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Integration of partial differential equations with quadratures*

by

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§1. Integration with quadratures

Lagrange solved a partial differential equation of the first order

(1) $V(x, y, z, p, q) = 0, \quad p = \partial z/\partial x, \quad q = \partial z/\partial y$

as follows (1772): If we have a complete integral

$$f(x, y, z; a, b) = 0, \quad a = \text{constant}, \quad b = \text{constant},$$

of the equation (1), then the general integral is obtained by elimination of a from

$$f = 0, \quad \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a} = 0, \quad b = \phi(a),$$

where ϕ is an arbitrary function of a. To get the complete integral, we take a solution F of $[F, V] = 0$, where

$$[F, V] = \frac{\partial F}{\partial p}\frac{dV}{dx} + \frac{\partial F}{\partial q}\frac{dV}{dy} - \frac{dF}{dx}\frac{\partial V}{\partial p} - \frac{dF}{dy}\frac{\partial V}{\partial q},$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z}, \quad \frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z}.$$

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Then the system of partial differential equations of the first order $[G, V] = [G, F] = 0$ with the unknown function G is complete, and hence we can solve it. The complete integral is obtained by elimination of p and q from $V = 0$, $F = a$, $G = b$. A function F of (x, y, z, p, q) is a solution of $[F, V] = 0$, if and only if it is constant along every integral curve of the Lagrange-Charpit system

(2) $$\frac{dx}{\partial V/\partial p} = \frac{dy}{\partial V/\partial q} = \frac{dz}{p\,\partial V/\partial p + q\,\partial V/\partial q} = \frac{-dp}{dV/dx} = \frac{-dq}{dV/dy}.$$

Monge tried to reduce a partial differential equation of the second order

(3) $$F(x, y, z, p, q, r, s, t) = 0,$$

$$r = \partial^2 z/\partial x^2, \quad s = \partial^2 z/\partial x \partial y, \quad t = \partial^2 z/\partial y^2$$

to an equation of the first order $V_1 + \phi(V_2) = 0$ containing an arbitrary function ϕ (1784). It is possible, if and only if the equation (3) has the form

(4) $$Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$$

and the system of linear equations of the first order

$$N\frac{dV}{dx} - L\frac{\partial V}{\partial p} + \lambda_1\frac{\partial V}{\partial q} = 0,$$

$$N\frac{dV}{dy} + \lambda_2\frac{\partial V}{\partial p} - H\frac{\partial V}{\partial q} = 0$$

has two functionally independent solutions V_1 and V_2. Here H, K, L, M, N are functions of (x, y, z, p, q), and λ_1, λ_2 are the roots of the quadratic equation

$$\lambda^2 - 2K\lambda + HL - MN = 0.$$

An equation of the form (4) is called Monge-Ampère's equation.

Ampère tried to integrate partial differential equations of the second order which can not be solved by Monge's method, and constructed the general integral on various examples (1819). Generalizing his method, Darboux gave the following method of integration (1870): For a given equation (3) find two equations $G_i = 0$ of the second order containing an arbitrary function ϕ_i of one variable, $i = 1, 2$ so that the three equations $F = G_1 = G_2 = 0$ forms a completely integrable system. Suppose that it is possible. Then, integrating the completely integrable system, we have the general integral of (3) containing two arbitrary functions ϕ_1 and ϕ_2.

§2. Involutive systems

Lagrange and Jacobi found a method of prolonging a system of partial differential equations of the first order with one unknown function to a complete system. Any system is prolonged to either a complete system or an incompatible system by their method. Generalizing the notion of a complete system, Cartan and Kuranishi defined an involutive system. They gave a method of prolonging a system of partial differential equations of general type to an involutive system. By this method, unfortunately, there exists a system which is prolonged neither to an involutive system nor an incompatible system. Combining Cartan-Kuranishi's prolongation and that of Lagrange and Jacobi, we have a method of prolongation by which any system of general type can be prolonged to an involutive system or to an incompatible system ([11], [12]).

§3. Transformation of equations

It is Laplace who tried to transform an equation to an integrable equation with quadratures, if the given equation can not be solved with quadratures. For this purpose he devised the following Laplace transformation which can be applied to a linear hyperbolic equation

$$s + ap + bq + cz = 0, \tag{5}$$

where a, b, c are functions of x, y (1777, [3, Chap.II]): The equation (5) is expressed in the form

$$\frac{\partial}{\partial x}(q + az) + b(q + az) - H_o z = 0,$$

where

$$H_o = \frac{\partial a}{\partial x} + ab - c.$$

Hence the equation (5) is solved with quadratures, if $H_o = 0$. Suppose $H_o \neq 0$. Then we can transform (5) by the Laplace transformation

$$z_1 = q + az, \quad H_o z = p_1 + bz_1$$

to

$$\frac{\partial}{\partial x}(q_1 + a_1 z_1) + b(q_1 + a_1 z_1) - H_1 z_1 = 0,$$

where

$$H_1 = \frac{\partial a_1}{\partial x} - \frac{\partial b}{\partial y} + H_o, \quad a_1 = a - \frac{\partial \log H_o}{\partial y}.$$

The (n+1)-st Laplace invariant H_n is defined inductively by

$$H_n = \frac{\partial a_n}{\partial x} - \frac{\partial b}{\partial y} + H_{n-1},$$

unless $H_{n-1} = 0$, where

$$a_n = a_{n-1} - \frac{\partial \log H_{n-1}}{\partial y}.$$

The given equation (5) is transformed to

$$\frac{\partial}{\partial x}(q_n + a_n z_n) + b(q_n + a_n z_n) - H_n z_n = 0$$

by succesive application of the Laplace transformation. Hence, if $H_n = 0$, then we can solve (5) with quadratures.

A transformation which acts on any partial differential equation was found by Lie (1875, [10]). This is a contact transformation defined as follows: A transformation

$$X_i = X_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \quad 1 \leqq i \leqq n$$

$$Z = Z(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$$

$$P_i = P_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \quad 1 \leqq i \leqq n$$

is called a contact transformation, if it satisfies

$$dZ - \sum_{i=1}^{n} P_i d X_i = \rho (dz - \sum_{i=1}^{n} p_i dx_i)$$

identically, where ρ is a function of $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$. The fundamental theorem of Lie is as follows:

Given n functions X_i, $1 \leqq i \leqq n$, satisfying

$$[X_j, X_k] = 0, \quad 1 \leqq j, k \leqq n.$$

Then we can find $n+1$ functions Z, P_h, $1 \leqq h \leqq n$ so that they form a contact transformation with X_i, $1 \leqq i \leqq n$. By this theorem any equation of the first order

$$V(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

$$p_i = \partial z / \partial x_i, \quad 1 \leqq i \leqq n,$$

is transformed to $p_1 = 0$ by a contact transformation.

Bäcklund found the following transformation, investigating transformations of a surface of constant negative curvature (1882, [4, p.438]): Two partial differential equations $\Phi = 0$ and $\Psi = 0$ are called the original and the transformed equation of a Bäcklund transformation

$$F_i(x, y, z, p, q; x_1, y_1, z_1, p_1, q_1) = 0, \quad 1 \leqq i \leqq 4$$

respectively, if for any integral surface $z = \varphi(x, y)$ of $\Phi = 0$ we have a completely integrable system of the first order eliminating x and y from

$$F_i(x, y, \varphi, \varphi_x, \varphi_y; x_1, y_1, z_1, p_1, q_1) = 0, \quad 1 \leqq i \leqq 4,$$

and if each solution of this system is an integral surface of $\Psi = 0$. For example, take

$$F_1 = x_1 - x, \quad F_2 = y_1 - y, \quad F_3 = z_1 - q - az,$$

$$F_4 = p_1 + bq - \left(\frac{\partial a}{\partial x} - c\right)z.$$

Then it is the Laplace transformation ([9]). To apply this Bäcklund transformation for integrating Monge-Ampère's equations, Imschenetsky generalized the Laplace transformation as follows ([2]): Take

$$F_1 = x_1 - x, \quad F_2 = y_1 - y, \quad F_3 = z_1 - h(x, y, z, q),$$

$$F_4 = p_1 - k(x, y, z, q),$$

and assume that

$$\frac{\partial h}{\partial q} \neq 0, \quad \frac{\partial (h, k)}{\partial (z, q)} \neq 0.$$

Then the equation

$$\frac{\partial h}{\partial q} s + \frac{\partial h}{\partial z} p + \frac{\partial h}{\partial x} - k = 0$$

is transformed to

$$\frac{\partial h}{\partial q} s_1 = \frac{\partial k}{\partial q} q_1 + \frac{dk}{dy} \frac{\partial h}{\partial q} - \frac{dh}{dy} \frac{\partial k}{\partial q},$$

where we replace x, y, z, q by their values obtained from $x_1 = x$, $y_1 = y$, $z_1 = h$, $p_1 = k$.

§4. Cauchy's problem

Cauchy proved that the initial value problem of an equation of the first order V = 0 is solved by integrating the Lagrange-Charpit system (2). To prove this theorem he showed that the Lagrange-Charpit system has the following property: For any initial curve satisfying $dz - pdx - qdy = 0$ and $dV = 0$, the integral surface of the Lagrange-Charpit system satisfies $dz - pdx - qdy = 0$.

The Cauchy problem of Monge-Ampère's equation is solved by integrating the Lagrange-Charpit system of an intermediate integral of the first order, if and only if Monge's method of integration is applied to the given equation with success.

Goursat [6] tried to solve the Cauchy problem of a partial differential equation of the second order which is not of Monge-Ampère's type by integrating the Lagrange-Charpit system of an intermediate integral of the first order. In this process he obtained the notion of characteristics, the original idea of which is in the work of Ampère. The equation of the second order whose Cauchy problem is solved by integrating the Lagrange-Charpit system of an intermediate integral is called the Goursat equation ([1]).

§5. Integrable systems

Generalizing the Lagrangr-Charpit system, we defined an integrable system in [13]. Consider a system of ordinary differential equations

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{Ap + Bq} = \frac{dp}{C} = \frac{dq}{D}$$

in the space of (x, y, z, p, q), where A, B, C, D are functions of (x, y, z, p, q). Then it is called an integrable system, if for any initial curve satisfying $dz - pdx - qdy = 0$ and $Adp + Bdq - Cdx - Ddy = 0$, the integral surface of the system satisfies $dz - pdx - qdy = 0$. We showed that the Cauchy problem of the linear hyperbolic equation (5) is solved by integrable systems, if and only if $H_1 = 0$.

§6. Darboux's method

From this point of view, an equation of the form

(6) $$s + f(x, y, z, p, q) = 0$$

is solved by Darboux's method, if and only if the Cauchy problem is solved by integrable systems along each of the two characteristics. Vessiot [15] showed that if an equation of the second order (3) is solved by Darboux's method, it is transformed to an equation of the form (6) by a contact transformation. The classification of equations of the form (6) solved by Darboux's method was made by Goursat [8]. This result is a generalization of the following theorem of Lie ([5, vol.6, p.295]): If an equation of the second order is solved by Monge's method with respect to each of the two characteristics, then it is transformed to $s = 0$ by a contact transformation.

§7. Integrable systems of higher order

In this section and the following one we shall state the results obtained in [14]. Expanding the notion of an integrable system, we define an integrable system of order n to solve the Cauchy problem of an equation of the form (6) in the space of $(x, y, z, p, q_1, \ldots, q_n)$ containing the derivatives of higher order $q_i = \partial^i z/\partial y^i$. In this space consider a system of ordinary differential equations

$$(7) \qquad \frac{dx}{dy} = 0, \quad \frac{dz}{dy} = q_1, \quad \frac{dp}{dy} = -f, \quad \frac{dq_i}{dy} = q_{i+1}, \quad 1 < i \leq n,$$

$$\frac{dq_n}{dy} = u,$$

and differential forms

$$\omega_0 = dz - pdx - qdy,$$

$$\omega_i = dq_i + f_{i-1}dx - q_{i+1}dy, \quad 1 \leqq i < n,$$

$$\omega = dq_n + f_{n-1}dx - udy,$$

where u is a function of $(x, y, z, p, q_1, \ldots, q_n)$ to be determined later, and f_i is the function of $(x, y, z, p, q_1, \ldots, q_{i+1})$ defined inductively by

$$f_{i+1} = \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial z} q_1 - \frac{\partial f_i}{\partial p} f + \sum_{j=1}^{i+1} \frac{\partial f_i}{\partial q_j} q_{j+1}, \quad 0 \leqq i < n-1,$$

$$f_0 = f.$$

We say that the system (7) is integrable, if for any initial curve satisfying $\omega_0 = \omega_i = \omega = 0$, $1 \leqq i < n$, the integral surface of the system (7) satisfies $\omega_0 = \omega_i = \omega = 0$, $1 \leqq i < n$. In this sense the integrble system defined in §5 is the integrable system of the first order. We also say that the equation (6) is solved by integrable systems of order n, if for any initial curve satisfying $\omega_0 = \omega_i = 0$, $1 \leqq i < n$, we can find such a function u that satisfies $\omega = 0$ along the given initial curve and makes the system (7) integrable. If we apply this method of integration to the linear hyperbolic equation, then we obtain the following theorem: The linear hyperbolic equation (5) is solved by integrable systems of order n, if and only if $H_n = 0$.

§8. Equations of Laplace type

An equation of the form (6) is solved by Monge's method, if and only if it has the form

(8) $$s + M(x, y, z, q)p + N(x, y, z, q) = 0,$$

and if its first invariant

$$H = \frac{\partial M}{\partial x} - N\frac{\partial M}{\partial q} - \frac{\partial N}{\partial z} + M\frac{\partial N}{\partial q}$$

vanishes. An Imschenetsky transformation can be applied to the equation (8), if and only if $H \neq 0$. In this case let us say that it is an equation of Imschenetsky type. The equation of Imschenetsky type is solved by integrable systems of the first order, if and only if each of the following four second invariants vanishes:

$$H_{11} = \frac{\partial L_1}{\partial q}, \quad H_{12} = \frac{\partial L_2}{\partial q},$$

$$H_{13} = \frac{dL_1}{dy} - ML_1 - Z_1M, \quad H_{14} = \frac{dL_2}{dy} - NL_1 - Z_1N - 2H,$$

where

$$X_1 = \frac{\partial}{\partial x} - N\frac{\partial}{\partial q}, \quad Z_1 = \frac{\partial}{\partial z} - M\frac{\partial}{\partial q},$$

$$L_1 = Z_1 \log H + \frac{\partial M}{\partial q}, \quad L_2 = X_1 \log H + \frac{\partial N}{\partial q}.$$

We defined in [13] an equation of Laplace type as an equation of Imschenetsky type whose transformed equation has the form

$$s_1 - (\gamma p_1 + \delta) q_1 - (\alpha p_1 + \beta) = 0,$$

where α, β, γ, δ are functions of x_1, y_1, z_1. The equation of Imschenetsky type is of Laplace type, if and only if $H_{11} = H_{13} = 0$. We proved in [13] that the equation of Laplace type is solved by integrable systems of the first order, if and only if the transformed equation is solved by Monge's method. Expanding this theorem we have the following theorem: The equation of Laplace type is solved by integrable systems of the second order, if and only if the transformed equation is solved by integrable systems of the first order.

§9. Remarks

1. The equation (6) has the two characteristics $dx = 0$, $dp + f dy = 0$ and $dy = 0$, $dq + f dx = 0$. The same argument as above can be made with respect to the other characteristics.

2. Lagrange's method of obtaining the general integral from a complete integral is applied with success to an equation of the first order with n independent variables. However,

it is not powerful for integratig an equation of higher order.

3. A contact transformation is prolonged uniquely to the transformation which acts on the space of (x_i, z, $p_{i_1 \ldots i_r}$; $1 \leqq i, i_1, \ldots, i_r \leqq n$, $1 \leqq r \leqq m$) containing

$p_{i_1 \ldots i_r} = \partial^r z / \partial x_{i_1} \ldots \partial x_{i_r}$, and satisfies identically

$$dP_{i_1 \ldots i_r} - \sum_{i=1}^{n} P_{i_1 \ldots i_r i} dX_i$$

$$= \sum_{(j_1 \ldots j_s)} \rho_{i_1 \ldots i_r, j_1 \ldots j_s} \cdot$$

$$\cdot (dp_{j_1 \ldots j_s} - \sum_{j=1}^{n} p_{j_1 \ldots j_s j} dx_j),$$

$$1 \leqq i_1, \ldots, i_r, j_1, \ldots, j_s \leqq n, \quad 1 \leqq r < m.$$

Conversely the transformation satisfying this identity is obtained as the prolongation of a contact transformation.

4. The Imschenetsky transformation is a bijective transformation. In general, the Bäcklund transformation is not bijective. Results on general Bäcklund transformations are contained in Goursat's report [9].

5. Results on integration with quadratures are contained in Forsyth's book [5]. Goursat tried to treat them

systematically by his theory of characteristics in his book [7].

6. An integrable system of higher order can be defined in the space of $(x, y, z, p_{ij};\ 0 \leq i,\ j \leq n,\ 1 \leq i+j \leq n)$ containing $p_{ij} = \partial^{i+j} z / \partial x^i \partial y^j$, for integrating Monge-Ampère's equation of general type.

7. The Bäcklund transformation of Laplace type defined in [13] is the same as the Imschenetsky transformation applied to an equation of Laplace type.

§10. Problems on the Galois theory of Monge-Ampère's equations

1. Determine the method of integration by which we can define the solvability of an equation.

The method by integrable systems of order n is an answer to this problem.

2. Define the Galois group of an equation. Apply the Bäcklund transformation for constructing the Galois group.

If we define the Galois group as the subgroup of the group of contact transformations which leaves the given equation invariant, then it is not always an infinite Lie group, but sometimes a finite Lie group.

3. Define the solvability of the Galois group, and clarify the relation between the two solvability of the group and the equation.

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