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Some remarks on the Kirby-Siebenmann class

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§1. Statement of results.

Let $k \in H^4(BTop;Z_2)$ be the Kirby-Siebenmann class, i.e. the unique obstruction to stable PL reducibility of Top bundles. In this note we remark some elementary properties of k and using them we construct a few non-triangulable manifolds of dimension 5 and 6.

First we show

Proposition 1. k is primitive, i.e. if $\mu : BTop \times BTop \to BTop$ is the natural H-space structure on BTop, then

$$\mu^*(k) = k \times 1 + 1 \times k.$$

For a topological manifold M, we define the Kirby-Siebenmann class of M, k(M), to be that of the tangent micro-bundle of M. Then we have

Corollary 2. (i) $k(M) = k(\nu(M))$, where $\nu(M)$ is the stable normal bundle of M.

(ii) $k(M \times N) = k(M) \times 1 + 1 \times k(N)$.

Next we consider the following commutative diagram.

$$\begin{array}{ccccccc} & \overset{i'}{\nearrow} & G/PL & \xrightarrow{p'} & G/Top & \overset{m}{\searrow} & \\ Top/PL & & \downarrow j & & \downarrow j' & & B(Top/PL) = K(Z_2,4) \\ & \underset{i}{\searrow} & BPL & \xrightarrow[p]{} & BTop & \underset{k}{\nearrow} & \end{array}$$

And we show

Proposition 3. $m = k_2{}^2 + x \bmod 2 \in H^4(G/Top;Z_2)$, where k_2

is the first Kervaire obstruction and x mod 2 is the mod 2 of the fundamental class of $K(Z_{(2)},4)$. (Recall that G/Top localized at 2 = $\prod_{i\geq 0} K(Z_2,4i+2) \times \prod_{i\geq 1} K(Z_{(2)},4i)$, Sullivan [1] and Kirby-Siebenmann.)

As a corollary, we obtain

Corollary 4. Let $I_j = (i^j_{n_j}, \cdots, i^j_1)$ be admissible $(j=1,\cdots,m)$ such that, $e(I_j) < 4$ and $i^j_1 \neq 1$, then

$$P(Sq^{I_1}(k), \cdots, Sq^{I_m}(k)) \neq 0$$

for any polynomial $P(x_1, \cdots, x_m) \neq 0$.

On the other hand, it is easy to show

Proposition 5. $Sq^1(k) \neq 0$.

By using Proposition 3. and the surgery theory in Top category (C.T.C. Wall [2]), we obtain

Theorem 6. Let M^5 be an oriented closed PL manifold with $\pi_1(M) = Z_2$, then there is a non-triangulable topological manifold N^5 having the same homotopy type as M.

Theorem 7. Let M^5 be a non-orientable closed topological manifold with $\pi_1(M) = Z_2$. Then for any homotopy equivalence

$$f : N^5 \to M^5$$

we have

$$k(N) = f^*(k(M)).$$

Theorem 8. There is a non-triangulable closed topological manifold M^6 having the same homotopy type as PR^6.

§2. Proofs.

Proof of Proposition 1. Consider the following commutative diagram.

$$\begin{array}{ccc} BPL \times BPL & \xrightarrow{\mu} & BPL \\ \downarrow p\times p & & \downarrow p \\ BTop \times BTop & \xrightarrow[\mu]{} & BTop \end{array}$$

Clearly $p^*(k) = 0$, hence $(p\times p)^*\mu^*(k) = 0$. But since $H^i(BTop;Z_2) \simeq H^i(BPL;Z_2)$ for $i \leq 3$, we have the result.

Q.E.D.

Proof of Proposition 3. By Sullivan [1] and Kirby-Siebenmann

$$G/PL \text{ localized at } 2 = K(Z_2,2) \underset{\delta Sq^2}{\times} K(Z_{(2)},4) \times \prod_{i\geq 1} K(Z_2,4i+2) \times \prod_{i\geq 2} K(Z_{(2)},4i)$$

$$G/Top \text{ localized at } 2 = \prod_{i\geq 0} K(Z_2,4i+2) \times \prod_{i\geq 1} K(Z_{(2)},4i).$$

Therefore $H^4(G/Top;Z_2) \cong Z_2 \oplus Z_2$ generated by $k_2{}^2$ and $x \bmod 2$. The Serre exact sequence of the fibering $Top/PL \to G/PL \to G/Top$ yields

$$0 \to H^3(Top/PL;Z_2) \overset{\tau}{\to} H^4(G/Top;Z_2) \overset{p^*}{\to} H^4(G/PL;Z_2) \to \cdots$$

Thus $m = \tau(u)$ is the non-zero element of Ker p^* ($u \in H^3(Top/PL;Z_2)$ is the fundamental class).

Now clearly $p^*(k_2{}^2) \neq 0$ and $p^*(x \bmod 2) \neq 0$. Hence we have

$$m = k_2{}^2 + x \bmod 2.$$

Q.E.D.

Corollary 4 is an immediate consequence of Proposition 3.

Proof of Proposition 5. This follows from the Serre exact sequence of the fibering

$$Top/PL \to BSpinPL \to BSpinTop.$$

Proof of Theorem 6. According to Wall [2], the surgery theory is valid in Top category. So we use it.

We first recall that $L_5(Z_2,+) = 0$ ([2]) .

Now there is a fibering sequence

$$\cdots \to Top/PL \to G/PL \to G/Top \xrightarrow{m} B(Top/PL).$$

Thus we have an exact sequence

$$\cdots \to [M,G/PL] \to [M,G/Top] \xrightarrow{m_*} H4(M;Z_2)$$

Now $[M,G/Top] \cong H^4(M;Z) \oplus H^2(M;Z_2)$ and by Proposition 3, m_* is given by

$$m_*(y \oplus z) = y \bmod 2 + z^2$$

where $y \in H^4(M;Z)$ and $z \in H^2(M;Z_2)$. Since M is orientable and $\pi_1(M) = Z_2$, we have

$$H^4(M;Z) \xrightarrow[\bmod 2]{\sim} H^4(M;Z_2).$$

therefore m_* is epimorphic. Hence the map

$$[M,G/PL] \to [M,G/Top]$$

is not epimorphic. Since the surgery obstruction is trivial, we obtain

$$\mathcal{S}_{PL}(M) \to \mathcal{S}_{Top}(M)$$

is not epimorphic. This proves the proposition.

Q.E.D.

Proof of Theorem 7. Let $f: N \to M$ be a homotopy equivalence. It suffices to show that if M is triangulable, then so is N. Thus assume that M is a PL manifold. Since M is non-orientable, the map

$$H^4(M;Z) \xrightarrow[\text{mod } 2]{} H^4(M;Z_2)$$

is the zero map.

Let $z \in H^2(M;Z_2)$ be any element and assume $z^2 \neq 0$. Then since $H^4(M;Z_2) \cong Z_2$, z^2 is the unique non-zero element of $H^4(M;Z_2)$. Since M is non-orientable, we have

$$Sq^1(z^2) \neq 0.$$

On the other hand $Sq^1(z^2) = Sq^1Sq^2(z) = Sq^3(z) = 0$. This is a contradiction. Hence $z^2 = 0$ and we have $m_* = 0$. Therefore the map $[M,G/PL] \to [M,G/Top]$ is epimorphic. Since $L_5(Z_2,-) = 0$, it follows that

$$\mathcal{S}_{PL}(M) \to \mathcal{S}_{Top}(M)$$

is epimorphic. In particular N is triangulable.

Q.E.D.

Proof of Theorem 8. According to Wall [2], $L_7(Z_2,-) = 0$ and $L_6(Z_2,-) = Z_2$ given by the Kervaire invariant. Now we have

$$[PR^6,G/Top] \cong H^2(PR^6;Z_2) \oplus H^4(PR^6;Z) \oplus H^6(PR^6;Z_2)$$

$$\cong Z_2 \oplus Z_2 \oplus Z_2 .$$

Let $u \in H^1(PR^6;Z_2)$ and $x \in H^4(PR^6;Z)$ be the generators, thus $x \bmod 2 = u^4$. Then it is easy to show

$$(u^2,0,0),\ (u^2,0,u^6),\ (0,x,0),\ (0,x,u^6)$$

are not in $\mathrm{Im}([PR^6,G/PL] \to [PR^6,G/Top])$.

Now a simple calculation shows that the surgery obstruction of (0,x,0) is zero. Hence the map

$$\mathcal{S}_{PL}(PR^6) \rightarrow \mathcal{S}_{Top}(PR^6)$$

is not epimorphic. This proves the theorem.

Q.E.D.

References

[1] Sullivan, D., Triangulating and Smoothing Homotopy Equivalences and Homeomorphisms. Geometric Topology seminar notes, Princeton University, 1967.

[2] Wall, C.T.C., Surgery on compact manifolds, Academic Press, 1970.

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