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SPIN-COBORDISM INVARIANTS OF SOME S^1-MANIFOLDS

By

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Introduction.

Let Y and Y' be differentiable manifolds which have S^{2n-1}-bundle structures associated to differentiable complex $n(\geq 2)$ vector bundles over S^2 and also have unique spin-structures. By computing the Atiyah-Hirzebruch invariants [2, 3] for some natural S^1-actions of Y and Y', we conclude differentiable isomorphisms of the bundle structures of Y and Y' including S^1-actions, from spin-cobordisms of the S^1-manifolds (cf. Theorem (2. 2) and Corollary (2. 3)). By the Reidemeister torsion invariants [4], G. de Rham [7] proved that diffeomorphic rotations of the p dimensional sphere S^p are isomorphic. Our conclusion seems to be an analogy of this result in a certain sense.

1. Constructions of manifolds.

Let S^p be the standard p dimensional sphere and $\xi = (E, \pi, S^p)$ be differentiable n dimensional complex vector bundles over S^p, where $p \geq 1$ and $n \geq 2$. We denote by X_ξ the 2n-disk bundle space associated to ξ and by Y_ξ the (2n-1)-sphere bundle space associated to ξ. X_ξ, Y_ξ are compact connected oriented differentiable manifolds, $\dim X_\xi = 2n+p$, $\dim Y_\xi = 2n+p-1$ and Y_ξ is the boundary manifold of X_ξ; $\partial X_\xi = Y_\xi$.

We present S^1 as the unit circle of the complex number plane; $S^1 = \{z \mid |z| = 1\}$. Let

$$F: S^1 \times E(\xi) \longrightarrow E(\xi)$$

be differentiable S^1-actions such that $F(z, \ ): E(\xi) \longrightarrow E(\xi)$ are differentiable vector bundle maps for each $z \in S^1$. On each fibre, $F(z, \ )$ are non-singular linear maps with characteristic roots $\{z^{m_i} \mid m_i$, positive integers for $1 \leq i \leq n\}$. F define differentiable S^1-actions on the manifolds X_ξ, Y_ξ and clearly these S^1-actions are compactible. The set of n positive integers (admitting repeatitions), $\{m_1, \cdots, m_n\}$ is called a <u>type</u> of the S^1-action.

Lemma (1. 1). $H^1(X_\xi; Z_2) = H^1(Y_\xi; Z_2) = 0$.

Proof. It is clear that $H^1(X_\xi, Z_2) = 0$. Since $n \geq 2$, it follows that $H^2(X_\xi, Y_\xi; Z_2) = 0$. By the exact sequence

$$\cdots \to H^1(X_\xi; Z_2) \longrightarrow H^1(Y_\xi; Z_2) \longrightarrow H^2(X_\xi, Y_\xi; Z_2) \longrightarrow \cdots,$$

we have also $H^1(Y_\xi; Z_2) = 0$.

Lemma (1. 2). <u>If</u> $c_1(\xi) \equiv 0 \bmod 2$, <u>then we have</u>

$$W_2(X_\xi) = W_2(Y_\xi) = 0.$$

Proof. We denote tangent bundles by τ. It follows that

$$\tau(X_\xi) \cong \pi^*(\tau(S^p)) \oplus \text{the tangent bundle along the fibre of } X_\xi$$
$$\cong \pi^*(\tau(S^p) \oplus \xi),$$
$$W_2(X_\xi) = W_2(\tau(X_\xi)) = \pi^*(W_2(S^p) + W_2(\xi))$$
$$= \pi^* c_1(\xi) \qquad \bmod 2$$
$$= 0,$$

and

$$\tau(Y_\xi) \cong \pi^*(\tau(S^p)) \oplus \text{ the tangent bundle along the fibre of } Y_\xi,$$

$$\tau(Y_\xi) \oplus 1 \cong \pi^*(\tau(S^p) \oplus \xi),$$

$$W_2(Y_\xi) = W_2(\tau(Y_\xi)) = W_2(\tau(Y_\xi) \oplus 1$$
$$= 0.$$

2. The case where the base space of ξ is S^2.

Let ξ be differentiable n dimensional complex vector bundles over differentiable manifolds M. The actions of $z \in S^1$ are commutative with the coordinate transformations $g_{\alpha\beta}(x)$ of ξ (for $x \in U_\alpha \cap U_\beta$, where U_α, U_β are the coordinate neighborhoods of ξ) and hence, for the characteristic vectors v_i of characteristic values z^{m_i}, we have

$$\begin{aligned} z \circ (g_{\alpha\beta}(x)v_i) &= g_{\alpha\beta}(x)(z \circ v_i) \\ &= g_{\alpha\beta}(x)(z^{m_i} v_i) \\ &= z^{m_i}(g_{\alpha\beta}(x)v_i), \end{aligned}$$

that is, $g_{\alpha\beta}(x)v_i$ are also characteristic vectors for z^{m_i}. If $m_1, \cdots, m_n$ are all different positive integers, the set of all characteristic vectors for z^{m_i} in each fibre of ξ makes differentiable complex line (sub)bundles ξ_i and gives a Whitney sum decomposition of ξ;

$$\xi = \bigoplus_{i=1}^{n} \xi_i.$$

The actions of $z \in S^1$ on ξ_i are the multiplications by z^{m_i}.

For a space X with an action of a group G, the set of points which are left fixed by all elements of G is denoted by X^G. For S^1-actions of X_ξ and Y_ξ defined in §1, $(X_\xi)^{S^1}$ is diffeomorphic to S^p, and $(Y_\xi)^{S^1} = \phi$. For the Whitney sum decompositions of ξ determined by S^1-actions, it is natural to consider the case where the base space of ξ is S^2.

Theorem (2. 1). Let ξ be a differentiable n (≥ 2) dimensional complex vector bundle over S^2 such that $c_1(\xi) \equiv 0 \bmod 2$. Suppose that Y_ξ has an S^1-action of the type $\{m_1, \cdots, m_n\}$ where m_i are all different positive integers and $\sum_{i=1}^{n} m_i = 2m$. If we denote by $\xi = \bigoplus_{i=1}^{n} \xi_i$ the decomposition of ξ into the Whitney sum of differentiable complex line bundles, induced by the S^1-action, then we have the Atiyah-Hirzebruch invariants,

$$\rho(z, Y_\xi) = \frac{(-1)^n}{2} z^m \left(\prod_{i=1}^{n} \frac{1}{1-z^{m_i}}\right) \cdot \sum_{i=1}^{n} \left(\frac{1+z^{m_i}}{1-z^{m_i}}\right) c_1(\xi_i)[S^2],$$

for any $z \in S^1$ which are not m_ith roots of unity ($1 \leq i \leq n$).

Proof. Since we have $H^1(X_\xi; Z_2) = H^1(Y_\xi; Z_2) = 0$ and $W_2(X_\xi) = W_2(Y_\xi) = 0$ by (1. 1) and (1. 2), X_ξ and Y_ξ have spin-structures which are unique upto isomorphisms (cf. [1]). It is clear that

$$(X_\xi)^{S^1} \cong S^2.$$

We have, therefore, the Atiyah-Hirzebruch invariants [2, 3],

$$\begin{aligned}\rho(z, Y_\xi) &= \mathrm{spin}(z, (X_\xi)^{S^1}) \\ &= (-1)^{n+1} \hat{\mathfrak{A}}(S^2) \prod_{i=1}^{n} \left(z^{\frac{-m_i}{2}} e^{\frac{c_1(\xi_i)}{2}} - z^{\frac{m_i}{2}} e^{\frac{-c_1(\xi_i)}{2}}\right)^{-1} [S^2].\end{aligned}$$

By straight-forward calculations of the right side of this equation, we obtain the formula of the theorem and completes the proof.

Theorem (2. 2). Let ξ and ξ' be differentiable n (≥ 2) dimensional complex vector bundles over S^2 such that $c_1(\xi) \equiv c_1(\xi') \equiv 0 \bmod 2$. Let $m_1, \cdots, m_n$ be positive integers such that any m_i are not sums of other m_{i_j} and $\sum_{i=1}^{n} m_i = 2m$.

Suppose that Y_ξ and $Y_{\xi'}$ have S^1-actions of type $\{m_1, \cdots, m_n\}$. If we have

$$\rho(z, Y_\xi) = \rho(z, Y_{\xi'})$$

for any $z \in S^1$ which are not m_ith roots of unity $(1 \leq i \leq n)$, then there is a differentiable bundle isomorphism between ξ and ξ', including S^1-actions.

Proof. Let

$$\xi' = \bigoplus_{i=1}^{n} \xi_i'$$

be the decomposition of ξ into the Whitney sum of complex line bundles as that of ξ in Theorem (2. 1). For any $z \in S^1$ which are not m_ith roots of unity $(1 \leq i \leq n)$, we have

$$\sum_{i=1}^{n} (1-z^{m_1}) \cdots (1-z^{m_{i-1}})(1+z^{m_i})(1-z^{m_{i+1}}) \cdots (1-z^{m_n}) c_1(\xi)$$
$$= \sum_{i=1}^{n} (1-z^{m_1}) \cdots (1-z^{m_{i-1}})(1+z^{m_i})(1-z^{m_{i+1}}) \cdots (1-z^{m_n}) c_1(\xi'),$$

because of the equality $\rho(z, Y_\xi) = \rho(z, Y_{\xi'})$. From the assumption on m_i $(1 \leq i \leq n)$, it follows that

$$\sum_{j=1}^{n} c_1(\xi_j) = \sum_{j=1}^{n} c_1(\xi_j'),$$
$$-c_1(\xi_1) + \sum_{j=2}^{n} c_1(\xi_j) = -c_1(\xi_1') + \sum_{j=2}^{n} c_1(\xi_j'),$$
$$\cdots$$
$$-c_1(\xi_i) + \sum_{j \neq i} c_1(\xi_j) = -c_1(\xi_i') + \sum_{j \neq i} c_1(\xi_j'),$$
$$\cdots$$
$$-c_1(\xi_n) + \sum_{j=1}^{n-1} c_1(\xi_j) = -c_1(\xi_n') + \sum_{j=1}^{n-1} c_1(\xi_j').$$

Since $H^2(S^2; Z) \cong Z$ (torsion free), we obtain the equations

$$c_1(\xi_i) = c_1(\xi_i'), \qquad 1 \leq i \leq n,$$

and, therefore, bundle isomorphisms

$$\xi_i \cong \xi_i', \qquad 1 \leq i \leq n.$$

Moreover, by differentiable approximations [5] of homotopies of classifying maps and by the method of parallelisms for connections in principal fibre bundles [6], we have differentiable isomorphisms between ξ_i and ξ_i'.

Since $\xi = \bigoplus_{i=1}^{n} \xi_i$ and $\xi' = \bigoplus_{i=1}^{n} \xi'_i$, it follows that there is a differentiable isomorphism between ξ and ξ', including S^1-actions. Thus we complete the proof of the theorem.

Corollary (2. 3). Let ξ and ξ' be differentiable n (≥ 2) dimensional complex vector bundles over S^2 such that $c_1(\xi) \equiv c_1(\xi') \equiv 0 \bmod 2$. Let $m_1, \cdots, m_n$ be positive integers such that any m_i are not sums of other m_{i_j} and $\sum_{i=1}^{n} m_i = 2m$. Suppose that Y_ξ and $Y_{\xi'}$ have S^1-actions of type $\{m_1, \cdots, m_n\}$. There is a differentiable bundle isomorphism between Y_ξ and $Y_{\xi'}$ including S^1-actions if and only if they are spin-cobordant with respect to the S^1-actions.

Proof. If Y_ξ and $Y_{\xi'}$ are spin-cobordant with respect to S^1-actions, we have

$$\rho(z, Y_\xi) = \rho(z, Y_{\xi'})$$

by Atiyah-Hirzebruch [2]. The differentiable bundle isomorphism between Y_ξ and $Y_{\xi'}$ follows directly from Theorem (2. 2).

The converse is trivial.

3. The case where the base spaces of ξ are S^p for p=1 or p>2.

Proposition (3. 1). Let ξ be differentiable n (≥ 2) dimensional complex vector bundles over S^p (p=1 or p>2) and let $m_1, \cdots, m_n$ be all different positive integers. If Y_ξ have S^1-actions of types $\{m_1, \cdots, m_n\}$, then ξ and hence Y_ξ are differentiably isomorphic to product bundles.

Proof. By the first part of the proof of Theorem (2. 1), ξ split into

Whitney sums of differentiable complex line bundles ξ_i; $\xi = \bigoplus_{i=1}^{n} \xi_i$. Since we have $H^2(S^p; Z) = 0$ for $p=1$ or $p>2$, ξ_i are all topologically trivial. By the last part of the proof of Theorem (2. 2), ξ_i are differentiably trivial and hence ξ are differentiably isomorphic to product bundles.

Corollary (3. 2). Let ξ be differentiable n (≥ 2) dimensional complex vector bundles over S^p ($p=1$ or $p>2$) and let $m_1, \cdots, m_n$ be all different positive integers. If Y_ξ admit S^1-actions of type $\{m_1, \cdots, m_n\}$, then we have

$$\rho(z, Y_\xi) = 0.$$

In the case, $p=2$, we obtain the following theorem by Theorem 2 [9]:

Theorem (3. 3). Let ξ and ξ' be differentiable n (≥ 2) dimensional complex vector bundles over S^4. The bundle structures of Y_ξ and $Y_{\xi'}$ are differentiably isomorphic if and only if they are spin-cobordant with respect to S^1-actions of the type $\{2, \cdots, 2\}$.

Remark. In the above Theorem, the type $\{2, \cdots, 2\}$ of S^1-actions on Y_ξ and $Y_{\xi'}$ can be replaced by the type $\{m, \cdots, m\}$ for any positive integers m. (Cf. The proof of Theorem 2. 2 [8].)

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