int_0^x f^* dt)\gamma_o(x)\}$, $f \in X$ satisfies the required conditions. An alternative proof of this is given in [21].

Now we put for $s > 0$

(2.3) $$\delta(s) = \delta(s ; X) = \sup_{t\in I,\, st\in I} \gamma(st)/\gamma(t) .$$

Since $\delta(s_1 s_2) \leq \delta(s_1)\,\delta(s_2)$, $s_1, s_2 > 0$, putting $\bar{\gamma} = \bar{\gamma}_X = \inf_{s>1}\{\log\delta(s)/\log s\}$ and $\underline{\gamma} = \underline{\gamma}_X = \sup_{0<s<1}\{\log\delta(s)/\log s\}$, we obtain

(2.4) $$\begin{cases} \bar{\gamma} = \lim_{s\to\infty} \log\delta(s)/\log s , \\ \underline{\gamma} = \lim_{s\to 0} \log\delta(s)/\log s . \end{cases}$$

The indices $\bar{\gamma}$ and $\underline{\gamma}$ are defined in this way in [21]. Since $1 \leq \delta(s) \leq s$, $s > 1$ and $s \leq \delta(s) \leq 1$, $0 < s < 1$, and $\delta(s) \geq \delta(s^{-1})^{-1}$, $s > 1$ hold, we can prove easily

(2.5) i) $0 \leq \underline{\gamma} \leq \bar{\gamma} \leq 1$.

ii) $\bar{\gamma}_X + \underline{\gamma}_{X'} = \underline{\gamma}_X + \bar{\gamma}_{X'} = 1$.

Now we present two applications of the γ-indices. First of them concerns with the Lebesgue-Orlicz points of functions of symmetric spaces, and the results are obtained by D. V. Salehov in [15]. Let I_1 be the interval (0, 1), and let X be a symmetric Banach function space over I_1. A point $t_o \in I_1$ is called a <u>Lebesgue-Orlicz point</u> of a function f of X, whenever

(2.6) $\lim_{h \to 0} \| (f - f(t_o))\chi_{(t_o\ ;\ h)}\| \big/ \gamma(2h) = 0,$

where $\chi_{(t_o\ ;\ h)} = \chi_{(t_o-h,\ t_o+h)}$. This is a generalizarion of the notion of the usual Lebesgue point or that of the Lebesgue point of the order p, $1 < p < \infty$. For every $f \in X$ let $\mathcal{E}(f)$ denote the totality of the Lebesgue-Orlicz points of f. I_1 being a finite interval, it is easy to see that $0 < \underline{\gamma}$ is equivalent to the fact that $\gamma(t\alpha(t)) \big/ \gamma(t) \to 0$ as $t \to 0$ for any function $\alpha(t) \to 0$. Thus, Salehov's results are stated as follows :

(2.7) *If* $\underline{\gamma} > 0$, *then* $mes(\mathcal{E}(\chi_e)) = 1$ *for any measurable set* $e \subset I_1$.

In fact, the union of the set of all accumulation points of e and the set of those points of e^c coincides with the set $\mathcal{E}(\chi_e)$. Furthermore, the following characterization for $\underline{\gamma} > 0$ in terms of Lebesgue- Orlicz points is given :

(2.8) Theorem (Salehov). $\underline{\gamma} > 0$ *if and only if* $mes(\mathcal{E}(f)) = 1$ *for every bounded function* f *of* X.

The necessity follows from (2.7) and the Lusin's theorem. The proof of the converse part is established by a construction of measurable sets e_1, $e_2 \subset I_1$ such that $mes(e_1) = \frac{1}{2}$ and each point of e_1 is not a Lebesgue-Orlicz point of the function χ_{e_2}, provided that $\gamma(t\alpha(t)) \big/ \gamma(t) \not\to 0$ as $t \to 0$ for a function $\alpha(t) \to 0$ [14].

Another application of the γ-indices concerns with interpolation theorems of the Marcinkiewicz type, and is discussed by E. M. Semenov [12, 13] and by M. Zippin [21]. In the remainder of this section we assume that Σ, the set of all simple functions on $I = (0, \infty)$, is dense in X. For any α,β with $0<\alpha\leq\beta<1$, we define $\Omega(\alpha,\beta)$ to be the set of all measurable functions f integrable on every finite interval and satisfying the inequality:

(2.9) $$\operatorname{Min}\{t^{\alpha},t^{\beta}\}\leq\int_0^{st}f^*(u)\,du\Big/\int_0^{s}f^*(u)\,du\leq\operatorname{Max}\{t^{\alpha},t^{\beta}\},\quad s,t>0.$$

The following theorem is due to Semenov (c.f. (1.8)):

(2.10) Theorem [12]. _If_ $0<\alpha<\underline{\gamma}\leq\overline{\gamma}<\beta<1$, _then there exists an_ $K>0$ _such that_ $\|f\|\leq K\cdot\sup_{\|g\|_{X'}\leq1,\ g\in\Omega(\alpha,\beta)}\|f\|_{\Lambda(g^*)}$ _for all_ $f\in X$.

For symmetric Banach function spaces X_1 and X_2, a linear operator T, which maps Σ into M, is called _of weak type_ $\{X_1,X_2\}$, if there exists an $K>0$ such that, for every $f\in\Sigma$,

(2.11) $$(Tf)^*(t)\,\gamma_{X_2}(t)\leq K\int_0^{\infty}f^*(t)\,d\gamma_{X_1}(t)$$

holds a.e. $t\in I$. The least number K satisfying (2.11) is denoted by $\|T\|_{\omega\{X_1\to X_2\}}$. The usual norm of the operator T is written as $\|T\|_{X_1\to X_2}$. For given two pairs $\Delta=(\{X_1,Y_1\},\{X_2,Y_2\})$ of symmetric spaces with concave fundamental functions such that $\underline{\gamma}_{X_1}>\overline{\gamma}_{X_2}$, the Calderón's operator $S(\Delta)$, which play a fundamental role in [3], can be also defined, and holds the following proposition which is a general version of a result in [3]:

(2.12) _The Calderon's operator_ $S(\Delta)$ _is of weak type_ $\{X_i,Y_i\}$ _for_ $i=1,2$, _if_ $\underline{\gamma}_{X_1}>\overline{\gamma}_{X_2}$.

On the basis of (2.10) and (2.12) the following theorem is proved [13, 21]:

(2.13) Theorem(Semenov-Zippin). _Let_ X_1, X_2 _and_ X _be symmetric Banach function spaces over_ $(0,\infty)$. _if_ $\overline{\gamma}_{X_2}<\underline{\gamma}_X\leq\overline{\gamma}_X<\underline{\gamma}_{X_1}$, _then there exists an_ $K>0$ _such that_ $\|T\|_{X\to X}\leq K$ _holds for every linear operator_ T _of weak type_ $\{X_i,X_i\}$ $i=1,2$ _with_ $\|T\|_{\omega\{X_i\to X_i\}}\leq1$.

§ 3. The indices $\overline{\sigma}$ and $\underline{\sigma}$.

For every $a>0$, let σ_a denote the compression operator on X : $\sigma_a f = f_a$, $f \in X$, where f_a is given by $f_a(x) = f(ax)$, if $ax \in I$, and $f_a(x) = 0$, otherwise. σ_a is then a bounded linear operator on X and

(3.1) $\quad \mathrm{Min}(a^{-1}, 1) \le \| \sigma_a \| \le \mathrm{Max}(a^{-1}, 1), \quad a>0.$

The values of $\|\sigma_a\|$ play an important role to determine the majorants for interpolation theorems [7,18,19]. The following definition of the σ-indices is due to D. W. Boyd [1,2]. If we put $\overline{\sigma} = \overline{\sigma}_X = \inf_{a>1} \{\log \|\sigma_{a^{-1}}\| / \log a\}$ and $\underline{\sigma} = \underline{\sigma}_X = \sup_{0<a<1} \{\log \|\sigma_{a^{-1}}\| / \log a\}$, we obtain in the same way as (2.4)

(3.2) $$\begin{cases} \overline{\sigma} = \lim_{a\to\infty} \{\log \|\sigma_{a^{-1}}\| / \log a\}, \\ \underline{\sigma} = \lim_{a\to 0} \{\log \|\sigma_{a^{-1}}\| / \log a\}. \end{cases}$$

Since $\delta(a;X) \le \|\sigma_{a^{-1}}\|$, $a>0$ and $\|\sigma_a\|_X = a^{-1}\|\sigma_{a^{-1}}\|_{X'}$, $a>0$, the following relations hold :

(3.3) i) $0 \le \underline{\sigma} \le \underline{\gamma} \le \overline{\gamma} \le \overline{\sigma} \le 1.$

ii) $\overline{\sigma}_X + \underline{\sigma}_{X'} = \underline{\sigma}_X + \overline{\sigma}_{X'} = 1.$

Two indices, γ and σ, do not coincide in general. In fact, we can prove

(3.4) **For any** α, $0<\alpha<1$, there exists a symmetric Banach function space X such that $\underline{\gamma} = \overline{\gamma} = \alpha$ but $\underline{\sigma} = 0$ (or $\overline{\sigma} = 1$).

For $\alpha = \frac{1}{2}$, a symmetric Banach function space Y is constructed in [20], for which $\gamma(x) = \gamma_Y(x) = x^{\frac{1}{2}}$, $x>0$, hence $\underline{\gamma} = \overline{\gamma} = \frac{1}{2}$, but $\underline{\sigma} = 0$ (or $\overline{\sigma} = 1$) holds. If $0<\alpha<\frac{1}{2}$, then consider a linear space Y_α, the set of all measurable functions such that $|f|^{\frac{1}{2\alpha}} \in Y$, with the norm $\|f\|_\alpha = \|f^{\frac{1}{2\alpha}}\|_Y^{2\alpha}$, $f \in Y_\alpha$. It is easy to see that Y_α is then a symmetric

Banach function space with $\overline{\gamma}_{Y_\alpha} = \underline{\gamma}_{Y_\alpha} = \alpha$. On the other hand, $\underline{\sigma}_{Y\alpha} = 0$ or $\overline{\sigma}_{(Y_\alpha)} = 1$ holds according as $\underline{\sigma}_Y = 0$ or $\overline{\sigma}_Y = 1$. For α with $\frac{1}{2} < \alpha < 1$, it suffices to consider the conjugate space of $Y_{1-\alpha}$ constructed above. Using a similar argument as in[20], we can also construct, for any α, $0 < \alpha < 1$, a symmetric space such that $\underline{\gamma} = \overline{\gamma} = \alpha$, but both $\underline{\sigma} = 0$ and $\overline{\sigma} = 1$ hold.

In terms of the σ-indices we can characterize many interesting properties of symmetric spaces. Now we introduce two theorems concerning with the Hardy-Littlewood maximal functions and the Hilbert transform.

For any real measurable function f on the interval $I = (0, \ell)$ let θf denote the Hardy-Littlewood maximal function of f : $\theta f(x) = \sup_{y \in I} \int_x^y f(t) / (y-x)\, dt$, $x \in I$. If X has the property that $f \in X$ implies $\theta f \in X$, then we write $X \in HLP$. The property $X \in HLP$ is characterized by $\overline{\sigma}$ as follows :

(3.5) Theorem [17]. $X \in HLP$ <u>if and only if</u> $\overline{\sigma} < 1$.

Let f be a locally integrable function on $R = (-\infty, \infty)$, and let H denote the Hibert transform : $(Hf)(x) \underset{a.e.}{=} \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \right) f(t)/(x - t)\, dt$ $= \frac{1}{\pi}(P) \int_{-\infty}^{\infty} f(t)/(x - t)\, dt$. Let $X \in [H]$ denote the statement that H is a bounded linear operator acting on X. It is known that $L^p \in [H]$ for $1 < p < \infty$. Generalizing this fact to an arbitrary symmetric Banach function space, Boyd showed the following theorem :

(3.6) Theorem [1]. $X \in [H]$ <u>if and only if</u> $0 < \underline{\sigma} \leq \overline{\sigma} < 1$, <u>or equivalently</u>, <u>both</u> $\overline{\sigma}_X < 1$ <u>and</u> $\overline{\sigma}_{X'} < 1$ <u>hold</u>.

Another necessary and sufficient conditions for $\underline{\sigma} > 0$ or $\overline{\sigma} < 1$ are given in [18, 19] in terms of the complete continuity of operators or of the continuity of operators of a weak type.

For familiar symmetric function spaces such as the Lebesgue spaces L^p, $1 \leq p \leq \infty$, the Orlicz spaces L_M, Lorents spaces $\Lambda(\varphi)$ and $M(\varphi)$, both $\underline{\gamma} = \underline{\sigma}$ and $\overline{\gamma} = \overline{\sigma}$ hold. For the Orlicz spaces L_M, these equalitis are derived from the inequality $\|\sigma_a - 1\| \leq 2\, \delta(a)$, $a > 0$, which is a direct consequence of Theorem 6 in [7].

Here we remark that although we have defined indices γ and σ for a symmetric Banach function space over I, these indices can be also defined for symmetric spaces over any (adequate) measure space $(E, \mathfrak{m}, \mu)$, and their meanings are described in terms of distribution functions.

§ 4. The indices $\overline{\tau}$ and $\underline{\tau}$.

The indices discussed in this section are defined by the author in [16]. Being different from the indices γ and σ defined in the preceding sections, the indices $\overline{\tau}$ and $\underline{\tau}$ can be defined on an arbitrary (not necessary symmetric) Banach function space over any σ-finite measure space $(E, \mathfrak{m}, \mu)$.

Let X be a Banach function space over $(E, \mathfrak{m}, \mu)$. For each ε, $0 < \varepsilon < 1$ let $N(\varepsilon)$ denote the least natural number, if it exists, satisfying the condition :

(4.1) $\|f_i\| \geq \varepsilon$, $f_i \perp f_j$, $i \neq j$, $i, j = 1,2,\ldots,n$ <u>and</u> $N(\varepsilon) \leq n$

<u>imply</u> $\|\sum_{i=1}^{n} f_i\| > 1$,

where $f \perp g$ means that $f(x)\, g(x) = 0$ μ-a.e.. If there is no such natural number let $N(\varepsilon) = \infty$. Putting $p(\varepsilon) = \log N(\varepsilon) / -\log \varepsilon$, $0 < \varepsilon < 1$, and $\tau^u = \inf_{0<\varepsilon<1} p(\varepsilon)$, we obtain $\tau^u = \lim_{\varepsilon \to 0} p(\varepsilon)$. τ^u is called the <u>upper exponent of</u> $(X, \|\cdot\|)$, and we put

(4.1) $\underline{\tau} = \underline{\tau}_X = 1/\tau^u$.

Also, for each ε , $0 < \varepsilon < 1$, let $N'(\varepsilon)$ be the greatest natural number (or ∞) satisfying the condition :

(4.3) $\| f_i \| \leq \varepsilon$, $f_i \perp f_j$, $i \neq j$, $i, j = 1,2,\ldots,n$ and $n \leq N'(\varepsilon)$ imply $\| \sum_{i=1}^{n} f_i \| < 1$.

Putting $P'(\varepsilon) = \log N'(\varepsilon) / -\log \varepsilon$, $0 < \varepsilon < 1$ and $\tau^{\ell} = \sup_{0<\varepsilon<1} p'(\varepsilon)$, we obtain $\tau^{\ell} = \lim_{\varepsilon \to 0} p'(\varepsilon)$. τ^{ℓ} is called the lower exponent of $(X, \|\cdot\|)$, and we put

(4.4) $\bar{\tau} = \bar{\tau}_X = 1 / \tau^{\ell}$.

Then the following propositions hold [16]:

(4.5) i) $0 \leq \underline{\tau} \leq \bar{\tau} \leq 1$.

ii) $\bar{\tau}_X + \underline{\tau}_{X'} = \underline{\tau}_X + \bar{\tau}_{X'} = 1$.

The indices γ and σ have no relation to the reflexivity of the Banach space concerned. For the τ-indices, the following proposition holds.

(4.6) If $0 < \underline{\tau} \leq \bar{\tau} < 1$, then $(X, \|\cdot\|)$ is reflexive as a Banach space.

The converse of (4.6) is not true in general. If X is a symmetric Banach function space over I, we can prove without difficulty that the following inequalities hold :

(4.7) $0 \leq \underline{\tau} \leq \underline{\sigma} \leq \underline{\gamma} \leq \bar{\gamma} \leq \bar{\sigma} \leq \bar{\tau} \leq 1$.

The values of the τ-indices and those of σ-indices do not coincide in general as will be shown below. Even for symmetric spaces, the converse of (4.6) is not valid. In fact, we have

(4.8) There exists a reflexive symmetric Banach function space for which $\underline{\gamma} = 0$ (hence $\underline{\tau} = 0$) holds.

To show (4.8) we consider the space $\Lambda(\varphi;p)$ over the interval $I_1 = (0, 1)$, the set of all measurable functions f for which $(\int_0^1 \varphi f^{*p} \, dt)^{\frac{1}{p}} < \infty$, where φ is positive and decreasing on I_1, and $1 < p < \infty$. We assume also that the function $\Phi(u) = \int_0^u \varphi(t)\,dt$, $u \in I_1$ satisfies the condition : $\varliminf_{u\to 0} \Phi(2u)/\Phi(u) = 1$. It is known [6] that the space $\Lambda(\varphi; p)$ is reflexive as a Banach space, if $1 < p < \infty$. On the other hand, it is shown in [18] that $\underline{\gamma}_{\Lambda(\varphi)} = 0$ holds under the additional assumption above. This, however, implies also $\underline{\gamma}_{\Lambda(\varphi;p)} = 0$.

As an application of the τ-indices we have [16]

(4.9) Let X_i, i = 1, 2 be Banach function spaces over measure spaces $(E_i, \mathfrak{m}_i, \mu_i)$. Then every integral operator T from X_1 into X_2 is compact, if $\bar{\tau}_{X_1} < \underline{\tau}_{X_2}$.

For the Lebesgue spaces L^p, $1 \leq p \leq \infty$ all the indices coincide and equal to p^{-1}. For the Orlicz spaces L_M over a finite measure space E it is known [16] that $\tau^{*} = \underline{\tau}^{-1}$ and $\tau^{\ell} = \bar{\tau}^{-1}$ coincide with the exponents σ_M and s_M of the convex function M respectively, where σ_M and s_M are defined by W. Matsuzewska and W. Orlicz [8] as follows :

$$(4.10)\quad \begin{cases} \sigma_M = \lim\limits_{\lambda\to\infty}\left[\log\left\{\varlimsup\limits_{u\to\infty}\dfrac{M(\lambda u)}{M(u)}\right\}\Big/\log\lambda\right]; \\ s_M = \lim\limits_{\lambda\to\infty}\left[\log\left\{\varliminf\limits_{u\to\infty}\dfrac{M(\lambda u)}{M(u)}\right\}\Big/\log\lambda\right]. \end{cases}$$

Recently the author was informed of a result of Boyd [Indices for the Orlicz spaces, Preprint] that shows that $\underline{\sigma} = \sigma_M^{-1}$ and $\bar{\sigma} = s_M^{-1}$. Therefore, for the Orlicz space over a finite masurespace we have $\sigma_M^{-1} = \underline{\tau} = \underline{\sigma} = \underline{\gamma}$ and $s_M^{-1} = \bar{\tau} = \bar{\sigma} = \bar{\gamma}$. For the Lorentz spaces $\Lambda(\varphi)$ and $M(\varphi)$ the σ-indices and τ-indices are not the same in general, as is easily verified. For the further results or applications for the indices, which could not be refered to here, see [2, 13, 14].

REFERENCES

1. D. W. Boyd, The Hilbert transform on rearrangement invariant spaces, Canad. J. Math. 19, 599-616 (1967).

2. __________, Indices of Function spaces and their relationship to interpolation, Canad. J. Math. 21, 1245-1254 (1969).

3. A. P. Calderón, Spaces between L^1 and L^∞ and theorem of Marcinkiewicz, Studia Math. 26, 273-299 (1966).

4. I. Halperin, Function spaces, Canad. J. Math. 5, 273-288 (1953).

5. G. G. Lorentz, Some new functional spaces, Ann Math. 51, 37-55 (1950).

6. ____________, On the theory of spaces Λ, Pacific J. Math. 1, 411-429 (1950).

7. G. G. Lorentz and T. Shimogaki, Majorants for interpolation theorems, Publ. Ramanujan Inst. 1, 115-122 (1969).

8. W. Matuszewska and W. Orlicz, On certain properties of φ-functions, Bull Acad. Polon. Sci. , Sér. sci. math. astr. et phys., 8, 439-443 (1960).

9. W. A. J. Luxemburg and A. C. Zaanen, Notes on Banach function spaces I-V, Proc. Acad. Sci. Amsterdam A66, 135-147,148-153,239-250,251-263, 496-504 (1963).

10. W. A. J. Luxemburg, Rearrangement invariant Banach Function spaces, Proc. Symposium in Analysis, 1967, Queen's Univ., 83-144.

11. E. M. Semenov, Imbedding theorems for Banach spaces of measurable functions, Soviet Math. Dokl. 5, 831-834 (1964).

12. __________, A new interpolation theroem, Functional Analysis and its applications 2, 158-168 (1968).

13. __________, A method for establishing interpolation theorems in symmetric spces, Soviet Math. Dokl. 10, 507-510 (1969).

14. D.V. Salehov, More about Lebesgue Orlicz points, Ukrain. Mat. J. 17, 72-81 (1965).

15. __________, Convergence of singular integrals at the Lebesgue-Orlicz points of functions in symmetric spaces, Soviet Math. Dokl. 8, 975-979 (1967).

16. T. Shimogaki, Exponents of norm in semi-ordered linear spaces, Bull. Acad. Polon. Sci. Sér. sci. math. astr. et phys., 8, 135-140 (1964).

17. __________, Hardy-Littlewood majorants in function spaces, J. Math. Soc. Japan 17, 365-373 (1965).

18. __________, On the complete continuity of operators in an interpolation theorem, J. Fac. Sci. Hokkaido Univ. I, 20, 109-114 (1968).

19. __________, An interpolation theorem on Banach function spaces, Studia Math. 31, 233-240 (1968).

20. __________, A note on norms of compression operators on function spaces, Proc. Japan Acad. 46, 239-242 (1970).

21. M. Zippin, Interpolation of operators of weak type between rearrangement invariant function spaces (Preprint).

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