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AN EXTENSION OF GAUSS' FORMULA FOR HYPERGEOMETRIC SERIES.

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1. Introduction. A system of linear ordinary differential equations with rational coefficients:

$$dx/dt=A(t)x$$

is in Fuchsian class if all the poles of $A(t)$ are regular singular points of the system. A fundamental problem for such a system is the computation of the group of the system. But the computation has scaresly been carried out in closed form except for the case of hypergeometric system

$$(t-B)dx/dt=Ax \tag{1}$$

where B is the diagonal matrix

$$B=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and A has the form

$$A=\begin{pmatrix} 1-c & 1 \\ (c-1-a)(c-1-t) & c-1-a-b \end{pmatrix}$$

A crucial role was played in the computation of the group of the system in Riemann's paper "Beitrage zur Theorie der durch die Gauss'sche Reihe F(a,b,c,x) darstellbaren Functionen", by Gauss' Formula

$$F(a,b,c,1)=\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b) \tag{2}$$

A generalization of the formula for higher dimensional systems (1) has implicitly been used for the computations of Stokes multipliers of the system

$$tdx/dt=(A+tB)x$$

in a paper by M.Kohno ([1]). The object of the paper is to present this generalization by modifying slightly the result of him.

2. Statement of the Theorem.

Let B be a diagonal matrix with constant diagonal elements $\lambda_1,\lambda_2,\ldots,\lambda_n$ and let A be the matrix with (j,k)-th element $a_{j,k}$ $(j,k=1,2,\ldots,n)$. We consider a system of differential equations

$$(1)\quad (t-B)dx/dt= Ax$$

The system (1) has n+1 regular singular points: $\lambda_1, \ldots,\lambda_n,\infty$. The characeristic exponents at $t=\lambda_k$ are

$$0,0,\ldots,a_{kk},\ldots,0$$

So, if we denote by e_k the constant n-vector with all the elements zero except the k-th which is 1, there is a solution $x_k(t)$ which behaves like $(t-\lambda_k)^{a_{kk}}e_k$.

Theorem. *If on none of the numbers* $a_{11},\ldots,a_{nn}$; $\rho_1, \ldots,\rho_n$ $(\det(A-\rho I)=0)$ *is an integer, then we have*

$$(3)\quad \det(x_1(t),x_2(t),\ldots,x_n(t)) = (t-\lambda_1)^{a_{11}}\cdots(t-\lambda_n)^{a_{nn}}\frac{\prod_{k=1}^{n}\Gamma(a_{kk}+1)}{\prod_{j=1}^{n}\Gamma(\rho_j+1)}$$

where the branches of multi-valued factors $(t-\lambda_k)^{a_{kk}}$ *are suitably chosen in a simply connected domain D whose boundary is a closed Jordan curve* J *carrying all the singularities* $\lambda_1,\ldots,\lambda_n$.

Our proof is a direct continuation of the argument of Gauss' original proof of the formula (2). But it will be carried out only for the special case

$$(4)\qquad |\lambda_j-\lambda_k| > |\lambda_k| > 0 \qquad (j\neq k)$$

The method of analytic continuation of functions beyond the circle of convergence across an arc of finite non-zero width, used in [2], may be used to avoid the unnecessary condition (4).

3. Proof of the Theorem.

If we expand the solution $x_k(t)$ into power series in $(t-\lambda_k)$;

(5) $$x_k(t) = \sum_{s=o}^{\infty} G_k(s)(t-\lambda_k)^{a_{kk}+s}$$

the coefficeints $G_k(s)$ satisfy

(6) $$(a_{kk}+s-A)G_k(s)= (B-\lambda_k)(a_{kk}+s+1)G_k(s+1)$$

(7) $$G_k(o)=e_k$$

A simple change of (6) by the transformation

(8) $$G_k(s)=H_k(s)\frac{\Gamma(a_{kk}+1)}{\Gamma(a_{kk}+s+1)}$$

takes (6) and (7) into

(6)* $$(a_{kk}+s-A)H^k(s)=(B-\lambda_k)H^k(s+1)$$

(7)* $$H^k(o)=e_k$$

Now we introduce a new parameter m which takes non-negative integral values into the system and define new vector valued functions

(9) $$x_k(t,m)= \sum_{s=o}^{\infty} H^k(s)\cdot\frac{\Gamma(a_{kk}+1)}{\Gamma(a_{kk}+m+s+1)}(t-\lambda_k)^{a_{kk}+s+m}$$

and the matrix $X(t,m)$ whose k-th vertical vector is $x_k(t,m)$.

By an easy computation using the property of the Gamma-function

$$x\Gamma(x) = \Gamma(x+1)$$

we have

(10) $$(t-\lambda_k)dx_k(t,m)/dt=(t-\lambda)x_k(t,m-1)$$

Similarly, by using (6)*, we have

(11) $$(t-\lambda_k)dx_k(t,m)/dt=(A+m)x_k(t,m)+(B-\lambda_k)x_k(t,m-1)$$

Combining (10) and (11) together, we have systems of difference equations

(12) $(t-B)x_k(t,m-1)=(A+m)x_k(t,m)$

The most important feature of the system (12) is that their coefficents are independent of suffix k. Consequently, the matrix $X(t,m)$ satisfies:

(13) $(t-B)X(t,m-1)=(A+m)X(t,m)$

Since we are interested only in computing the determinant of $X(t,m)$, we write $\det X(t,m)=c(t,m)$ and deduce

$$(14)\quad c(t,m)=c(t,o)(t-\lambda_1)^m\cdots(t-\lambda_n)^m\frac{\Gamma(\rho_1+1)\cdots\Gamma(\rho_n+1)}{\Gamma(m+\rho_1+1)\cdots\Gamma(m+\rho_n+1)}$$

On the other hand, for a fixed t, $c(t,m)$ is given asymtptotically by

$$(15)\quad c(t,m)=(t-\lambda_1)^{m+a_{11}}\cdots(t-\lambda_n)^{m+a_{nn}}\frac{\Gamma(a_{11}+1)\cdots\Gamma(a_{nn}+1)}{\Gamma(m+a_{11}+1)\cdots\Gamma(m+a_{nn}+1)}\left\{1+O\left(\frac{1}{m}\right)\right\}$$

since the expansion (9) is a convergent inverse factorial series, for t which is in the intersection of the circles of convergence of the solutions $x_k(t)$ $k=1,2,\dots,n$. The condition (4) shows $t=o$ is such a point. And since our domain D is a simply connected doamin, we have

$$(16)\quad X(t^*)=\quad c(t^*)=\det X(t^*)=c(t)\exp\left[\int_t^{t^*}\Sigma\, a_{kk}(t-\lambda_k)\,dt\right]$$

that is, (3) is valid uniformly in D. We have

$$C(t,o)=\lim_{m\to\infty}(t-\lambda_1)^{a_{11}}\cdots(t-\lambda_n)^{a_{nn}}\frac{\prod_{k=1}^{n}\Gamma(a_{kk}+1)}{\prod_{j=1}^{n}\Gamma(\rho_j+1)}$$

$$\times\frac{\Gamma(m+\rho_1+1)\cdots\Gamma(m+\rho_n+1)}{\Gamma(m+a_{11}+1)\cdots\Gamma(m+a_{nn}+1)}\left\{1+O\left(\frac{1}{m}\right)\right\}$$

$$=\lim_{m\to\infty}(t-\lambda_1)^{a_{11}}\cdots(t-\lambda_n)^{a_{nn}}\frac{\prod_{k=1}^{n}\Gamma(a_{kk}+1)}{\prod_{j=1}^{n}\Gamma(\rho_j+1)}\cdot m^{\Sigma\rho_j-\Sigma a_{kk}}\left\{1+O\left(\frac{1}{m}\right)\right\}$$

which complete the proof of (3) since

$$\Sigma\rho_j-\Sigma a_{kk}=0$$

by invariance of the trace of matrix A (Fuch's relation).

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4. An Application of the Theorem.

Consider the case n=2. Let $x_1(t)^*$, $x_2(t)^*$ be solutions

$$x_1^*(t) = \sum_{s=o}^{\infty} G_1^*(s)(t-\lambda_1)^s \qquad G_1^*(o)=e_2$$

$$x_2^*(t) = \sum_{s=o}^{\infty} G_2^*(s)(t-\lambda_2)^s \qquad G_2^*(o)=e_1$$

We have

$$\det(x_1,x_1^*)=(t-\lambda_1)^{a_{11}}[\ 1+O(t-\lambda_1)\ ]$$

$$\det(x_2,x_2^*)=(t-\lambda_2)^{a_{22}}[\ 1+O(t-\lambda_2)\ ]$$

If we assume

$$x_1(t)= px_2+p^*x_2^*$$

The connection constants p^*_2 can easily computed with the help of the theorem by taking the limit $t \to \lambda_2$.

Reference

1. M.Kohno: The convergence condition of a series appearing in connection problems and the determination of Stokes multipliers, Publ. R.I.M.S. Kyoto univ., 3(1968), 337-350.

2. K.Okubo: A global representation of a fundamental set of solutions and a Stokes phenomenon fro a system of linear ordinary differential equation, J.Math.Soc.Japan, 15(1963) 268-288.