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LIMITING ABSORPTION METHOD AND ABSOLUTE CONTINUITY FOR THE SCHRÖDINGER OPERATOR

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§0. Introduction

The present paper is concerned with a spectral property of the Schrödinger operator

$$(0.1)\qquad L = -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j} + ib_j\right)^2 + V$$

in $\mathbb{R}^n$, where b_j and V denote the operators of multiplication by real-valued functions $b_j(x)$ and $V(x)$. $b_j(x)$ is the j-th component of the magnetic vector potential, while $V(x)$ represents the electric scalar potential. $V(x)$ is usually assumed to diminish at infinity, which corresponds to the situation in a two-particle problem that the interaction between the particles dies off as their mutual distance becomes large.

The spectral structure of the operator L has been investigated by many authors with various degrees of the smallness assumption on $V(x)$ at infinity. For the sake of convenience of explanation we consider, for the moment, the case $n = 3$ and $b_j \equiv 0$ in (0.1), so that $L = -\Delta + V$, Δ being the n-dimensional Laplacian. A most interesting problem in the spectral theory for L is that of absolute continuity. Namely, let H be the unique self-adjoint restriction in $L_2(\mathbb{R}^3)$ of L (the existence of H is guaranteed, e.g., by Ikebe-Kato [8]), and E the associated spectral measure: $H = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$. Then the problem is: Is $(E(\lambda)f,f)$ [1], $f \in L_2(\mathbb{R}^3)$, absolutely continuous (with respect to the ordinary Lebesgue measure) on $(0,\infty)$? Or, equivalently, is H restricted to $E((0,\infty))L_2(\mathbb{R}^3)$ an absolutely continuous operator? (Here it should be noted that if the problem is affirmatively answered, then the absence of the singular spectrum on $(0,\infty)$ follows.) Let us now assume that $V(x) = O(|x|^{-\alpha})$, $\alpha > 0$. The above problem has been solved, to cite a few, by Povzner [14] for $\alpha > 3.5$, by Ikebe [7] for $\alpha > 2$, by Jäger [10] for $\alpha > 1.5$ [2], by Rejto [15] for $\alpha > 4/3$, by Kato [11] for $\alpha > 5/4$, and by Agmon [2] and Saitō [16] for $\alpha > 1$. For the repulsive potential case $(\partial V/\partial |x| \leqslant 0)$ we note work of Lavine [12] and Arai [3]. Attention should also be paid to the result of Dollard [4] for Coulomb type potentials $(\alpha = 1)$.

Recently, R. Lavine obtained the following result (lecture given at the (1971) Oberwolfach symposium on Mathematical Theory of Scattering): If $V = V_1 + V_2$ with $V_1(x) = o(1)$, $\partial V_1/\partial |x| = O(|x|^{-\beta})$, $\beta > 1$ and $V_2(x) = O(|x|^{-\gamma})$, $\gamma > 1$, then H is absolutely continuous on $(0,\infty)$.[3] His method of proof is similar to the one employed in [12].

In this paper we shall establish the same result as Lavine's by a different method, where the condition $V_1(x) = o(1)$ will be replaced by $V_1(x) = O(|x|^{-\delta})$, $\delta > 0$, however.

The spectral measure E is, roughly speaking, determined by the boundary values of the resolvent $(H - z)^{-1}$ on the reals, the resolvent being well-defined for z non-real. This leads us to the study of the asymptotic behavior of $(H - \lambda \pm i\varepsilon)^{-1}$ as ε tends to 0 through positive values. We cannot, however, expect that the limit of $(H - \lambda \pm i\varepsilon)^{-1}f$ for $\varepsilon \downarrow 0$ exists in the L_2 sense for $f \in L_2(\mathbb{R}^3)$, and, therefore, we have to choose appropriate classes of functions so that the limiting procedure in question may be justified. This forms the contents of the so-called limiting absorption method or principle [4). Once the limiting absorption method proves applicable, the absolute continuity of H on $(0,\infty)$ readily follows.

The greater part of the present paper will be devoted to the justification of the limiting absorption method for the more general Schrödinger operator (0.1) in which $b_j \not\equiv 0$, but we shall impose on b_j some asymptotic condition at infinity.

Section 1 states and proves all the theorems related with the limiting absorption method, while several lemmas needed for proving the theorems are stated without proof. These lemmas are proved in Section 2. Finally in Section 3 the absolute continuity of the Schrödinger operator is verified.

§1. Limiting absorption method

Consider the inhomogeneous Schrödinger equation

$$(1.1) \qquad Lu - \kappa^2 u \equiv -\sum_{j=1}^{n} D_j D_j u - V(x)u - \kappa^2 = f$$

in $\mathbb{R}^n$, where

$$(1.2) \qquad D_j u = (\partial_j + ib_j(x))u \qquad (\partial_j = \frac{\partial}{\partial x_j}).$$

$b_j(x)$ and $V(x)$ are real-valued functions whose more precise properties will be specified soon. The complex parameter κ is assumed to vary in the closed upper half-plane. The inhomogeneous term f is assumed to lie in a suitable Hilbert space contained in $L_2(\mathbb{R}^n)$.

Equation (1.1) may be solved rather easily if κ is non-real and κ^2 is not an eigenvalue of L. Denoting the solution by $u(\kappa, f)$, the following question arises : Does there exist a limit in some sense or other of $u(\kappa, f)$ when κ tends to a real limit ($\neq 0$) ? If such a limit exists, then it may be easily imagined that the limit function satisfies (1.1) also (with κ replaced by the real limit). We want to solve the above problem by first establishing some a priori estimates for solutions of (1.1) with non-real κ, and then carrying out the limiting procedure preserving the obtained a priori estimates. This way of constructing solutions (1.1) for κ real is what is called the limiting absorption method.

Before giving the assumption on $V(x)$ and $b_j(x)$ we shall list the notation which will be employed in the sequel without further reference.

$\mathbb{R}$: real numbers.

$\mathbb{C}$: complex numbers.

$D_j = \partial_j + \sqrt{-1}\, b_j(x) = \partial_j + ib_j(x) \quad (\partial_j = \frac{\partial}{\partial x_j},\ j = 1, 2, \cdots, n)$.

$\mathcal{D}_j = \mathcal{D}_j^{(\kappa)} = D_j + \frac{n-1}{2|x|}\tilde{x}_j - i\kappa\tilde{x}_j \quad (\tilde{x}_j = x_j/|x|,\ \kappa \in \mathbb{C})$.

$Du = (D_1 u, D_2 u, \cdots, D_n u)$.

$\mathcal{D}u = (\mathcal{D}_1 u, \mathcal{D}_2 u, \cdots, \mathcal{D}_n u)$.

$D_r u = \sum_{j=1}^{n} D_j u \cdot \tilde{x}_j \quad (r = |x|)$.

$\mathcal{D}_r u = \sum_{j=1}^{n} \mathcal{D}_j u \cdot \tilde{x}_j \quad (r = |x|)$.

$E_r = \{x \,/\, |x| \geqslant r\} \quad (r > 0)$.

$B_r = \{x \,/\, |x| \leqslant r\} \quad (r > 0)$.

$B_{rs} = \{x \,/\, r \leqq |x| \leqq s\} \quad (0 < r < s)$.

$L_{2,\beta}(G)$ $(\beta \in \mathbb{R})$ denotes the class of all functions f on G such that $(1 + |x|)^{\beta} f$ is square integrable over G. The norm and inner product of $L_{2,\beta}(G)$ are denoted by $\| \ \|_{\beta,G}$ and $(\ ,\ )_{\beta,G}$, respectively. We set $L_{2,\beta}(\mathbb{R}^n) = L_{2,\beta}$, $\| \ \|_{\beta,\mathbb{R}^n} = \| \ \|_{\beta}$ and $(\ ,\ )_{\beta,\mathbb{R}^n} = (\ ,\ )_{\beta}$. When $\beta = 0$, we shall omit the subscript 0 as in $L_2(G)$, $\| \ \|_G$ etc.

H_2 is all L_2 functions with distribution derivatives up to the second order, inclusive.

C^m is the class of m-times continuously differentiable functions.

C_0^∞ is the class of infinitely continuously differentiable functions with compact support in $\mathbb{R}^n$.

$$M_p(x) = \int_{|x-y| \leq 1} \frac{|p(x)|^2}{|x-y|^{n-4+\alpha}} dy \qquad (\alpha > 0).$$

Q_α denotes the class of locally L_2 functions $p(x)$ such that $M_p(x)$ is uniformly bounded in $\mathbb{R}^n$.

N_{loc} is the class of all locally N functions.

Now let us make the following assumption on the coefficients $V(x)$ and $b_j(x)$.

Assumption 1.1.

(V) $V(x)$ can be decomposed as $V(x) = V_1(x) + V_2(x)$ such that V_1, V_2 are real-valued $Q_{\alpha, loc}$ functions for some $\alpha > 0$, and there exist positive constants C, δ, R_0 such that

(V_1) the radial derivative $\frac{\partial V_1}{\partial |x|}$ exists for $|x| \geq R_0$,

$$|V_1(x)| \leq C|x|^{-\delta}, \quad \frac{\partial V_1}{\partial |x|} \leq C|x|^{-1-\delta} \quad \text{for} \quad |x| \geq R_0,$$

(V_2) and $|V_2(x)| \leq C|x|^{-1-\delta}$ for $|x| \geq R_0$.

(B) $b_j(x)$ is a real-valued C^1 function satisfying $|B_{jk}(x)| \leq C|x|^{-1-\delta}$ for $|x| \geq R_0$, $j,k = 1, 2, \cdots, n$ with the same C, δ, R_0 as in (V), where $B_{jk}(x) = \partial_j b_k(x) - \partial_k b_j(x)$.

(UC) The unique continuation property holds for the differential operator L in $\mathbb{R}^n$.

The main results of this section are summarized in the following four theorems.

Theorem 1.2. Let K be an open set in the upper half-plane of $\mathbb{C}$ of the form

(1.3) $$K = \{\kappa = \kappa_1 + i\kappa_2 \in \mathbb{C} \,/\, \kappa_1 \in (a,\ b),\ \ \kappa_2 \in (0, \alpha)\},$$

where $0<a<b<\infty$ and $0<\alpha<\infty$. Choose an $\varepsilon>0$ sufficiently small (so that $\varepsilon \leq \delta/2$ and $\varepsilon<1$). Then under Assumption 1.I there exists a constsnt $C = C(K, \varepsilon)$ [5] (which is independent of κ_2 however small it may be) such that the following a priori inqualities hold for any $u \in C_0^\infty$ and any $\kappa \in K$:

(1.4) $$\|u\|_{-\frac{1+\varepsilon}{2}} \leq C\,\|(L - \kappa^2)u\|_{\frac{1+\varepsilon}{2}},$$

(1.5) $$\|\mathcal{D}u\|_{-\frac{1+\varepsilon}{2}, E_1} \leq C\,\|(L - \kappa^2)u\|_{\frac{1+\varepsilon}{2}},$$

(1.6) $$\|u\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leq C\rho^{-\varepsilon}\,\|(L - \kappa^2)u\|^2_{\frac{1+\varepsilon}{2}} \qquad (\rho \geq 1).$$

Theorem 1.3. Let Assumption 1.1 be fulfilled and let K and ε be as in Theorem 1.2. Then for any pair $(\kappa, f) \in K \times L_{2,\frac{1+\varepsilon}{2}}$ there exists a unique solution $u = u(\kappa, f) \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ of

(1.7) $$(L - \kappa^2)u = f.$$

Moreover, the solution u satisfies

(1.8) $$\| u \|_{-\frac{1+\varepsilon}{2}} \leq C \| f \|_{\frac{1+\varepsilon}{2}} ,$$

(1.9) $$\| \mathcal{D} u \|_{-\frac{1+\varepsilon}{2}, E_1} \leq C \| f \|_{\frac{1+\varepsilon}{2}} ,$$

(1.10) $$\| u \|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leq C \rho^{-\varepsilon} \| f \|^2_{\frac{1+\varepsilon}{2}} \qquad (\rho \geq 1)$$

with the same constant $C = C(K, \varepsilon)$ as given in Theorem 1.2.

For $\kappa \in \overline{K}$, where $\overline{K}$ is the closure of K in $\mathbb{C}$, we can construct a solution $u = u(\kappa, f)$ as the limit of a sequence of solutions $\{u_m = u(\kappa_m, f)\}$ $(\kappa_m \in K, \ \kappa_m \to \kappa)$ obtained in the preceding theorem.

Theorem 1.4 (limiting absorption principle). Let Assumption 1.1 be fulfilled and let K and ε be as in Theorem 1.2. Let $\kappa \in \overline{K}$ and let $f \in L_{2,\frac{1+\varepsilon}{2}}$. Let $\{\kappa_m\} \subset K$ be a sequence tending to κ. Let $u_m = u(\kappa_m, f)$. Then $\{u_m\}$ converges in $L_{2,-\frac{1+\varepsilon}{2}}$ to a $u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ which solves

(1.11) $$(L - \kappa^2)u = f.$$

The limit $u = u(\kappa, f)$ thus obtained is independent of the choice of the sequence $\{\kappa_m\}$ and is determined as a unique solution of the equation $(L - \kappa^2)u = f$ with the boundary condition at infinity $\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_I} < \infty$. (The last condition replaces the usual outgoing radiation condition ; cf. Saitō[16].) Moreover, $u(\kappa, f)$ is $L_{2,-\frac{1+\varepsilon}{2}}$-strongly continuous in $\kappa \in \overline{K}$.

As is easily checked, the above theorem is an immediate consequence of the following more general assertion.

Theorem 1.5. Let Assumption 1.1 be fulfilled and let K and ε be as in Theorem 1.2. Then for any pair $(\kappa, f) \in \overline{K} \times L_{2,\frac{1+\varepsilon}{2}}$ there exists a unique solution $u = u(\kappa, f) \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ of

(1.12) $$(L - \kappa^2)u = f, \quad \|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_I} < \infty.$$

In this case the estimates (1.8), (1.9) and (1.10) hold good. The mapping

(1.13) $$\overline{K} \times L_{2,\frac{1+\varepsilon}{2}} \ni (\kappa, f) \longmapsto u(\kappa, f) \in L_{2,-\frac{1+\varepsilon}{2}}$$

is continuous on $\overline{K} \times L_{2,\frac{1+\varepsilon}{2}}$.

Remark 1.6. In Theorems 1.2, 1.3, 1.4 and 1.5

one may replace K by any of the following :

(1.14) $\{\kappa = \kappa_1 + i\kappa_2 \in \mathbb{C} \ / \ \kappa_1 \in (-b, -a), \ \ \kappa_2 \in (0, \alpha)\}$,

(1.15) $\{\kappa = \kappa_1 - i\kappa_2 \in \mathbb{C} \ / \ \kappa_1 \in (a, b), \ \ \kappa_2 \in (0, \alpha)\}$,

(1.16) $\{\kappa = \kappa_1 - i\kappa_2 \in \mathbb{C} \ / \ \kappa_1 \in (-b, -a), \ \ \kappa_2 \in (0, \alpha)\}$.

In the latter two cases where one considers in the lower half-plane, one has to make an obvious change in the definition of $\mathscr{D}u$, i.e., one should put

(1.17) $$\mathscr{D}_j u = D_j u + \frac{n-1}{2|x|}\tilde{x}_j u + i\kappa\tilde{x}_j u.$$

We shall list a series of lemmas which will be used to show these theorems and will be proved in the following section.

Lemma 1.7. There exists a constant $C = C(K, \varepsilon) > 0$ such that

(1.18) $$\|\mathscr{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} \leq C\{\|u\|_{-\frac{1+\varepsilon}{2}} + \|(L - \kappa^2)u\|_{\frac{1+\varepsilon}{2}}\}$$

is valid for any $u \in C_0^\infty$ and any $\kappa \in K$.

Lemma 1.8. There exists a constant $C = C(K, \varepsilon) > 0$ such that

$$\|u\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leqq C\rho^{-\varepsilon}\left\{\|u\|^2_{-\frac{1+\varepsilon}{2}} - \|(L - \kappa^2)u\|^2_{\frac{1+\varepsilon}{2}}\right\} \qquad (\rho \gtrless 1) \tag{1.19}$$

holds for any $u \in C_0^\infty$ and any $\kappa \in K$.

The next lemma shows the uniqueness of the solution of $(L - \kappa^2)u = f$ with $\kappa \in \overline{K}$ satisfying some boundary condition at infinity.

Lemma 1.9. (i) Let $u \in H_{2,loc}$ be a solution of $(L - \kappa^2)u = 0$ with $\kappa \in K$ such that $u \in L_{2,-\frac{1+\varepsilon}{2}}$. Then u is identically zero.

(ii) Let $u \in H_{2,loc}$ be a solution of $(L - \kappa^2)u = 0$ with $\kappa \in \overline{K}$ such that $u \in L_{2,-\frac{1+\varepsilon}{2}}$ and $\|\mathcal{C}u\|_{-\frac{1+\varepsilon}{2}, E_1} < \infty$. Then u is identically zero.

The following two lemmas are related to the existence of the solution $u(\kappa, f)$ of $(L - \kappa^2)u = f$ and the continuity of $u(\kappa, f)$ in κ and f.

Lemma 1.10. Let $\kappa \in K$. Then the set $\{(L - \kappa^2)u \,/\, u \in C_0^\infty\}$ is dense in $L_{2,\frac{1+\varepsilon}{2}}$.

Lemma 1.11. Let $\{u_m\}$ be a sequence in $L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ and let $\{\kappa_m\}$ be a convergent sequence in $\overline{K}$: $\kappa_m \to \kappa \in \overline{K}$ $(m \to \infty)$. Assume that

$$f_m \equiv (L - \kappa_m^2)u_m \in L_{2,\frac{1+\varepsilon}{2}}, \tag{1.20}$$

(1.21) $$f_m \to f \quad \text{in} \quad L_{2,\frac{1+\varepsilon}{2}} \quad (m\to\infty),$$

and there exists a constant C_0 such that

(1.22) $$\begin{cases} \|u_m\|_{-\frac{1+\varepsilon}{2}} \leq C_0, \\ \|\mathcal{D}^{(\kappa_m)}u_m\|_{\frac{-1+\varepsilon}{2}, E_1} \leq C_0, \\ \|u_m\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leq C_0^2\rho^{-\varepsilon} \qquad (\rho \geq 1) \end{cases}$$

for all $m = 1, 2, \cdots$. Then $\{u_m\}$ has a strong limit u in $L_{2,-\frac{1+\varepsilon}{2}}$. u satisfies

(1.23) $$u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc},$$

(1.24) $$(L - \kappa^2)u = f,$$

(1.25) $$\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty,$$

(1.26) $$\mathcal{D}^{(\kappa_m)}u_m \to \mathcal{D}u \quad \text{in} \quad L_2(E_1)_{loc} \quad (m\to\infty).$$

Using these lemmas we can now prove Theorems 1.2, 1.3, 1.4 and 1.5.

Proof of Theorem 1.2. Since (1.5) and (1.6) follow from (1.4) and Lemmas 1.7 and 1.8, it is sufficient to show (1.4) alone. Let us assume that

(1.4) is false. Then for each positive integer m we can find $\kappa_m \in K$, $u_m \in C_0^\infty$ such that

$$(1.27) \qquad \begin{cases} \|u_m\|_{-\frac{1+\varepsilon}{2}} = 1, \\ \|(L - \kappa_m)u_m\|_{\frac{1+\varepsilon}{2}} \leqq 1/m. \end{cases}$$

Since $\{\kappa_m\}$ is a bounded set in $\mathbb{C}$, we may assume, with no loss of generality, that $\kappa_m \to \kappa$ with $\kappa \in \overline{K}$ as m tends to ∞. It follows from (1.27) and Lemmas 1.7 and 1.8 that we have for all $m = 1, 2, \cdots$

$$(1.28) \qquad \begin{cases} \|u_m\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leqq C\rho^{-\varepsilon}(1 + 1/m^2) \leqq 2C\rho^{-\varepsilon} \qquad (\rho \geq 1), \\ \|\mathcal{D}^{(\kappa_m)} u_m\|_{\frac{-1+\varepsilon}{2}, E_1} \leqq C(1 + 1/m) \leqq 2C. \end{cases}$$

Therfore, we can apply Lemma 1.11 with $f_m = (L - \kappa_m^2)u_m$ and $f = \lim_{m\to\infty} f_m = 0$, which follows from (1.27), to see that there exists a limit $u = \lim_{m\to\infty} u_m$ in $L_{2,-\frac{1+\varepsilon}{2}}$ which is a solution of $(L - \kappa^2)u = 0$ satisfying $\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$. Since $\|u_m\|_{-\frac{1+\varepsilon}{2}} = 1$, $\|u\|_{-\frac{1+\varepsilon}{2}} = 1$. But this is a contradiction, because we have, on the other hand, $u \equiv 0$ by Lemma 1.9, (ii). Thus we have shown (1.4) . Q. E. D.

Proof of Theorem 1.3. Let $f \in L_{2,\frac{1+\varepsilon}{2}}$. Since $\{(L - \kappa^2)u \;/\; u \in C_0^\infty\}$ is dense in $L_{2,\frac{1+\varepsilon}{2}}$ by Lemma 1.10, there exists a sequence $\{u_m\} \subset C_0^\infty$ such that

$$(1.29) \qquad (L - \kappa^2)u_m = f_m \longrightarrow f \quad \text{in } L_{2,\frac{1+\varepsilon}{2}}$$

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as $m \to \infty$. Applying Theorem 1.2, we obtain

$$
(1.30) \qquad \begin{cases} \|u_m\|_{-\frac{1+\varepsilon}{2}} \leqq C\|f_m\|_{\frac{1+\varepsilon}{2}} \\ \|\mathcal{D}u_m\|_{\frac{-1+\varepsilon}{2}, E_1} \leqq C\|f_m\|_{\frac{1+\varepsilon}{2}} \\ \|u_m\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leqq C\rho^{-\varepsilon}\|f_m\|^2_{\frac{1+\varepsilon}{2}} \qquad (\rho \geqq 1) \end{cases}
$$

for all $m = 1, 2, \cdots$. Since it follows from (1.29) that $\{\|f_m\|_{\frac{1+\varepsilon}{2}}\}$ is a bounded sequence, we can see that (1.22) is satisfied with $C_0 = (C + \sqrt{C}) \cdot \sup_m \|f_m\|_{\frac{1+\varepsilon}{2}}$ and $\kappa_m \equiv \kappa$. Hence we can apply Lemma 1.11 to obtain a solution $u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ of $(L - \kappa^2)u = f$ by taking u to be the strong limit in $L_{2,-\frac{1+\varepsilon}{2}}$ of $\{u_m\}$. By Lemma 1.9, (i) the u is a unique solution of $(L - \kappa^2)u = f$, $u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$. By letting $m \to \infty$ in the first and third inequalities of (1.30), (1.8) and (1.10) of the theorem follow directly. To show (1.9) let G be a bounded measurable set in E_1. Then we have from the second inequality of (1.30)

$$
(1.31) \qquad \|\mathcal{D}u_m\|_{\frac{-1+\varepsilon}{2}, G} \leqq C\|f_m\|_{\frac{1+\varepsilon}{2}} .
$$

Letting m to ∞ in (1.31), we obtain

$$
(1.32) \qquad \|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, G} \leqq C\|f\|_{\frac{1+\varepsilon}{2}},
$$

since $\mathcal{D}^{(\kappa_m)}u_m \to \mathcal{D}u$ in $L_2(E_1)_{loc}$ by (1.26) of Lemma

1.11. Since $G \subset E_1$ is arbitrary, and the right side of (1.32) is independent of G, we obtain (1.9). Q. E. D.

Remark 1.12. Under the assumption of Theorem 1.3 the unique existence of the solution in $L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ of $(L - \kappa^2)u = f$, $f \in L_{2,\frac{1+\varepsilon}{2}}$ follows rather easily if one notes that L determines a unique self-adjoint restriction H (in L_2) (see, e.g., Ikebe-Kato [8]), and that κ^2 is not real. In fact, for any $f \in L_{2,\frac{1+\varepsilon}{2}}$ $(\subset L_2)$ $u = (H - \kappa^2)^{-1}f \in L_2$ is seen to be a unique solution of equation (1.1) satisfying $u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$. But in the above proof we have had no recourse to the self-adjointness that does not seem powerful enough to derive the uniform estimates (1.8)~(1.10) for $\kappa \in K$.

Proof of Theorem 1.5. Let $\kappa \in \overline{K}$ and $f \in L_{2,\frac{1+\varepsilon}{2}}$. Take $\{\kappa_m\} \subset K$ such that $\kappa_m \to \kappa$ as $m \to \infty$. By Theorem 1.3 there exists a unique solution $u_m \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$ of the equation $(L - \kappa_m^2)u_m = f$ which satisfies

$$(1.33) \qquad \begin{cases} \|u_m\|_{-\frac{1+\varepsilon}{2}} \leq C\|f\|_{\frac{1+\varepsilon}{2}} \\ \|\mathcal{D}(\kappa_m)u_m\|_{\frac{-1+\varepsilon}{2},E_1} \leq C\|f\|_{\frac{1+\varepsilon}{2}} \\ \|u_m\|^2_{-\frac{1+\varepsilon}{2},E_\rho} \leq C\rho^{-\varepsilon}\|f\|^2_{\frac{1+\varepsilon}{2}} \qquad (\rho \geq 1) \end{cases}$$

for all $m = 1, 2, \cdots$. Then one can see from Lemma 1.11

with $f_m \equiv f$ that $\{u_m\}$ has a strong limit u in $L_{2,-\frac{1+\varepsilon}{2}}$, $u \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$, which is a solution of equation (1.1). Arguments similar to those used in the proof of Theorem 1.3 to derive the estimates (1.8)~(1.10) from (1.30) show that the same estimates follow from (1.33) in the present case. Since $\|\mathcal{B}u\|_{\frac{-1+\varepsilon}{2},E_1} < \infty$, it follows by Lemma 1.9, (ii) that the u obtained above is a unique solution of equation (1.1), $\|\mathcal{B}u\|_{\frac{-1+\varepsilon}{2},E_1} < \infty$.

Finally let us prove the asserted continuity of the mapping (1.13). Now that the unique existence has been established, one can apply Lemma 1.11 again to the solutions $u_m = u(\kappa_m, f_m)$, where $\{\kappa_m\}$ and $\{f_m\}$ are assumed to be convergent sequences in $\overline{K}$ and $L_{2,\frac{1+\varepsilon}{2}}$, respectively. The required continuity follows from the fact that $\{u_m\}$ is a Cauchy sequence in $L_{2,-\frac{1+\varepsilon}{2}}$, which is a conclusion of Lemma 1.11. Q. E. D.

§2. Proof of the lemmas

This section is devoted to giving the proof of Lemmas 1.7∼1.11.

First we shall prepare a lemma which is a well-known elliptic estimate for the case that L has smooth coefficients. In our case where the coefficients are allowed to have certain singularities, an additional consideration will be required.

Lemma 2.1. Assume that $V \in Q_{\alpha,\mathrm{loc}}$ with some $\alpha > 0$ and b_j are real-valued C^1 functions on $\mathbb{R}^n$. Let K be a bounded set in $\mathbb{C}$. Then for each $R > 0$ there exists a constant $C = C(K, R)$ such that the estimate

$$(2.1) \qquad \int_{B_R} \sum_{j=1}^{n} |\partial_j u(x)|^2 dx \leqq C \int_{B_{R+1}} \left\{ |u(x)|^2 + |(L - \kappa^2)u(x)|^2 \right\} dx$$

holds for any $u \in H_{2,\mathrm{loc}}$ and any $\kappa \in K$.

Proof. Let $\psi \in C_0^\infty$ such that $0 \leqq \psi \leqq 1$ and

$$(2.2) \qquad \psi(x) = \begin{cases} 1 & (|x| \leqq R), \\ 0 & (|x| \geqq R + 1/2). \end{cases}$$

Since $u \in H_{2,\mathrm{loc}}$ satisfies

$$(2.3) \qquad (L - \kappa^2)(\psi u) = \psi (L - \kappa^2) u - \left\{ u \cdot \Delta\psi + 2 \sum_{j=1}^{n} D_j u \cdot \partial_j \psi \right\},$$

we obtain

$$(2.4)\qquad ((L-\kappa^2)(\psi u),\ \psi u) = (\psi(L-\kappa^2)u,\ \psi u) - ((\Delta\psi)u,\ \psi u) - 2\sum_{j=1}^{n}((\partial_j\psi)D_j u,\ \psi u),$$

whence follows by integration by parts

$$(2.5)\qquad \|D(\psi u)\|^2 = ((\kappa^2 - V)\psi u,\ \psi u) - (\psi(L-\kappa^2)u,\ \psi u) - ((\Delta\psi)u,\ \psi u) - 2\sum_{j=1}^{n}((\partial_j\psi)D_j u,\ \psi u)$$

$$= ((\kappa^2 - V)\psi u,\ \psi u) + (\psi(L-\kappa^2)u,\ \psi u) - ((\Delta\psi)u,\ \psi u) - 2\sum_{j=1}^{n}(D_j(\psi u),\ (\partial_j\psi)u) + 2\|(\partial_j\psi)u\|^2.$$

Hence we obtain the estimate

$$(2.6)\qquad \|D(\psi u)\|^2 \leq C_1 \int_{B_{R+1/2}} \left\{ |V(x)|\,|u(x)|^2 + |(L-\kappa^2)u(x)|^2 + |u(x)|^2 \right\} dx$$

with a constant $C_1 = C_1(K, R)$. Since $D_j = \partial_j + ib_j(x)$ and b_j are locally bounded on $\mathbb{R}^n$, it follows from (2.6) that

$$(2.7)\qquad \sum_{j=1}^{n}\|\partial_j(\psi u)\|^2 \leq C_2 \int_{B_{R+1/2}} \left\{ |V(x)|\,|u(x)|^2 + |(L-\kappa^2)u(x)|^2 + |u(x)|^2 \right\} dx$$

with $C_2 = C_2(K, R)$. Since $V \in Q_{\alpha,loc}$, we can make use of Lemma 2 of Ikebe-kato [8] to show

$$(2.8) \qquad \int_{B_{R+1/2}} |V(x)||u(x)|^2 \, dx$$
$$= \eta \int_{B_{R+1}} \sum_{j=1}^{n} |\partial_j u(x)|^2 dx + C_3 \int_{B_{R+1}} |u(x)|^2 dx,$$

where η is an arbitrary positive number and $C_3(\eta, R)$ is a positive constant. (2.1) then follows from (2.7) and (2.8). Q. E. D.

For later purpose let us rewrite equation (1.1)

$$(2.9) \qquad (L - \kappa^2)u = -\sum_{j=1}^{n} D_j D_j u + Vu - \kappa^2 u = f$$

in the form

$$(2.10) \qquad -\sum_{j=1}^{n} D_j \mathcal{D}_j u + \frac{n-1}{2r} \mathcal{D}_r u - i\kappa \mathcal{D}_r u + \tilde{V}(x)u = f,$$

where

$$(2.11) \qquad \tilde{V}(x) = V(x) + \frac{1}{4r^2}(n-1)(n-3).$$

Let us proceed to the proof of Lemma 1.7. To this end we need two lemmas.

Lemma 2.2. Let $u \in C_0^\infty$ and let $f = (L - \kappa^2)u$ with $\kappa \in \mathbb{C}$. Further, let $\varphi(r)$ be a C^1 function on $[R_0, \infty)$ such that $\varphi(R_0) = 0$, and let us put

$$\text{(2,12)} \qquad \tilde{V}_2(x) = V_2(x) + \frac{1}{4r^2}(n-1)(n-3).$$

Then we have

$$\text{(2.13)} \qquad \int_{E_{R_0}} \left\{ (\kappa_2\varphi + \frac{1}{2}\cdot\frac{\partial\varphi}{\partial r})|\mathcal{D}u|^2 + (\frac{\varphi}{r} - \frac{\partial\varphi}{\partial r})(|\mathcal{D}u|^2 - |\mathcal{D}_r u|^2)\right\} dx$$

$$= \int_{E_{R_0}} \left\{ \frac{1}{2}(\frac{\partial\varphi}{\partial r}V_1 + \frac{\partial V_1}{\partial r}\varphi) - \kappa_2 \varphi V_1 \right\}|u|^2\, dx$$

$$- \operatorname{Re}\left[\int_{E_{R_0}} \varphi \tilde{V}_2 u\cdot\overline{\mathcal{D}_r u}\, dx\right]$$

$$- \operatorname{Im}\left[\int_{E_{R_0}} \sum_{j,k=1}^{n} \varphi B_{jk}\, \mathcal{D}_j u\cdot \tilde{x}_k \bar{u}\, dx\right]$$

$$+ \operatorname{Re}\left[\int_{E_{R_0}} \varphi f\, \overline{\mathcal{D}_r u}\, dx\right],$$

where $\kappa_2 = \operatorname{Im}\kappa$, $r = |x|$ and $B_{jk}(x) = \partial_j b_k(x) - \partial_k b_j(x)$.

Proof. First note that (2.10) holds with our u and f. Multiply both sides of (2.10) by $\varphi\overline{\mathcal{D}_r u}$, integrate over E_{R_0}, and take the real part. Then we have

$$(2.14)\qquad \mathrm{Re}\,\Big[\int_{E_{R_0}} \varphi f\,\overline{\mathcal{D}_r u}\,dx\Big] = -\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \sum_{j=1}^{n} \varphi D_j\mathcal{D}_j u\cdot\overline{\mathcal{D}_r u}\,dx\Big]$$

$$+\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \varphi\Big(\frac{n-1}{2r} - i\kappa\Big)|\mathcal{D}_r u|^2 dx\Big]$$

$$+\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \varphi V_1 u\cdot\overline{\mathcal{D}_r u}\,dx\Big]$$

$$+\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \varphi \widetilde{V}_2 u\cdot\overline{\mathcal{D}_r u}\,dx\Big]$$

$$= I_1 + I_2 + I_3 + I_4.$$

Let us compute I_s $(s = 1,2,3)$ by (repeated, if necessary) application of integration by parts as follows : Noting that

$$(2.15)\qquad D_j\mathcal{D}_k u - D_k\mathcal{D}_j u = \Big(\frac{n-1}{2r} - i\kappa\Big)(\tilde{x}_k\mathcal{D}_j u - \tilde{x}_j\mathcal{D}_k u) + iB_{jk}u,$$

we have

$$(2.16)\qquad I_1 = \mathrm{Re}\,\Big[\int_{E_{R_0}} \sum_{j,k=1}^{n} \mathcal{D}_j u\cdot\overline{D_j\mathcal{D}_k u}\cdot\varphi\tilde{x}_k\,dx\Big]$$

$$+\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \sum_{j=1}^{n} \mathcal{D}_j u\cdot(\partial_j\varphi)\cdot\overline{\mathcal{D}_r u}\,dx\Big]$$

$$+\,\mathrm{Re}\,\Big[\int_{E_{R_0}} \sum_{j,k=1}^{n} \mathcal{D}_j u\cdot(\partial_j\tilde{x}_k)\varphi\overline{\mathcal{D}_k u}\,dx\Big]$$

$$= \int_{E_{R_0}} \left(\frac{\partial \varphi}{\partial r} - \frac{n-1}{2r}\varphi - \kappa_2\varphi\right) |\mathcal{D}_r u|^2 \, dx$$

$$+ \int_{E_{R_0}} \left(\frac{\varphi}{r} - \frac{1}{2}\cdot\frac{\partial \varphi}{\partial r} + \kappa_2\varphi\right) |\Theta u|^2 \, dx$$

$$+ \operatorname{Im} \left[\int_{E_{R_0}} \sum_{j,k=1}^{n} \varphi B_{jk} (\mathcal{D}_j u) \tilde{x}_k \bar{u} \, dx \right].$$

(2.17) $$I_2 = \int_{E_{R_0}} \varphi\left(\frac{n-1}{2r} + \kappa_2\right) \cdot |\mathcal{D}_r u|^2 \, dx.$$

(2.18) $$I_3 = \operatorname{Re} \left[\int_{E_{R_0}} \varphi V_1 u \left\{ \sum_{j=1}^{n} (\tilde{x}_j (\partial_j \bar{u}) - i\tilde{x}_j b_j \bar{u}) + \frac{n-1}{2r}\bar{u} + i\bar{\kappa}\bar{u} \right\} dx \right]$$

$$= - \operatorname{Re} \left[\int_{E_{R_0}} \sum_{j=1}^{n} \partial_j (\varphi V_1 u \tilde{x}_j) \cdot \bar{u} \, dx \right]$$

$$+ \int_{E_{R_0}} \varphi V_1 \left(\frac{n-1}{2r} + \kappa_2\right) |u|^2 \, dx$$

$$= - I_3 - \int_{E_{R_0}} \left(\frac{\partial \varphi}{\partial r} V_1 + \varphi \frac{\partial V_1}{\partial r} + \varphi V_1 \frac{n-1}{r}\right) |u|^2 \, dx$$

$$+ 2 \int_{E_{R_0}} \varphi V_1 \left(\frac{n-1}{2r} + \kappa_2\right) |u|^2 \, dx$$

$$= \int_{E_{R_0}} \left\{ \kappa_2 \varphi V_1 - \frac{1}{2}\left(\frac{\partial \varphi}{\partial r} V_1 + \varphi \frac{\partial V_1}{\partial r}\right) \right\} |u|^2 \, dx.$$

Thus (2.13) follows from (2.14), (2.16), (2.17) and (2.18).

Q. E. D.

Lemma 2.3. Let K be as in Theorem 1.2 and let $\varepsilon > 0$. Then there exists a constant $C = C(K, \varepsilon)$ such that

$$(2.19)\qquad \kappa_2 \|u\|_{\frac{1-\varepsilon}{2}} \leq C\left\{\|u\|_{-\frac{1+\varepsilon}{2}} + \|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} + \|(L-\kappa^2)u\|_{\frac{1+\varepsilon}{2}}\right\}$$

holds for any $u \in C_0^\infty$ and any $\kappa = \kappa_1 + i\kappa_2 \in K$.

Proof. First let us show (2.19) for $u \in C_0^\infty$ with support in E_1. Integration over E_1 after multiplying both sides of equation (2.9) by $\varphi(x)\bar{u} = (1+|x|)^{1-\varepsilon}\bar{u}$ yields, on taking the imaginary part,

$$(2.20)\qquad -\operatorname{Im}\int_{E_1}\varphi(D_jD_ju)\cdot\bar{u}\,dx - 2\kappa_1\kappa_2\int_{E_1}\varphi|u|^2\,dx = \operatorname{Im}\int_{E_1}\varphi f\bar{u}\,dx,$$

where we put $f = (L-\kappa^2)u$. The first term on the ~~right~~ left-hand side of (2.20) being integrated by parts, we obtain

$$(2.21)\qquad \kappa_2\int_{E_1}\varphi|u|^2\,dx = \frac{1}{2\kappa_1}\left\{\operatorname{Im}\int_{E_1}\sum_{j=1}^{n} D_ju\cdot\tilde{x}_j\frac{\partial\varphi}{\partial r}\bar{u}\,dx - \operatorname{Im}\int_{E_1}\varphi f\bar{u}\,dx\right\}$$

$$= \frac{1}{2\kappa_1}\left\{\operatorname{Im}\int_{E_1}\frac{\partial\varphi}{\partial r}\left(\mathcal{D}_r u - \frac{n-1}{2r}u + i\kappa u\right)\bar{u}\,dx - \operatorname{Im}\int_{E_1}\varphi f\bar{u}\,dx\right\},$$

where we should note $\kappa_1 \neq 0$. Since

$$(2.22) \qquad \frac{\partial \varphi}{\partial r} = (1-\varepsilon)(1+r)^{-\varepsilon} = \frac{1-\varepsilon}{1+r}\varphi,$$

it follows from (2.21) and Schwarz' inequality that

$$(2.23) \qquad \kappa_2 \int_{E_1} \varphi |u|^2 \, dx \leq \frac{1}{2|\kappa_1|} \left[\int_{E_1} \varphi |u|^2 dx \right]^{\frac{1}{2}}$$

$$\times \left\{ \left[\int_{E_1} \varphi \frac{(1-\varepsilon)^2}{(1+r)^2} |\mathcal{D}_r u|^2 \, dx \right]^{\frac{1}{2}} + \left[\int_{E_1} \varphi |\kappa_1|^2 \cdot \frac{(1-\varepsilon)^2}{(1+r)^2} |u|^2 \, dx \right]^{\frac{1}{2}} + \left[\int_{E_1} \varphi |f|^2 \, dx \right]^{\frac{1}{2}} \right\}.$$

Hence, noting that u is supported by the set E_1, we have

$$(2.24) \qquad \kappa_2 \|u\|_{\frac{1-\varepsilon}{2}} \leq \frac{1}{2|\kappa_1|} \left\{ (1-\varepsilon) \|\mathcal{D} u\|_{\frac{-1+\varepsilon}{2}, E_1} + (1-\varepsilon)|\kappa_1| \, \|u\|_{-\frac{1+\varepsilon}{2}} + \|f\|_{\frac{1+\varepsilon}{2}} \right\}.$$

Thus (2.19) follows with

$$(2.25) \qquad C = \frac{1}{2t} \left\{ (1-\varepsilon)(1+T) + 1 \right\}$$

$$\left(t = \inf_{\kappa \in K} |\kappa_1|, \quad T = \sup_{\kappa \in K} |\kappa_1| \right)$$

for $u \in C_0^\infty$ with support in E_1.

Next let us proceed to the general case, where no restriction is made on the support of $u \in C_0^\infty$ except it is compact. Let $\alpha(x)$ be a C^∞ function such that $0 \leqq \alpha \leqq 1$ and

$$(2.26) \qquad \alpha(x) = \begin{cases} 1 & (|x| \geqq 3), \\ 0 & (|x| \leqq 2). \end{cases}$$

Then we can decompose u as $u = (1-\alpha)u + \alpha u$. For $(1-\alpha)u$ the estimate

$$(2.27) \qquad \|(1-\alpha)u\|_{\frac{1-\varepsilon}{2}} \leqq \left[\int_{B_3} (1+|x|)^{1-\varepsilon} |u|^2 \, dx \right]^{\frac{1}{2}}$$

$$\leqq C\|u\|_{-\frac{1+\varepsilon}{2}}$$

is valid with a positive constant $C = C(\varepsilon)$. Let us estimate the term αu. αu is a C_0^∞ function with support in E_1, and we have

$$(2.28) \qquad (L - k^2)(\alpha u) = \alpha(L - k^2)u - (\Delta\alpha)u - 2\sum_{j=1}^{n}(\partial_j\alpha)(D_j u)$$

$$\equiv \alpha f + g.$$

Therefore, by what we have proved for C_0^∞ functions with support in E_1, it follows that

$$(2.29) \qquad k_2\|\alpha u\|_{\frac{1-\varepsilon}{2}} \leqq C\left\{ \|u\|_{-\frac{1+\varepsilon}{2}} + \|\partial(\alpha u)\|_{\frac{-1+\varepsilon}{2}, E_1} + \|f\|_{\frac{1+\varepsilon}{2}} + \|g\|_{\frac{1+\varepsilon}{2}} \right\},$$

where C is as defined by (2.25). Notice that

$$(2.30) \qquad \mathcal{L}_j(\alpha u) = \alpha \mathcal{L}_j u + (\partial_j \alpha) u$$

and the support of g is contained in $B_{2,3}$. Then, using Lemma 2.1, we obtain

$$(2.31) \qquad \begin{cases} \|\mathcal{L}(\alpha u)\|_{\frac{-1+\varepsilon}{2}, E_1} \leqq \|\mathcal{L}u\|_{\frac{-1+\varepsilon}{2}, E_1} + C_1 \|u\|_{-\frac{1+\varepsilon}{2}}, \\ \|g\|_{\frac{1+\varepsilon}{2}} \leqq C_2 \left\{ \|u\|_{-\frac{1+\varepsilon}{2}} + \|f\|_{\frac{1+\varepsilon}{2}} \right\}, \end{cases}$$

where $C_j = C_j(K, \varepsilon)$ $(j = 1,2)$. (2.29) together with (2.31) yields

$$(2.32) \qquad k_2 \|\alpha u\|_{\frac{1-\varepsilon}{2}} \leqq C_3 \left\{ \|u\|_{-\frac{1+\varepsilon}{2}} + \|\mathcal{L}u\|_{\frac{-1+\varepsilon}{2}, E_1} + \|f\|_{\frac{1+\varepsilon}{2}} \right\}$$

with $C_3 = C_3(K, \varepsilon)$. Now (2.19) follows from (2.27) and (2.32). Q. E. D.

Proof of Lemma 1.7. Let us put $\varphi(x) = \alpha(|x|)(1+|x|)^{\varepsilon}$, where $\alpha(r)$ is a C^1 function on $[R_0, \infty)$ such that $0 \leqq \alpha \leqq 1$, $\alpha'(r) \geqq 0$ and

$$(2.33) \qquad \alpha(r) = \begin{cases} 0 & (r = R_0), \\ 1 & (r \geqq R_0 + 1), \end{cases}$$

R_0 being the constant specified in (V) of Assumption 1.1.

Then we have

$$(2.34)\qquad \frac{\varphi}{r} - \frac{\partial\varphi}{\partial r} = \frac{1+r}{r}(1+r)^{-1+\varepsilon} - \varepsilon(1+r)^{-1+\varepsilon}$$

$$\geqq (1-\varepsilon)(1+r)^{-1+\varepsilon} > 0 \qquad (r \geqq R_0+1),$$

and

$$(2.35)\qquad \frac{1}{2}\cdot\frac{\partial\varphi}{\partial r} = \frac{\varepsilon}{2}\cdot\alpha(r)(1+r)^{-1+\varepsilon} + \frac{1}{2}\cdot\alpha'(r)(1+r)^{\varepsilon}$$

$$= \frac{\varepsilon}{2}\alpha(r)(1+r)^{-1+\varepsilon} \geqq 0 \qquad (r \geqq R_0).$$

With the φ defined above we can apply Lemma 2.2. Taking note of (2.34), (2.35) and Assumption 1.1, for $u \in C_0^\infty$ and $f = (L - \kappa^2)u$ $(\kappa \in K)$ we obtain from (2.13)

$$(2.36)\qquad \frac{\varepsilon}{2}\cdot\int_{E_{R_0}} \alpha(1+r)^{-1+\varepsilon}|\mathscr{D}u|^2 dx \leqq C_1\Big\{\int_{B_{R_0,R_0+1}} |\mathscr{D}u|^2\,dx$$

$$+ \int_{E_{R_0}} (1+r)^{-1-\varepsilon}|u|^2 dx + \kappa_2\int_{E_{R_0}} (1+r)^{-\varepsilon}|u|^2 dx$$

$$+ \int_{E_{R_0}} \alpha(1+r)^{-1-\varepsilon}|\mathscr{D}u||u|\,dx$$

$$+ \int_{E_{R_0}} \alpha(1+r)^{\varepsilon}|f||\mathscr{D}u|\,dx$$

with a constant $C_1 = C_1(\varepsilon)$, whence follows by Schwarz' inquality

$$(2.37)\quad \|\mathcal{D}u\|^2_{\frac{-1+\varepsilon}{2},E_{R_0+1}} \leqq C_2\Big\{\|\mathcal{D}u\|^2_{B_{R_0,R_0+1}} + \|u\|^2_{-\frac{1+\varepsilon}{2}} + K_2\|u\|_{\frac{1-\varepsilon}{2}}\|u\|_{-\frac{1+\varepsilon}{2}} + \|f\|^2_{\frac{1+\varepsilon}{2}}\Big\}$$

with a constant $C_2 = C_2(\varepsilon)$. The third term on the right-hand side of (2.37) is estimated by Lemma 2.3 to [redacted] get

$$(2.38)\quad \|\mathcal{D}u\|^2_{\frac{-1+\varepsilon}{2},E_{R_0+1}} \leqq C_3\Big\{\|\mathcal{D}u\|^2_{B_{R_0,R_0+1}} + \|u\|^2_{-\frac{1+\varepsilon}{2}} + \|u\|_{-\frac{1+\varepsilon}{2}}\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2},E_1} + \|f\|^2_{\frac{1+\varepsilon}{2}}\Big\}$$

with a constant $C_3 = C_3(K, \varepsilon)$. Thus one can derive from (2.38)

$$(2.39)\quad \|\mathcal{D}u\|^2_{\frac{-1+\varepsilon}{2},E_{R_0+1}} \leqq C_4\Big\{\|\mathcal{D}u\|^2_{B_{1,R_0+1}} + \|u\|^2_{-\frac{1+\varepsilon}{2}} + \|f\|^2_{\frac{1+\varepsilon}{2}}\Big\}$$

with a constant $C_4 = C_4(K, \varepsilon)$. Now Lemma 2.1 can be applied to estimate the first term on the right-side of (2.39) as follows :

$$(2.40)\quad \|\mathcal{D}u\|^2_{B_{1,R_0+1}} \leqq C_5(\|u\|^2_{B_{R_0+2}} + \|f\|^2_{B_{R_0+2}}) \leqq C_6(\|u\|^2_{-\frac{1+\varepsilon}{2}} + \|f\|^2_{\frac{1+\varepsilon}{2}}),$$

where $C_j = C_j(K, \varepsilon)$ $(j = 5, 6)$, which together with (2.39) completes the proof. Q. E. D.

Proof of Lemma 1.8. Let $u \in C_0^\infty$ and $\kappa \in K$. The definition of $\mathcal{D}_r$ enables one to write

$$(2.41) \qquad |\mathcal{D}_r u(x)|^2 = |D_r u(x) + \frac{n-1}{2r}u(x) - i(\kappa_1 + i\kappa_2)u(x)|^2$$

$$= |D_r u(x) + \frac{n-1}{2r}u(x) - \kappa_2 u(x)|^2 + \kappa_1^2 |u(x)|^2$$

$$- 2\kappa_1 \mathrm{Im}\left[D_r u(x) \cdot \overline{u(x)}\right],$$

which, integrated over the sphere S_r, gives

$$(2.42) \qquad \kappa_1^2 \int_{S_r} |u(x)|^2 \, dS \leqq \int_{S_r} |\mathcal{D}_r u(x)|^2 \, dS$$

$$+ 2\kappa_1 \mathrm{Im} \int_{S_r} D_r u(x) \cdot \overline{u(x)} \, dS.$$

Multiplying equation (2.9) by $\overline{u(x)}$, integrating over the ball B_r, and taking the imaginary part yield

$$(2.43) \qquad - \mathrm{Im} \int_{S_r} D_r u \cdot \bar{u} \, dS - 2\kappa_1\kappa_2 \int_{B_r} |u|^2 \, dx = \mathrm{Im} \int_{B_r} f\bar{u} \, dx.$$

Employing (2.43) in the last term of (2.42), one obtains

$$(2.44) \qquad \kappa_1^2 \int_{S_r} |u|^2 \, dS \leqq \int_{S_r} |\mathcal{D}_r u|^2 \, dS - 4\kappa_1^2\kappa_2 \int_{B_r} |u|^2 \, dx$$

$$- 2\kappa_1 \mathrm{Im} \int_{B_r} f\bar{u} \, dx$$

$$\leqq \int_{S_r} |\mathcal{Q}_r u|^2 \, dS + 2|\kappa_1| \, \|f\|_{\frac{1+\varepsilon}{2}} \|u\|_{-\frac{1+\varepsilon}{2}} .$$

Now one can multiply (2.44) by $(1+r)^{-1-\varepsilon}$ and integrate from ρ $(\geqq 1)$ to ∞ with respect to r to obtain

$$(2.45) \qquad \kappa_1^2 \|u\|^2_{-\frac{1+\varepsilon}{2}, E_\rho} \leqq \int_{E_\rho} (1+r)^{-1-\varepsilon} |\mathcal{Q}_r u|^2 \, dx + \frac{2|\kappa_1|}{\varepsilon} \rho^{-\varepsilon} \|f\|_{\frac{1+\varepsilon}{2}} \|u\|_{-\frac{1+\varepsilon}{2}}$$

$$\leqq \rho^{-2\varepsilon} \|\mathcal{D}u\|^2_{\frac{-1+\varepsilon}{2}, E_1} + \frac{2|\kappa_1|}{\varepsilon} \rho^{-\varepsilon} \|f\|_{\frac{1+\varepsilon}{2}} \|u\|_{-\frac{1+\varepsilon}{2}},$$

from which follows (1.19) by the use of Lemma 1.7 and Schwarz' inequality. Q. E. D.

The following lemma will be used to prove Lemmas 1.9 and 1.10.

Lemma 2.4. (i) Let $\kappa \in \mathbf{C}$, $\beta \in \mathbf{R}$. Let $v \in L_{2,\beta} \cap H_{2,loc}$ satisfy $(L - \kappa^2)v = 0$. Then we have $Dv \in L_{2,\beta}$.

(ii) Let $\kappa = \kappa_1 + i\kappa_2$, $\kappa_1 \kappa_2 \neq 0$ and $\beta \in \mathbf{R}$. Let $v \in H_{2,loc}$ satisfy $(L - \kappa^2)v = 0$ and $v, Dv \in L_{2,\beta}$. Then we have $v \in L_{2,\beta+1/2}$.

Proof. Let us first show (i). Multiply $(L - \kappa^2)v = 0$ by $\varphi\bar{v} = (1+|x|)^{2\beta}\bar{v}$ and integrate over B_{R_0R} with $R_0 < R < \infty$. Then we have by integration by parts

(2.46)
$$\int_{B_{R_0R}} \varphi|Dv|^2\,dx + \int_{B_{R_0R}} \frac{\partial\varphi}{\partial r}(D_rv)\bar{v}\,dx - \Big[\int_{S_R} - \int_{S_{R_0}}\Big]\varphi(D_rv)\bar{v}\,dS + \int_{B_{R_0R}} \varphi(V-\kappa^2)|v|^2dx = 0,$$

where

(2.47)
$$\Big[\int_{S_r} - \int_{S_t}\Big] f\,dS = \int_{S_r} f\,dS - \int_{S_t} f\,dS\ .$$

Here we should note that the surface integrals in (2.46) make sense and are continuous in R and R_0. This is because, on account of v being a locally H_2 function, v and D_rv can be regarded as $L_2(S^{n-1})$-valued continuous functions of $r = |x|$, where S^{n-1} denotes the $(n-1)$-sphere: $S^{n-1} = \{x \in R^n \ /\ |x| = 1\}$. By taking the real part of (2.46) we have

(2.48)
$$\int_{B_{R_0R}} \varphi|Dv|^2\,dx + \mathrm{Re}\int_{B_{R_0R}} 2\beta\varphi(1+r)^{-1}(D_rv)\bar{v}\,dx - \mathrm{Re}\Big[\int_{S_R} - \int_{S_{R_0}}\Big]\varphi(D_rv)\bar{v}\,dS + \int_{B_{R_0R}} \varphi(V-\kappa_1^2+\kappa_2^2)|v|^2dx = 0,$$

whence follows the inequality

$$\text{(2.49)} \qquad \frac{1}{2}\int_{B_{R_0R}} \varphi|Dv|^2\,dx \leq \mathrm{Re}\left[\int_{S_R} - \int_{S_{R_0}}\right]\varphi(D_r v)\bar{v}\,dS + \int_{B_{R_0R}} \varphi\left[\frac{|\beta|}{\eta} - V + \kappa_1^2 - \kappa_2^2\right]|v|^2\,dx,$$

where $\eta > 0$ has been chosen so small that $1-|\beta|\eta > 1/2$. Since, as can be easily verified by partial integration,

$$\text{(2.50)} \qquad \mathrm{Re}\int_{B_{R_0R}} \varphi(D_r v)\bar{v}\,dx = \frac{1}{2}\left[\int_{S_R} - \int_{S_{R_0}}\right]\varphi|v|^2\,dS - \frac{1}{2}\int_{B_{R_0R}} \left(\varphi\frac{n-1}{r} + \frac{\partial\varphi}{\partial r}\right)|v|^2\,dx,$$

it follows through differentiation in R that

$$\text{(2.51)} \qquad \mathrm{Re}\int_{S_R} \varphi(D_r v)\bar{v}\,dS = \frac{1}{2}\frac{d}{dR}\left[\int_{S_R} \varphi|v|^2\,dS\right] - \frac{1}{2}\int_{S_R} \left(\varphi\frac{n-1}{r} + \frac{\partial\varphi}{\partial r}\right)|v|^2\,dS$$

for $R > R_0$. Since $\varphi|v|^2$ is integrable over $\mathbb{R}^n$, we have

$$\text{(2.52)} \qquad \varliminf_{R\to\infty} \int_{S_R} \left(\varphi\frac{n-1}{r} + \frac{\partial\varphi}{\partial r}\right)|v|^2\,dS = 0.$$

Further, we have

(2.53) $$\varlimsup_{R\to\infty} \frac{d}{dR}\left[\int_{S_R} \varphi|v|^2\, dS\right] \leqq 0,$$

because it follows from

(2.54) $$\frac{d}{dR}\left[\int_{S_R} \varphi|v|^2\, dS\right] \geqq d > 0 \qquad (R \geqslant R_1)$$

with some $d > 0$ and some $R_1 \geqq R_0$ that

(2.55) $$\int_{S_R} \varphi|v|^2\, dS \geqq d(R - R_1) + \int_{S_{R_1}} \varphi|v|^2\, dS \qquad (R \geqq R_1),$$

which contradicts the fact that $\varphi|v|^2$ is integrable over $\mathbb{R}^n$. Thus from (2.51), (2.52) and (2.53) we obtain

(2.56) $$\varlimsup_{R\to\infty} \mathrm{Re} \int_{S_R} \varphi(D_r v)\bar{v}\, dS \leqq 0.$$

Therefore, it follows from (2.49) that

(2.57) $$\int_{E_{R_0}} \varphi|Dv|^2\, dx < \infty,$$

which implies that $Dv \in L_{2,\beta}$.

Next we shall show (ii). Put $\varphi = (1+|x|)^{2\beta+1}$ in (2.46), which is true in the present case, too, and take the imaginary part. Then we have

$$(2.58)\qquad \operatorname{Im}\int_{B_{R_0R}} \frac{\partial\varphi}{\partial r}(D_r v)\bar{v}\,dx - \operatorname{Im}\Big[\int_{S_R} - \int_{S_{R_0}}\Big]\varphi(D_r v)\bar{v}\,dS - 2\kappa_1\kappa_2\int_{B_{R_0R}}\varphi|v|^2\,dx = 0.$$

Since $v,\ D_r v \in L_{2,\beta}$ and $\frac{\partial\varphi}{\partial r} = O(|x|^{2\beta})$ $(|x|\to\infty)$, we have that $\frac{\partial\varphi}{\partial r}(D_r v)\bar{v}$ is absolutely integrable over $\mathbb{R}^n$ and

$$(2.59)\qquad \underline{\lim}_{R\to\infty}\int_{S_R}\varphi(D_r v)\bar{v}\,dS = 0.$$

In view of $\kappa_1\kappa_2 \neq 0$ it follows from (2.58)

$$(2.60)\qquad \int_{E_{R_0}}\varphi|v|^2\,dx < \infty,$$

i.e., $v \in L_{2,\beta+1/2}$. Q. E. D.

Proof of Lemma 1.9. Let us first prove (i). Since $\kappa\in K$, $(L - \kappa^2)u = 0$, and $u \in L_{2,-\frac{1+\varepsilon}{2}}$, we can apply Lemma 2.4, (i) and (ii) (repeatedly, if necessary) to see that $u,\ Du \in L_2$. Hence, multiplying $(L - \kappa^2)u = 0$ by $\bar{u}$ and integrating over $\mathbb{R}^n$, we have

$$(2.61)\qquad \sum_{j=1}^{n}(D_j u, D_j u) + ((V - \kappa^2)u, u) = 0.$$

By taking the imaginary part it follows that $-2\kappa_1\kappa_2\|u\|^2 = 0$, i.e., $u \equiv 0$.

Next let us show (ii). Since (i) has been shown, we have only to consider the case of $\operatorname{Im}\kappa = 0$. We note that we have the inequality

$$(2.62)\qquad \int_{S_r}\left\{|Du|^2 + \kappa^2|u|^2\right\} dS \leqq 2\int_{S_r}\left\{|\mathcal{D}u|^2 + \frac{(n-1)^2}{4r^2}|u|^2\right\} dS.$$

In fact, recalling the definition of Du and $\mathcal{D}u$, we obtain

$$(2.63)\qquad \begin{aligned} |\mathcal{D}u|^2 &= |Du|^2 + \kappa^2|u|^2 + \frac{(n-1)^2}{4r^2}|u|^2 \\ &\quad + \frac{n-1}{r}\operatorname{Re}\left[(D_r u)\cdot\bar{u}\right] - \kappa\operatorname{Im}\left[(D_r u)\cdot\bar{u}\right], \\ &\geqq \frac{1}{2}|Du|^2 + \kappa^2|u|^2 - \frac{(n-1)^2}{4r^2}|u|^2 \\ &\qquad - \kappa\operatorname{Im}\left[(D_r u)\cdot\bar{u}\right], \end{aligned}$$

where we have made use of

$$(2.64)\qquad \begin{aligned} \left|\frac{n-1}{r}\operatorname{Re}\left[(D_r u)\cdot\bar{u}\right]\right| &\leqq \frac{(n-1)^2}{2r^2}|u|^2 + \frac{1}{2}|D_r u|^2 \\ &\leqq \frac{(n-1)^2}{2r^2}|u|^2 + \frac{1}{2}|Du|^2 . \end{aligned}$$

Putting $f \equiv 0$ and $\kappa_2 = 0$ in (2.43), which is true in the present case, we have

(2.65) $$\operatorname{Im} \int_{S_r} D_r u \cdot \bar{u} \, dS = 0.$$

Integrating (2.63) over S_r, with (2.65) in regard, we obtain (2.62). On the other hand, it follows from $\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$ and $\|u\|_{-\frac{1+\varepsilon}{2}} < \infty$ that

(2.66) $$\lim_{r \to \infty} r^{\varepsilon} \int_{S_r} \left\{ |\mathcal{D}u|^2 + \frac{(n-1)^2}{4r^2} |u|^2 \right\} dS = 0.$$

Thus (2.62) and (2.66) are combined to give

(2.67) $$\lim_{r \to \infty} r^{\varepsilon} \int_{S_r} \left\{ |Du|^2 + \kappa^2 |u|^2 \right\} dS = 0.$$

Now we can apply Lemma 2.5 to be stated below to see that (2.67) implies that the solution u vanishes identically in E_{R_1} with some $R_1 > 0$. Hence by the unique continuation property (UC) $u \equiv 0$ on $\mathbb{R}^n$. Q. E. D.

Lemma 2.5. If $u \in H_{2,loc}$ is a solution of the equation $(L - \kappa^2)u = 0$ with κ real non-zero, and if u does not vanish identically in a neighborhood of the point at infinity, then for any $\varepsilon > 0$

(2.68) $$\lim_{r \to \infty} r^{\varepsilon} \int_{S_r} \left\{ |Du|^2 + \kappa^2 |u|^2 \right\} dS = \infty.$$

Remark on the proof. If we assume that $V(x)$

$= o(|x|^{-1})$, which case occurs, for instance, when $V_1 \equiv 0$, then the lemma reduces to Theorem 1.1 of Ikebe-Uchiyama [9]. Even in the present case, however, we can carry out the proof without essentially modifying the argument given in [9]. A remedy comes from techniques used by Odeh [13], Simon [17] or Agmon [1] utilizing the differentiability of $V_1(x)$. Namely, when one encounters an integral of the form

$$(2.69) \qquad \sum_{j=1}^{n} \int_{B_{sr}} V_1(x)\,|x|^{\alpha} u(x) \tilde{x}_j \overline{(D_j u)}\, dx,$$

this is estimated by the integrals $\int_{B_{sr}} |x|^{\alpha-\eta} |u|^2\, dx$ $\int_{B_{sr}} |x|^{\alpha-\eta} |Du|^2\, dx$ if one assumes only that $V_1(x) = O(|x|^{-\eta})$, but not the differentiability of $V_1(x)$. However, if it is assumed that $V_1(x) = O(|x|^{1-\eta})$ and $\frac{\partial V_1}{\partial |x|} = O(|x|^{-\eta})$, by carrying out integration by parts one may be convinced that the real part of (2.69) can be still estimated by $\int_{B_{sr}} |x|^{\alpha-\eta} |u|^2\, dx$ and additional surface integrals. (Here one should note that what is actually needed is not the estimation of the integral (2.69) itself but the one of its real part.) Roughly in this manner one can follow the line laid in [9] without drastic alteration.

We also remark that recently K. Masuda obtained a result (not yet published) of which our lemma is a consequence.

Proof of Lemma 1.10. Let $\kappa = \kappa_1 + i\kappa_2 \in K$ and let

$v_0 \in L_{2,-\frac{1+\varepsilon}{2}}$ satisfy

$$(2.70) \qquad (v_0, (L - \kappa^2)\varphi) = 0$$

for all $\varphi \in C_0^\infty$. It suffices to show $v_0 = 0$.

Let $\mathcal{X}^\pm = L_{2,\pm\frac{1+\varepsilon}{2}}$, and note that the adjoint spaces $(\mathcal{X}^\pm)^*$ can be identified with $\mathcal{X}^\mp$ by taking as the pairing between them the usual L_2 inner product. Define an operator A from $\mathcal{X}^-$ to $\mathcal{X}^+$ by

$$(2.71) \qquad D(A) = C_0^\infty, \qquad A\varphi = (L - \kappa^2)\varphi \quad \text{for } \varphi \in D(A).$$

A is densely defined in $\mathcal{X}^-$, and its adjoint A^* is an operator from $\mathcal{X}^- = (\mathcal{X}^+)^*$ to $\mathcal{X}^+ = (\mathcal{X}^-)^*$. By definition $v \in D(A^*)$ if and only if

$$(2.72) \qquad (v, A\varphi) = (w, \varphi)$$

for all $\varphi \in D(A)$ and for some $w \in \mathcal{X}^+$. Thus looking at (2.70) one can see that $v_0 \in D(A^*)$.

Now it is possible to imitate the argument in Ikebe-Kato [8], which has been used for proving Lemma 3 of [8],[6)] to show that $v_0 \in H_{2,loc}$. Then it follows from (2.70) that v_0 satisfies the equation $(L - \kappa^2)v_0 = 0$. Therefore, with the aid of Lemma 2.4 one can show $v_0 \in H_{2,loc} \cap L_2$. Hence, proceeding as in the proof of Lemma 1.9, (i), we have $v_0 = 0$. Q. E. D.

Finally we shall prove Lemma 1.11.

Proof of Lemma 1.11. Apply Lemma 2.1 with $u = u_m$. Then we have

$$(2.73) \qquad \int_{B_R} \sum_{j=1}^{n} |\partial_j u_m(x)|^2 \, dx \leq C \int_{B_{R+1}} \left\{ |u_m(x)|^2 + |f_m(x)|^2 \right\} dx,$$

which, together with the condition (1.21), and the first relation of (1.22), implies that $\left\{ \sum_{j=1}^{n} \|\partial_j u_m\|_{B_R} + \|u_m\|_{B_R} \right\}$ is a bounded sequence for each $R > 0$. Therefore, $\{u_m\}$ is relatively compact in $L_{2,loc}$, and hence there is a subsequence $\{u_{m_p}\}$ of $\{u_m\}$ such that

$$(2.74) \qquad \begin{cases} u_{m_p} \longrightarrow u & \text{in } L_{2,loc}, \\ \kappa_{m_p} \longrightarrow \kappa & \text{in } \bar{K} \end{cases}$$

as $p \to \infty$ with $u \in L_{2,loc}$. It follows from (2.74) and (1.21) that

$$(2.75) \qquad (u, (L - \bar{\kappa}^2)\varphi) = (f, \varphi) \qquad (\varphi \in C_0^\infty).$$

Therefore, as we have remarked in the proof of Lemma 1.10, u is seen to be an $H_{2,loc}$ function and satisfy $(L - \kappa^2)u = f$. From the convergence of $\{u_{m_p}\}$ to u in $L_{2,loc}$ and the third relation of (1.22) we have

$$(2.76) \qquad u_{m_p} \longrightarrow u \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}} \quad (p \to \infty).$$

Thus we have shown that u satisfies (1.23) and (1.24). Let us show that $\{\mathcal{D}^{(\kappa_{m_p})} u_{m_p}\}$ converges to $\mathcal{D}u$ in $L_2(E_1)_{loc}$. To this end we have only to verify $\partial_j u_{m_p} \longrightarrow \partial_j u$ in $L_{2,loc}$. Since we have

$$(2.77) \qquad g_p \equiv (L - \kappa^2)(u - u_{m_p})$$

$$= (\kappa^2 - \kappa_{m_p}^2) u_{m_p} + (f - f_{m_p}) \longrightarrow 0 \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}}$$

as $p \to \infty$, applying Lemma 2.1 with $u = u - u_{m_p}$ and $f = g_p$, we obtain for $R > 0$

$$(2.78) \qquad \sum_{j=1}^{n} \|\partial_j (u - u_{m_p})\|_{B_R}^2 \leq C\left\{\|u - u_{m_p}\|_{B_{R+1}}^2 + \|g_p\|_{B_{R+1}}^2\right\}$$

with $C = C(K, R)$, which implies that $\partial_j u_{m_p} \longrightarrow \partial_j u$ in $L_{2,loc}$ for each $j = 1,2,\cdots,n$, and hence $\{\mathcal{D}^{(\kappa_{m_p})} u_{m_p}\}$ converges to $\mathcal{D}u$ in $L_2(E_1)_{loc}$. Now let us show (1.25) : $\mathcal{D}u \in L_{2,\frac{-1+\varepsilon}{2}}(E_1)$. By the use of the third relation of (1.22) it follows that

$$(2.79) \qquad \|\mathcal{D}^{(\kappa_{m_p})} u_{m_p}\|_{\frac{-1+\varepsilon}{2}, G} \leq C_0$$

for any bounded measurable set G in E_1. Letting $p \to \infty$ in (2.79) we have $\|\mathcal{D}u\|_{\frac{-1+\varepsilon}{2}, G} \leq C_0$, and hence $\mathcal{D}u \in L_{2,\frac{-1+\varepsilon}{2}}(E_1)$. Finally we shall show that the sequence $\{u_m\}$ itself converges in $L_{2,-\frac{1+\varepsilon}{2}}$ to the u obtained above, which in turn implies that $\{\mathcal{D}^{(\kappa_m)}u_m\}$ converges to $\mathcal{D}u$ in $L_2(E_1)_{loc}$. In fact, let us assume that there exists a subsequence $\{m_q\}$ of $\{m\}$ such that

(2.80) $$\| u - u_{m_q} \|_{-\frac{1+\varepsilon}{2}} \geq \gamma \qquad (q = 1,2,\cdots)$$

with some $\gamma > 0$. Then, proceeding as above, we can find a subsequence $\{m_q'\}$ of $\{m_q\}$ which satisfies

(2.81) $$u_{m_q'} \longrightarrow u' \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}},$$

u' being a solution $\in H_{2,loc} \cap L_{2,-\frac{1+\varepsilon}{2}}$ of $(L - \kappa^2)u' = f$, $\| \mathcal{D}u' \|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$. By Lemma 1.9, (ii), asserting the uniqueness of the solution for $(L - \kappa^2)u = f$, $\| \mathcal{D}u \|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$, $u \in H_{2,loc} \cap L_{2,-\frac{1+\varepsilon}{2}}$, u and u' must coincide. Hence we have from (2.81)

(2.82) $$u_{m_q'} \longrightarrow u \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}},$$

which contradicts (2.80). Thus we have shown that $\{u_m\}$ converges to u in $L_{2,-\frac{1+\varepsilon}{2}}$, which completes the proof.

Q. E. D.

§3. Absolute continuity

First let us define a symmetric operator H_o acting in the Hilbert space L_2 by

$$(3.1)\qquad D(H_o) = C_0^\infty, \qquad H_o u = Lu \quad \text{for} \quad u \in D(H_o).$$

According to Theorem 1 of Ikebe-Kato [8], H_o admits a unique self-adjoint extension H. Let $E(B)$ be the spectral measure associated with H, where B varies over all Borel sets of the reals. In the present section we shall study a typical spectral property of H, that is, we shall show that $E((0,\infty))H$ is an absolutely continuous operator.

A characterization of $D(H)$, the domain of H, follows directly from Lemma 4 of [8]. Summarizing, we have the next

Lemma 3.1. H_o is essentially self-adjoint, and thus possesses a unique self-adjoint extension H. We have

$$(3.2)\qquad D(H) = \{u \in L_2 \,/\, u \in H_{2,loc} \quad \text{and} \quad Lu \in L_2\}.$$

Let $R(z) = (H - z)^{-1}$ denotes the resolvent of H, and recall that for $\kappa = \kappa_1 + i\kappa_2$ with $\kappa_1 \neq 0$ and $\kappa_2 > 0$, and for $f \in L_{2,\frac{1+\varepsilon}{2}}$, there can be determined by Theorem 1.5 and Remark 1.6 a unique solution $u(\kappa, f) \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$

of the equation $(L - \kappa^2)u = f$.

lemma 3.2. Let $f \in L_{2,\frac{1+\varepsilon}{2}}$, and let $z \in \mathbb{C} - \mathbb{R}$. Then

$$(3.3) \qquad R(z)f(x) = u(\sqrt{z}, f)(x)^{7)} \qquad \text{a.e.}$$

Proof. Since f necessarily belongs to L_2, $R(z)f$ makes sense as an element of L_2, and $u = R(z)f$ satisfies

$$(3.4) \qquad (L - z)u = f.$$

Moreover, the fact that $R(z)f \in \mathcal{D}(H)$ implies by Lemma 3.1 that $R(z)f \in L_2 \cap H_{2,loc}$, and hence $R(z)f \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$. On the other hand, since $\sqrt{z}$ lies in some K, where K is an open set of $\mathbb{C}$ of the type cosidered in Theorem 1.2 or Remark 1.6, it follows by Theorem 1.3 that equation (3.4) with $f \in L_{2,\frac{1+\varepsilon}{2}}$ has a <u>unique</u> solution $u(\sqrt{z}, f) \in L_{2,-\frac{1+\varepsilon}{2}} \cap H_{2,loc}$. By the uniqueness, therefore, $R(z)f$ must coincide with $u(\sqrt{z}, f)$ as an element of $L_{2,-\frac{1+\varepsilon}{2}}$, which implies that they are equal to each other almost everywhere. Q. E. D.

Now let $\lambda > 0$. It follows from Theorem 1.4 (cf. also Theorem 1.5) that for any $f \in L_{2,\frac{1+\varepsilon}{2}}$ a unique solution $u(\sqrt{\lambda}, f)$ of the equation

(3.5) $$(L - \lambda)u = f$$

can be constructed as the limit

(3.6) $$u(\sqrt{\lambda}, f) = \lim_{\mu \downarrow 0} u(\sqrt{\lambda + i\mu}, f) \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}}.$$

Similarly, if Remark 1.6 is taken into consideration, another unique solution $u(-\sqrt{\lambda}, f)$ of the same equation can be obtained :

(3.7) $$u(-\sqrt{\lambda}, f) = \lim_{\mu \downarrow 0} u(\sqrt{\lambda - i\mu}, f) \quad \text{in} \quad L_{2,-\frac{1+\varepsilon}{2}}.$$

(It may be also noted that $u(\pm\sqrt{\lambda}, f)$ can be determined as unique solutions of $(L - \lambda)u = f$ satisfying $\| \mathcal{D}^{(\pm\sqrt{\lambda})} u \|_{\frac{-1+\varepsilon}{2}, E_1} < \infty$.)

Let $\Delta = (\lambda_1, \lambda_2)$, where $0 < \lambda_1 < \lambda_2 < \infty$. Employing the well-known relation [8)]

(3.8) $$(E(\Delta)f, f) = \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} (\{R(\lambda+i\mu) - R(\lambda-i\mu)\} f, f) \, d\lambda$$
$$(f \in L_2),$$

we can represent $E(\Delta)f$ in terms of $u(\pm\sqrt{\lambda}, f)$ $(\lambda_1 \leq \lambda \leq \lambda_2)$ as follows.

Lemma 3.3. Let $\Delta = (\lambda_1, \lambda_2)$ be as above. Then for any $f \in L_{2,\frac{1+\varepsilon}{2}}$

(3.9) $$(E(\Delta)f, f) = \frac{1}{2\pi i} \int_{\Delta} (u(\sqrt{\lambda}, f) - u(-\sqrt{\lambda}, f), f) \, d\lambda.$$

Proof. Since by Lemma 3.2

(3.10) $$R(\lambda\pm i\mu)f(x) = u(\sqrt{\lambda\pm i\mu}\,, f)(x) \qquad \text{a.e.}$$

for $\lambda \neq 0$, $\mu > 0$, (3.8) can be rewritten in the form

(3.11) $$(E(\Delta)f, f) = \lim_{\eta\downarrow 0}\lim_{\mu\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\eta}^{\lambda_2-\eta}(u(\sqrt{\lambda+i\mu}, f)-u(\sqrt{\lambda-i\mu}, f), f)\,d\lambda.$$

By the use of the continuity of $u(\kappa, f)$, which has been stated in Theorem 1.4 or 1.5, it follows that $(u(\sqrt{\lambda+i\mu}, f)-u(\sqrt{\lambda-i\mu}, f), f)$ is uniformly bounded for $(\lambda,\mu)\in[\lambda_1, \lambda_2]\times[0, 1]$ and

(3.12) $$\lim_{\mu\downarrow 0}(u(\sqrt{\lambda+i\mu}, f) - u(\sqrt{\lambda-i\mu}, f), f)$$

$$= (u(\sqrt{\lambda}, f) - u(-\sqrt{\lambda}, f), f)$$

for $\lambda\in[\lambda_1, \lambda_2]$. With the aid of the Lebesgue dominated convergence theorem, therefore, (3.9) can be obtained from (3.11) and (3.12). Q. E. D.

Noting that $L_{2,\frac{1+\varepsilon}{2}}$ is dense in L_2 and $(u(\sqrt{\lambda}, f) - u(-\sqrt{\lambda}, f), f)$ is a continuous function of $\lambda\in(0, \infty)$, we obtain from Lemma 3.3 the following

Theorem 3.4. Let Assumption 1.1 be fulfilled. Then $E((0, \infty))H$ is an absolutely continuous operator.

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Footnotes

1) (,) denotes the usual L_2 inner product.

2) He has not directly treated the Schrödinger operator in R^n, however.

3) Agmon remarked in [2] that this would be the case.

4) For a general survey of the limiting absorption method see, e.g., Èidus [6].

5) Here and in the sequel we agree to mean by $C = C(A, B, \cdots)$ that C is a positive constant depending on $A, B, \cdots$. But very often symbols indicating obvious dependence will be left out. For instance, $C = C(K, \varepsilon)$, here, obviously depends on the differential operator L, but we do not insert L in the parentheses.

6) Lemma 3 of [8] asserts that $D(T_0^*) \subset H_{2,loc}$, where T_0 is the Schrödinger operator restricted to C_0^∞, but is regarded as acting in L_2.

7) ~~Here and in the sequel~~ By $\sqrt{z}$ is meant the branch of the square root of z with $\operatorname{Im}\sqrt{z} \geqq 0$.

8) See, for example, Dunford-Schwartz [5], p. 1602.

References

[1] Agmon, S., Lower bounds for solutions of Schrödinger equations, J. d'Anal. Math., 23 (1970), 1-25.

[2] Agmon, S., Spectral properties of Schrödinger operators, Congrès intern. Math., (1970) Tome 2, 679-683.

[3] Arai, M., Absolute continuity of Schrödinger operators with repulsive potentials, Publ. RIMS, 7 (1972).

[4] Dollard, J. D., Asymptotic convergence and the Coulomb interaction, J. Mathematical Phys., 5 (1964), 729-738.

[5] Dunford, N. and J. T. Schwartz, Linear operators, Part II, Interscience 1963.

[6] Èidus, D. M., The principle of limit amplitude, Usp. Mat. Nauk, 24 (1969), Vypusk 3 (147), 91-156. (Russian Math. Surveys, 24 (1969), No.3, 97-169).

[7] Ikebe, T., Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, Arch. Rational Mech. Anal., 5 (1960), 1-34.

[8] Ikebe, T. and T. Kato, Uniqueness of the self-adjoint extension of singular elliptic differential operators, Arch. Rational Mech. Anal., 9 (1962), 77-92.

[9] Ikebe, T. and J. Uchiyama, On the asymptotic behavior of eigenfunctions of second-order elliptic

differential operators, J. Math. Kyoto Univ., 11 (1971), 425-448.

[10] Jäger, J., Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum, Math. Z., 113 (1970), 68-98.

[11] Kato, T., Some results on potential scattering, Proc. of Tokyo International Conference on Functional Analysis and Related Topics, (1969), 206-215.

[12] Lavine, R., Absolute continuity of Hamiltonian operators with repulsive potential, Proc. Amer. Math. Soc., 22 (1969), 55-60.

[13] Odeh, F., Note on differential operators with purely continuous spectrum, Proc. Amer. Math. Soc., 16 (1965), 363-365.

[14] Povzner, A. Ya., On the expansions of arbitrary functions in terms of eigenfunctions of the operator $-\Delta u + cu$, Mat. Sbornik, 32 (1953), 109-156. (A. M. S. Translations, Series 2, 60 (1967), 1-49).

[15] Rejto, P. A., On partly gentle perturbations III, J. Math. Anal. Appl., 27 (1969), 21-67.

[16] Saitō, Y., Spectral and scattering theory for second-order differential operators with operator-valued coefficients, to appear in Osaka J. Math.

[17] Simon, B., On positive eigenvalues of one-body Schrödinger operators, Comm. Pure Appl. Math., 22 (1967), 531-538.