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# Oscillatory Property of Solutions of Second Order Differential Equations

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In this paper we shall discuss oscillatory property of solutions of second order differential equations by applying Liapunov's second method. Consider an equation

(1) $$(r(t)x')' + f(t, x, x') = 0 \quad \left( ' = \frac{d}{dt} \right),$$

where $r(t) > 0$ is continuous on $I = [0, \infty)$ and $f(t, x, u)$ is defined and continuous on $I \times R \times R$, $R = (-\infty, \infty)$. To discuss oscillatory property of solutions of (1), we consider an equivalent system

(2) $$x' = \frac{y}{r(t)}, \quad y' = -f(t, x, \frac{y}{r(t)}).$$

A solution $x(t)$ of (1) which exists in the future is said to be oscillatory if for every $T > 0$ there exists a $t_0 > T$ such that $x(t_0) = 0$. Moreover, the equation (1) is said to be oscillatory if every solution of (1) which exists in the future is oscillatory.

**Theorem 1.** Assume that there exist two continuous functions $V(t, x, y)$ and $W(t, x, y)$

which are defined on $t \geqq T$, $x > 0$, $|y| < \infty$ and $t \geqq T$, $x < 0$, $|y| < \infty$, respectively, where $T$ can be large, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

(i) $V(t, x, y) \to \infty$ uniformly for $x > 0$ and $-\infty < y < \infty$ as $t \to \infty$, and $W(t, x, y) \to \infty$ uniformly for $x < 0$ and $-\infty < y < \infty$ as $t \to \infty$,

(ii) $\dot{V}_{(2)}(t, x(t), y(t)) \leqq 0$ for all sufficiently large $t$, where $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large $t$ and

$$\dot{V}_{(2)}(t, x(t), y(t)) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{ V(t+h, x(t+h), y(t+h)) - V(t, x(t), y(t)) \},$$

(iii) $\dot{W}_{(2)}(t, x(t), y(t)) \leqq 0$ for all sufficiently large $t$, where $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) < 0$ for all large $t$ and

$$\dot{W}_{(2)}(t, x(t), y(t)) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{ W(t+h, x(t+h), y(t+h)) - W(t, x(t), y(t)) \}.$$

Then the equation (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) which is defined on $[t_0, \infty)$, and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large $t$. Now assume that $x(t) > 0$ for all $t \geqq \sigma$, where we can assume $\sigma$ to be sufficiently large. By the condition (i), if $t$ is sufficiently large, say $t \geqq t_1$, we have

$$V(\sigma, x(\sigma), y(\sigma)) < V(t, x, y)$$

for all $x > 0$, $|y| < \infty$. However, by the condition (ii), we have

$$V(t, x(t), y(t)) \leqq V(\sigma, x(\sigma), y(\sigma)) \quad \text{for all} \quad t \geqq \sigma,$$

which contradicts $V(\sigma, x(\sigma), y(\sigma)) < V(t_1, x(t_1), y(t_1))$. When we assume that $x(t) < 0$ for all large $t$, we have also a contradiction by considering $W(t, x(t), y(t))$. Thus we see that $x(t)$ is oscillatory.

To apply this theorem, the following lemmas play an important role. In the following, a scalar function $v(t, x, y)$ will be called a Liapunov function for (2), if $v(t, x, y)$ is continuous in $(t, x, y)$ in the domain of definition and is locally Lipschitzian in $(x, y)$. Moreover, we define $\dot{v}_{(2)}(t, x, y)$ by

$$\text{(3)} \qquad \dot{v}_{(2)}(t, x, y) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{ v(t+h, x+h\frac{y}{r(t)}, y+hf(t, x, \frac{y}{r(t)})) - v(t, x, y) \} .$$

If $\dot{v}_{(2)}(t, x, y) \leqq 0$, $v(t, x(t), y(t))$ is nonincreasing in $t$, where $\{ x(t), y(t) \}$ is a solution of (2), see [5].

Lemma 1. For $t \geqq T^*$, $x > 0$, $-\infty < y < \infty$, where $T^*$ can be large, we assume that there exists a Liapunov function $v(t, x, y)$ which satisfies the following conditions;

(i) $yv(t, x, y) > 0$ for $y \neq 0$, $t \geqq T^*$, $x > 0$,

(ii) $\dot{v}_{(2)}(t, x, y) \leqq -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on $t \geqq T^*$ and

$$\varliminf_{t\to\infty} \int_T^t \lambda(s)ds \geqq 0 \qquad \text{for all large } T. \tag{4}$$

Moreover, we assume that there exists a $\tau$ and a $w(t, x, y)$ for all large $T$ such that $\tau \geqq T$ and $w(t, x, y)$ is a Liapunov function defined on $t \geqq \tau$, $x > 0$, $y < 0$, which satisfies the following conditions;

(iii) $y \leqq w(t, x, y)$ and $w(\tau, x, y) \leqq b(y)$, where $b(y)$ is continuous, $b(0) = 0$ and $b(y) < 0$ $(y \neq 0)$,

(iv) $\dot{w}_{(2)}(t, x, y) \leqq -\rho(t)w(t, x, y)$, where $\rho(t) \geqq 0$ is continuous and

$$\int_\tau^\infty \frac{1}{r(t)} \exp\{-\int_\tau^t \rho(s)ds\}\, dt = \infty . \tag{5}$$

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large $t$, then $y(t) \geqq 0$ for all large $t$.

Proof. Suppose that there is a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $y(t_n) < 0$. We can assume that $t_n \geqq T^*$ and $t_n$ is sufficiently large so that

$$\varliminf_{t\to\infty} \int_{t_n}^t \lambda(s)ds \geqq 0, \quad x(t) > 0 \quad \text{for } t \geqq t_n. \tag{6}$$

Consider the function $v(t, x(t), y(t))$ for $t \geqq t_n$. Then we have

$$v(t, x(t), y(t)) \leqq v(t_n, x(t_n), y(t_n)) - \int_{t_n}^{t} \lambda(s)ds.$$

From (6) it follows that there is a $T_1 > 0$ such that for all $t \geqq T_1$,

$$\int_{t_n}^{t} \lambda(s)ds \geqq \frac{1}{2} v(t_n, x(t_n), y(t_n)),$$

because $v(t_n, x(t_n), y(t_n)) < 0$. Therefore, for $t \geqq T_1$, we have

$$v(t, x(t), y(t)) \leqq \frac{1}{2} v(t_n, x(t_n), y(t_n)) < 0,$$

which implies that $y(t) < 0$ for all $t \geqq T_1$.

For $T_1$, there is a $\tau$ such that $\tau \geqq T_1$ and there is a Liapunov function $w(t, x, y)$ defined on $t \geqq \tau$, $x > 0$, $y < 0$. For this $w(t, x, y)$, we have

$$y(t) \leqq w(t, x(t), y(t)) \leqq w(\tau, x(\tau), y(\tau))\, e^{-\int_{\tau}^{t} \rho(s)ds} \leqq b(y(\tau))\, e^{-\int_{\tau}^{t} \rho(s)ds}$$

for $t \geqq \tau$. Since $x'(t) = \frac{y(t)}{r(t)}$, we have

$$x'(t) \leqq b(y(\tau)) \frac{1}{r(t)} e^{-\int_{\tau}^{t} \rho(s)ds}, \tag{7}$$

and hence

$$x(t) \leqq x(\tau) + b(y(\tau)) \int_\tau^t \frac{1}{r(u)} e^{-\int_\tau^t \rho(s)\,ds} du.$$

Since $x(t) > 0$ for $t \geqq \tau$ and $b(y(\tau)) < 0$, there arises a contradiction by (5). Thus we see that $y(t) \geqq 0$ for all large $t$.

Remark. In the case where $r(t) \equiv 1$ and $\rho(t) \equiv 0$, condition (iii) can be replaced by

(iii)′ $a(y) \leqq w(t, x, y)$ and $w(\tau, x, y) \leqq b(y)$, where $a(y)$ is monotone, continuous, $a(0) = 0$, $a(y) < 0$ $(y \neq 0)$ and $b(y)$ is continuous, $b(0) = 0$, $b(y) < 0$ $(y \neq 0)$.

By the same argument, we can prove the following lemma.

Lemma 2. For $t \geqq T^*$, $x < 0$, $-\infty < y < \infty$, where $T^*$ can be large, we assume that there exists a Liapunov function $v(t, x, y)$ which satisfies the following conditions;

(i) $yv(t, x, y) < 0$ for $y \neq 0$, $t \geqq T^*$, $x < 0$,

(ii) $\dot{v}_{(2)}(t, x, y) \leqq -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on $t \geqq T^*$ and

$$\varliminf_{t\to\infty} \int_T^t \lambda(s)ds \geqq 0 \quad \text{for all large } T.$$

Moreover, we assume that there exists a $\tau$ and a $w(t, x, y)$ for all large $T$ such that $\tau \geqq T$ and $w(t, x, y)$ is a Liapunov function defined on $t \geqq \tau$, $x < 0$, $y > 0$, which satisfies the following conditions;

(iii) $-y \leqq w(t, x, y)$ and $w(\tau, x, y) \leqq b(y)$, where $b(y)$ is continuous, $b(0) = 0$ and

$b(y) < 0 \ (y \neq 0)$,

(iv) $\dot{w}_{(2)}(t, x, y) \leqq -\rho(t)w(t, x, y)$, where $\rho(t) \geqq 0$ is continuous and

$$\int_T^\infty \frac{1}{r(t)} \exp\{-\int_T^t \rho(s)ds\} = \infty.$$

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) < 0$ for all large $t$, then $y(t) \leqq 0$ for all large $t$.

If we can find Liapunov functions which satisfy the conditions in Lemmas 1 and 2, we can prove the following theorem by the same idea as in the proof of Theorem 1.

Theorem 2. Under the assumptions of Lemmas 1 and 2, we assume that for each $\delta > 0$, there exist a $T(\delta) > 0$ and Liapunov functions $V(t, x, y)$ and $W(t, x, y)$ which are defined on $t \geqq T(\delta)$, $x \geqq \delta$, $y \geqq 0$ and $t \geqq T(\delta)$, $x \leqq -\delta$, $y \leqq 0$, respectively, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

(i) $V(t, x, y)$ and $W(t, x, y)$ tend to infinity uniformly for $x$ and $y$ as $t \to \infty$,

(ii) $\dot{V}_{(2)}(t, x, y) \leqq 0$, as long as $\dot{V}_{(2)}$ is defined,

(iii) $\dot{W}_{(2)}(t, x, y) \leqq 0$. as long as $\dot{W}_{(2)}$ is defined.

Then the equation (1) is oscillatory.

Since we assume the existence of Liapunov functions satisfying the conditions in Lemmas 1 and 2, if $x(t) > 0$ in the future, then $x(t) \geqq \delta$ in the future for some $\delta > 0$, because $x'(t) = \frac{y(t)}{r(t)} \geqq 0$ in the future, and the similar for a solution $x(t) < 0$.

Example 1. Consider the equation (1) and assume that the following conditions are satisfied:

(a)

(8) $$\int_0^\infty \frac{dt}{r(t)} = \infty.$$

(b) For $t \geqq 0$ and $x \geqq 0$, there exists a continuous function $a(t)$ and an $\alpha(x)$ such that

(9) $$\varliminf_{t\to\infty} \int_T^t a(s)ds \geqq 0 \quad \text{for all large } T$$

and that $x\alpha(x) > 0$ $(x \neq 0)$, $\alpha'(x) \geqq 0$ and for all large t, $x \geqq 0$, $|u| < \infty$

(10) $$a(t)\alpha(x) \leqq f(t, x, u).$$

(c) For $t \geqq 0$ and $x \leqq 0$, there exists a continuous function $b(t)$ and a $\beta(x)$ such that

(11) $$\varliminf_{t\to\infty} \int_T^t b(s)ds \geqq 0 \quad \text{for all large } T$$

and that $x\beta(x) > 0$ $(x \neq 0)$, $\beta'(x) \geqq 0$ and for all large t, $x \leqq 0$, $|u| < \infty$

(12) $$f(t, x, u) \leqq b(t)\beta(x).$$

Under the assumptions above, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large $t$, then $y(t) \geqq 0$ for all large $t$. To see this, we can assume that (9) through (12) hold good for all $t \geqq T^*$ and all $T \geqq T^*$. For $t \geqq T^*$, $x > 0$, $|y| < \infty$, define $v(t, x, y)$ by

$$v(t, x, y) = \frac{y}{\alpha(x)} .$$

Then, we have

$$\dot{v}_{(2)}(t, x, y) = \frac{1}{\alpha^2(x)} \{ - f(t, x, \frac{y}{r(t)})\alpha(x) - y\alpha'(x)\frac{y}{r(t)} \}$$

$$\leqq - a(t) .$$

Hence this $v(t, x, y)$ satisfies the conditions in Lemma 1 with $\lambda(t) = a(t)$.

Since the condition (9) implies that for all $T \geqq T^*$, there is a $\tau$ such that $\tau \geqq T$ and

$$\int_\tau^t a(s)ds \geqq 0 \qquad \text{for all} \quad t \geqq \tau ,$$

a function $w(t, x, y) = y + \alpha(x) \int_\tau^t a(s)ds$ defined on $t \geqq \tau, x > 0, y < 0$ satisfies the conditions in Lemma 1 with $\rho(t) \equiv 0$. Thus the conclusion follows from Lemma 1.

If we consider functions

$$v(t, x, y) = \frac{y}{\beta(x)} , \quad t \geqq T^*, \ x < 0, \ |y| < \infty,$$

$$w(t, x, y) = -y - \beta(x) \int_{\tau}^{t} b(s)ds, \quad t \geqq \tau, \ x < 0, \ y > 0,$$

from Lemma 2 it follows that if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) < 0$ for all large $t$, then $y(t) \leqq 0$ for all large $t$.

Under the assumptions (a), (b) and (c), we shall discuss oscillatory property of solutions of (1). The following results contain Macki and Wong's result [3], Coles' result [2] and others.

(I) If we have

$$\int_0^{\infty} a(s)ds = \infty, \qquad \int_0^{\infty} b(s)ds = \infty, \tag{13}$$

then the equation (1) is oscillatory.

For $t \geqq T^*$, $x > 0$ and $-\infty < y < \infty$, set

$$V(t, x, y) = \begin{cases} \dfrac{y}{\alpha(x)} + \int_0^t a(s)ds & (y \geqq 0) \\ \int_0^t a(s)ds & (y < 0). \end{cases}$$

Then, clearly $V(t, x, y) \to \infty$ uniformly for $x > 0$ and $-\infty < y < \infty$, and we have

$$\dot{V}_{(2)}(t, x, y) = \frac{1}{\alpha^2(x)} \{ -f(t, x, \frac{y}{r(t)}) \alpha(x) - y\alpha'(x) \frac{y}{r(t)} \} + a(t)$$

$$\leqq -a(t) + a(t)$$

$$\leqq 0$$

for $t \geqq T^*$, $x > 0$ and $y \geqq 0$. Therefore, $V(t, x, y)$ satisfies the conditions in Theorem 1. Similarly,

$$W(t, x, y) = \begin{cases} \int_0^t b(s)ds & (y > 0) \\ \frac{y}{\beta(x)} + \int_0^t b(s)ds & (y \leqq 0) \end{cases}$$

satisfies the conditions in Theorem 1. Thus the conclusion follows from Theorem 1.

(II) If we have

$$(14) \quad \int_0^\infty a(s)ds < \infty, \quad \int_0^t \left(\frac{1}{r(s)} \int_s^\infty a(u)du\right)ds \to \infty \quad \text{as} \quad t \to \infty,$$

$$(15) \quad \int_0^\infty b(s)ds < \infty, \quad \int_0^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds \to \infty \quad \text{as} \quad t \to \infty,$$

$$(16) \quad \int_\epsilon^\infty \frac{du}{\alpha(u)} < \infty \quad \text{for some} \quad \epsilon > 0$$

and

$$(17) \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some} \quad \epsilon > 0,$$

the equation (1) is oscillatory.

For $t \geqq T^*$, $x > 0$, $|y| < \infty$, set

$$(18) \quad V(t, x, y) = \int_x^\infty \frac{du}{\alpha(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty a(u)du\right)ds.$$

For a solution $x(t)$ which satisfies $x(t) > 0$ for all large $t$, we can assume that $x(t) > 0$, $y(t) \geqq 0$ for $t \geqq \sigma$, $\sigma$ sufficiently large, and hence

$$\dot{V}(t, x(t), y(t)) = -\frac{1}{\alpha(x(t))}\frac{y(t)}{r(t)} + \frac{1}{r(t)}\int_t^\infty a(u)du$$

$$= \frac{1}{r(t)}\{-\frac{y(t)}{\alpha(x(t))} + \int_t^\infty a(u)du\}$$

If we set $V^*(t, x, y) = -\frac{y}{\alpha(x)} + \int_t^\infty a(u)du$, $V^*(t, x(t), y(t)) \leqq \int_t^\infty a(u)du$, and hence

$$\overline{\lim_{t\to\infty}}\, V^*(t, x(t), y(t)) \leqq 0.$$

On the other hand, we have.

$$\dot{V}^*_{(2)}(t, x, y) = -\frac{1}{\alpha^2(x)}\{-f(t, x, \frac{y}{r(t)})\alpha(x) - y\alpha'(x)\frac{y}{r(t)}\} - a(t)$$

$$\geqq 0.$$

Therefore $V^*(t, x(t), y(t)) \leqq 0$ and consequently $\dot{V}(t, x(t), y(t)) \leqq 0$ for $t \geqq \sigma$.

Similarly, if we define $W(t, x, y)$ by

$$W(t, x, y) = \int_x^{-\infty} \frac{du}{\beta(u)} + \int_0^t (\frac{1}{r(s)}\int_s^\infty b(u)du)ds,$$

this $W(t, x, y)$ satisfies the conditions in Theorem 1. Thus the conclusion follows from

Theorem 1.

Remark 1. It is clear that we can combine the conditions in (I) and (II). For example, if

$$\int_0^\infty a(s)ds = \infty, \quad \int_0^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds \to \infty \text{ as } t \to \infty$$

and

$$\int_\epsilon^\infty \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

then the equation (1) is oscillatory.

Remark 2. If a continuous function $a(t)$ satisfies (9), then $\int_0^\infty a(s)ds = \infty$ or $\int_0^\infty a(s)ds$ exists. Macki and Wong assumed $\alpha(x)$ and $\beta(x)$ to be nondecreasing, but we can find an $\alpha(x)$ and a $\beta(x)$ which have their derivatives, because $a(t)$, $b(t)$ are nonnegative in their case.

(III) If there exist a constant $m > 0$ and two positive differentiable functions $h(t)$ and $g(t)$ defined on $I$ such that $\alpha'(x) \geq m$, $\beta'(x) \geq m$ and

$$\int_0^t h(s)\{a(s) - \frac{1}{4}\frac{r(s)}{m}(\frac{h'(s)}{h(s)})^2\}\, ds \to \infty \text{ as } t \to \infty,$$

$$\int_0^t g(s)\{b(s) - \frac{1}{4}\frac{r(s)}{m}(\frac{g'(s)}{g(s)})^2\}\, ds \to \infty \text{ as } t \to \infty,$$

the equation (1) is oscillatory.

This is a generalization of a result of Opial [4], and in this case,

$$V(t,x,y) = \frac{y}{\alpha(x)} h(t) + \int_0^t h(s)\, \{a(s) - \frac{1}{4}\frac{r(s)}{m}(\frac{h'(s)}{h(s)})^2\}\, ds \qquad (x>0,\ y\geqq 0),$$

$$W(t,x,y) = \frac{y}{\beta(x)} g(t) + \int_0^t g(s)\, \{b(s) - \frac{1}{4}\frac{r(s)}{m}(\frac{g'(s)}{g(s)})^2\}\, ds \qquad (x<0,\ y\leqq 0)$$

satisfy the conditions in Theorem 1.

Lemma 3. In addition to the assumption of Lemma 1, assume that there exists a Liapunov function $u(t,x,y)$ defined on $t \geqq T^*$, $x>0$, $y>R$ ($R>0$: large), which satisfies

(i) $u(t,x,y) \to \infty$ uniformly for $t$, $x$ as $y \to \infty$, and $u(t,x,y) \leqq \gamma(y)$, where $\gamma(r)>0$ is continuous,

(ii) $\dot{u}_{(2)}(t,x,y) \leqq 0$.

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t)>0$ for all large $t$, then $y(t)$ is bounded for all large $t$.

Proof. Let $x(t)>0$ and $y(t) \geqq 0$ for $t \geqq \sigma$, $\sigma \geqq T^*$. By Lemma 1, there is such a $\sigma$. Let $K$ be such that $y(\sigma)<K$, $K>R$. There is a constant $\gamma^*>0$ such that $u(t,x,K) \leqq \gamma^*$, and there also exists an $M>0$ for which we have $\gamma^* < u(t,x,M)$ for all $t \geqq \sigma$ and $x>0$ by the condition (i). But there arises a contradiction by (ii), which shows that $0 \leqq y(t) < M$ for all $t \geqq \sigma$.

Lemma 4. In addition to the assumption of Lemma 2, assume that there exists a Liapunov function $u(t, x, y)$ defined on $t \geqq T^*$, $x < 0$, $y < -R$ ($R > 0$: large), which satisfies

(i) $u(t, x, y) \to \infty$ uniformly for $t$, $x$ as $y \to -\infty$, and $u(t, x, y) \leqq \gamma(|y|)$, where $\gamma(r) > 0$ is continuous,

(ii) $\dot{u}_{(2)}(t, x, y) \leqq 0$.

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) < 0$ for all large $t$, then $y(t)$ is bounded for all large $t$.

Theorem 3. Under the assumptions of Lemmas 3 and 4, we assume that for each $\delta > 0$ and $m > 0$, there exist a $T(\delta, m) > 0$ and two Liapunov functions $V(t, x, y)$ and $W(t, x, y)$ such that $V(t, x, y)$ is defined on $t \geqq T(\delta, m)$, $x > \delta$, $0 \leqq y < m$ and $W(t, x, y)$ is defined on $t \geqq T(\delta, m)$, $x < -\delta$, $-m < y \leqq 0$, and we assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

(i) $V(t, x, y)$ and $W(t, x, y)$ tend to infinity uniformly for $x$, $y$ as $t \to \infty$,

(ii) $\dot{V}_{(2)}(t, x, y) \leqq 0$ as long as $\dot{V}_{(2)}$ is defined,

(iii) $\dot{W}_{(2)}(t, x, y) \leqq 0$ as long as $\dot{W}_{(2)}$ is defined.

Then the equation (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) which exists in the future, and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large $t$. Now assume that $x(t) > 0$ for all large $t$. By Lemma 1, we can see that there is a $t_1 > 0$ such that $x(t) > 0$, $y(t) \geqq 0$ for all $t \geqq t_1$, where we can assume that $t_1 \geqq T^*$. By Lemma 3, there is an $m > 0$ such that $0 \leqq y(t) < m$ for all $t \geqq t_1$. Since $x'(t) = \frac{y(t)}{r(t)} \geqq 0$ for $t \geqq t_1$, we have $x(t) \geqq x(t_1) > 0$ for $t \geqq t_1$. Consider the Liapunov function $V(t, x, y)$ defined for $t \geqq T(\delta, m)$, $x > \delta$,

$0 \leqq y < m$, where $\delta = \frac{x(t_1)}{2}$ and we can assume $T \geqq t_1$. Then, by the same argument as in the proof of Theorem 1, there arises a contradiction. When $x(t) < 0$ for all large $t$, we have also a contradiction by using Lemma 4 and $W(t, x, y)$. Thus we can see that the equation (1) is oscillatory.

Example 2. (Bobisud [ 1 ] ). Consider an equation

$$x'' + a(t, x, x')x' + f(t, x, x') = 0 \tag{19}$$

and an equivalent system

$$x' = y, \quad y' = -a(t, x, y)y - f(t, x, y). \tag{20}$$

The following assumptions will be made;

(i) $f(t, x, y)$ is continuous on $I \times R \times R$ and $xf(t, x, y) > 0$ for $x \neq 0$,

(ii) $a(t, x, y)$ is continuous on $I \times R \times R$ and there exist continuous nonnegative functions $k(t)$ and $p(t)$ such that

$$-k(t) \leqq a(t, x, y) \leqq p(t) \quad \text{for} \quad t \in I, x \in R, y \in R,$$

(iii) for any $\delta > 0$ and $m > 0$, there exists a $T(\delta, m)$ and a $g(t; \delta, m) \geqq 0$ defined for $t \geqq T(\delta, m)$ such that

$$\int_{T(\delta, m)}^{t} g(s; \delta, m)ds \to \infty \text{ as } t \to \infty$$

and that $|x| \geqq \delta$, $|y| \leqq m$ and $xy \geqq 0$ imply $|f(t, x, y)| \geqq g(t; \delta, m)$,

(iv)

$$\int_0^{\infty} k(s)ds < \infty, \quad \lim_{t\to\infty} \int_0^t e^{-\int_0^s p(\sigma)d\sigma} ds = \infty.$$

Then the equation (19) is oscillatory.

For this equation, it is not difficult to find Liapunov functions which satisfy the conditions in Theorem 3. For $t \geqq 0$, $x > 0$, $|y| < \infty$, the function

$$v(t, x, y) = \begin{cases} e^{-\int_0^t k(s)ds} y & (y \geqq 0) \\ e^{\int_0^t p(s)ds} y & (y < 0) \end{cases}$$

satisfies the conditions in Lemma 1 with $\lambda(t) \equiv 0$. For any $\tau \geqq 0$, the function $w(t, x, y) = y$ defined for $t \geqq \tau$, $x > 0$, $y < 0$ satisfies the conditions in Lemma 1, since

$$\dot{w}_{(20)}(t, x, y) = - a(t, x, y)y - f(t, x, y)$$

$$\leqq - p(t)y$$

$$\leqq - p(t)w(t, x, y)$$

and

$$\int_{\tau}^{\infty} e^{-\int_{\tau}^{t} p(s)ds} dt = \infty .$$

Moreover, it is easily seen that $u(t, x, y) = y^2 \exp(-2\int_0^t k(s)ds)$ satisfies the conditions in Lemma 3, since $\int_0^{\infty} k(t)dt < \infty$ and $f(t, x, y) > 0$. Furthermore, we can see that

$$v(t, x, y) = \begin{cases} e^{\int_0^t p(s)ds}(-y) & (t \geqq 0,\ x < 0,\ y \geqq 0) \\ e^{-\int_0^t k(s)ds}(-y) & (t \geqq 0,\ x < 0,\ y < 0), \end{cases}$$

$$w(t, x, y) = -y \quad (t \geqq \tau,\ x < 0,\ y > 0)$$

and

$$u(t, x, y) = e^{-2\int_0^t k(s)ds} y^2 \quad (t \geqq 0,\ x < 0,\ y \leqq 0)$$

satisfy the conditions in Lemma 4. Next, for each $\delta > 0$ and $m > 0$, define $V(t, x, y)$ for $t \geqq T(\delta, m)$, $x \geqq \delta$, $0 \leqq y \leqq m$ by

$$V(t, x, y) = e^{-\int_0^t k(s)ds} y + L \int_{T(\delta, m)}^{t} g(s; \delta, m)ds,$$

where $L = e^{-\int_0^{\infty} k(s)ds} > 0$. Then we have

$$\dot{V}_{(20)}(t, x, y) = e^{-\int_0^t k(s)ds}\{-k(t)y - a(t, x, y)y - f(t, x, y)\} + Lg(t; \delta, m)$$

$$\leqq e^{-\int_0^t k(s)ds}\{-k(t)y + k(t)y - g(t; \delta, m)\} + Lg(t; \delta, m)$$

$$\leqq -e^{\int_0^t k(s)ds} g(t; \delta, m) + Lg(t; \delta, m) \leqq 0.$$

Thus we see that $V(t, x, y)$ satisfies the conditions in Theorem 3. Similarly,

$$W(t, x, y) = e^{-\int_0^t k(s)ds}(-y) + L\int_{T(\delta, m)}^t g(s; \delta, m)ds \cdot$$

is the desired one. Thus it follows from Theorem 3 that the equation (19) is oscillatory.

Example 3. For the equation (19), we assume (i) and (ii), and instead of (iii), (iv), we assume that

(iii)′ for any $\delta > 0$, there exists a $T(\delta) > 0$ and a $g(t; \delta) \geqq 0$ defined for $t \geqq T(\delta)$ such that

(21) $$e^{-\int_0^t k(s)ds} \int_{T(\delta)}^t g(s; \delta)ds \to \infty \quad \text{as} \; t \to \infty$$

and that $|x| \geqq \delta$, $xy \geqq 0$ imply $|f(t, x, y)| \geqq g(t; \delta)$,

(iv)′

$$\lim_{t\to\infty} \int_0^t e^{-\int_0^s p(\sigma)d\sigma} ds = \infty.$$

Then the equation (19) is oscillatory.

From conditions (i) and (ii), it follows that there are Liapunov functions which satisfy the conditions in Lemmas 1 and 2, as was seen in Example 2. For $t \geqq T(\delta)$, $x \geqq \delta$, $y \geqq 0$, define

$$V(t, x, y) = e^{-\int_0^t k(s)ds} y + e^{-\int_0^t k(s)ds} \int_{T(\delta)}^t g(s; \delta)ds.$$

Then we have

$$\begin{aligned} \dot{V}_{(20)}(t, x, y) &= e^{-\int_0^t k(s)ds} \{ -k(t)y - a(t, x, y)y - f(t, x, y) \} \\ &+ e^{-\int_0^t k(s)ds} \{ -k(t) \int_{T(\delta)}^t g(s; \delta)ds + g(t; \delta) \} \\ &\leqq e^{-\int_0^t k(s)ds} \{ -k(t)y + k(t)y - g(t; \delta) \} \\ &-k(t)e^{-\int_0^t k(s)ds} \int_{T(\delta)}^t g(s; \delta)ds + e^{-\int_0^t k(s)ds} g(t; \delta) \\ &\leqq 0. \end{aligned}$$

For $t \geqq T(\delta)$, $x \leqq -\delta$, $y \leqq 0$, if we define $W(t, x, y)$ by

$$W(t, x, y) = e^{-\int_0^t k(s)ds}(-y) + e^{-\int_0^t k(s)ds} \int_{T(\delta)}^t g(s; \delta)ds,$$

we have also $\dot{W}_{(20)}(t, x, y) \leqq 0$. Therefore we can conclude by Theorem 2 that the equation (19) is oscillatory.

Remark. For the equation (19), Bobisud claimed in [ 1 ] that the equation (19) is oscillatory under the assumptions (i), (ii) in Example 2 and

(iii)" given $\delta > 0$ there exists a $T(\delta) > 0$ and a $g(t; \delta) \geqq 0$ defined for $t \geqq T(\delta)$ with

(22) $$\frac{1}{t}\int_{T(\delta)}^{t}(t - s)g(s; \delta)ds \to \infty \text{ as } t \to \infty$$

and such that $|x| \geqq \delta$, $xy > 0$ imply $|f(t, x, y)| \geqq g(t; \delta)$,

(iv)" for any $t_1$, $t_2 > 0$,

(23) $$\frac{1}{t}\int_{t_1}^{t} k(\sigma)e^{\int_{t_2}^{\sigma} k(s)ds} d\sigma$$

is bounded from above and

$$\lim_{t\to\infty}\int_{t_1}^{t} e^{-\int_{t_1}^{s} p(\sigma)d\sigma} ds = \infty$$

However, there is a mistake in his proof, and actually his result is not necessarily true as the following example shows. Consider an equation

$$x'' - \frac{x'}{t+1} + f(t, x) = 0,$$

where

$$f(t,x)=\begin{cases}\dfrac{1}{t+1} & (x \geqq 1)\\[2ex] \dfrac{x}{t+1} & (|x|<1)\\[2ex] -\dfrac{1}{t+1} & (x \leqq -1)\end{cases}$$

This equation satisfies the conditions above, but it has solutions $x = t + 1$ and $x = -t - 1$ which are not oscillatory.

Under the condition (iv)″, which is equivalent to

(iv)‴ for some $t_0 > 0$

$$\frac{1}{t} e^{\int_{t_0}^{t} k(s)ds} < M \quad \text{for } t \geqq t_0 \tag{24}$$

and

$$\lim_{t\to\infty} \int_0^t e^{-\int_0^s p(\sigma)d\sigma} ds = \infty,$$

if we assume

$$\frac{1}{t}\int_{T(\delta)}^{t} g(s;\delta)ds \to \infty \text{ as } t \to \infty \tag{25}$$

in place of (22), the equation (19) is oscillatory, because (24) and (25) imply (21).

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