数理解析研究所講究録
第 162 巻 1972 年 67-74

On the infinitely multiple Markov property of stationary Gaussian processes with a multi-dimensional parmeter

by

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§1. Introduction

In this paper we discuss stationary Gaussian processes for which the parameter domain is a d-dimensional space rather than the usual real interval.

P. Lévy, H. P. Mckean, Jr., and etc. defined another Markov property of stochastic processes which contained the definition of usual one. Our interest is in the determination of all stationary Gaussian processes with the Markov property.

In one dimensional case, N. Levinson and H. P. Mckean, Jr. resolved this problem completely in their paper [1].

On the other hand, G. M. Molchan [2] introduced a Hilbert space associated with process to consider the problem in the field of the theory of differential equations. And recently Y. Okabe [3], G. M. Molchan [4] and L. D. Pitt [5] characterized the Markov property by a locality condition of the Hilbert space used by G. M. Molchan, and then Y. Okabe simplified and advanced the results of N. Levinson and H. P. Mckean, Jr. by using the theory of Sato's hyperfunctions [6] to understand an entire function of infra-exponential type as a local operator in the space of hyperfunctions.

In this paper following the idea in [3], we give a sufficient condition for the Markov property.

§2. Reproducing kernel Hilbert space and Markov property

Let $\mathbb{X} = (X(x); x \in \mathbb{R}^d)$ be a stationary Gaussian process with its parameter in $\mathbb{R}^d$ and its value in $\mathbb{R}$. We denote by R the correlation function of the process, i.e.

$$R(x-y) = EX(x)X(y)$$

, and introduce the Hilbert space $\mathcal{H}$ with the reproducing kernel R and following subspaces of $\mathcal{H}$; for each open subset D of $\mathbb{R}^d$,

$$\mathcal{H}_0(D) = \{u \in \mathcal{H} \mid u=0 \quad \text{on} \quad D\} \ ,$$

$$\mathcal{H}(D) = (\mathcal{H}_0(D))^{\perp} \quad \text{(the orthogonal complement)},$$

$$\mathcal{H}^+(D) = \bigcap_{n=1}^{\infty} \mathcal{H}((D^{-c})_n) \quad \text{(the future)} \ ,$$

$$\mathcal{H}^-(D) = \bigcap_{n=1}^{\infty} \mathcal{H}(D_n) \quad \text{(the past)} \ ,$$

$$\partial\mathcal{H}(D) = \bigcap_{n=1}^{\infty} \mathcal{H}((\partial D)_n) \quad \text{(the present)},$$

where $D_n = \{x \in \mathbb{R}^d \mid \text{dis}(x,D) < \frac{1}{n}\}$ etc.

When $\mathcal{P}_-$ is the projection of $\mathcal{H}$ onto $\mathcal{H}^-(D)$, the relation below is trivial:

$$(2.1) \qquad \partial\mathcal{H}(D) \subset \mathcal{H}^+(D) \cap \mathcal{H}^-(D) \subset \mathcal{P}_- \mathcal{H}^+(D) \ .$$

Now we can state the new definition of Markov property.

Definition

We say the process $\mathbb{X}$ has the Markov property in D if and only if

$$(2.2) \qquad \mathcal{P}_{-}\mathcal{H}^{+}(D) = \partial\mathcal{H}(D)$$

Remark 1

Roughly speaking, if $\mathbb{X}$ has the Markov property in D, then the least squares prediction of the future under the condition that we have known the past depends upon the informations of $\mathbb{X}$ in only nearest neighbourhood of the boundary ∂D.

Remark 2

We can also state the Markov property in terms of Borel fields [for example, see [5]].

§3. Proof of Markov property of $\mathbb{X}$

In this paper we assume that the correlation function R has a spectral density Δ .

Before proving our theorem, we quotate some notations and results of the theory of Fourier hyperfunctions introduced by M. Sato [6] T. Kawai [7].

We denote by $\mathbb{D}^d$ the radial compactification of $\mathbb{R}^d$ and by $\mathcal{R}$ the sheaf of Fourier hyperfunction which concides with the sheaf of hyperfunciton $\mathcal{B}$ on $\mathbb{R}^d$. Let $\mathcal{P}_*$ be the space of rapidly decreasing holomorphic functions, then the next identificaiton can be shown:

(3.1) $\mathcal{R}(\mathbb{D}^d) \cong (\mathcal{P}_*)'$

(in details, see T. Kawai [7])

We remark that we can represent the space $\mathcal{H}$ by the density Δ as follows:

(3.2) $\mathcal{H} = \{u;\ u=\widehat{f\Delta}\ ,\ f \in L^2(\Delta)\}$,

where $\wedge$ is the Fourier transformation, i.e.

$$\widehat{f\Delta}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\Delta(x)dx\ .$$

Because of (3.1), we can regard $\mathcal{H}$ as a subspace of $\mathcal{R}(\mathbb{D}^d)$.

Next, we impose the following assumption as to Δ through this paper.

<u>Assumption</u>: We assume that Δ satisfies the condition A and either one of the conditions B, C;

A: Δ is a reciprocal of an entire function P of infra-exponential type,

B: there exist a positive number $t_0 > 0$ and a monotone decreasing, nonnegative function g on $[t_0,\infty)$ such that

$$\int_{t_0}^{\infty} \frac{\log g(t)}{1+t^2}\, dt > -\infty$$

and $g(|x|) \leq \Delta(x)$ for any $x \in \mathbb{R}^d$, $|x| \geq t_0$,

C: there exist positive numbers t_0, $c_0 > 0$ and a continuous function h on $[t_0,\infty)$ such that

$$h(t_1+t_2) \geq c\ h(t_1)h(t_2) \quad \text{for any} \quad t_1,t_2 \geq t_0$$

and $h(|x|) \leq \Delta(x)$ for any $x \in \mathbb{R}^d$, $|x| \geq t_0$.

We denote by H_D the supporting function of a convex set D and by h_φ the p-indicator of an entire function φ on $\mathbb{C}^d$. Then, by the results proved by O.A. Prensjakova [8] and O. I. Orebkova [9] under the above assumption, we have, for any bounded convex open set D of $\mathbb{R}^d$,

(3.3) the closed linear hull of $\{e^{i\cdot x};\ x \in D\}$ in $L^2(\Delta)$

$= \{\varphi \in L^2(\Delta);\ \varphi$ can be extended to an entire function φ on $\mathbb{C}^d$ and there exists a convex set $D_\varphi \subset D$ such that $h_\varphi \leq H_{D_\varphi}\}$.

Therefore, by (3.2), (3.3) and the theorem of Paley-Wiener type in the space of hyperfunctions ([10]), we have

(3.4) $\mathcal{H}(D) \subset \{u \in \mathcal{H};\ p(i\partial)u=0$ on $\mathbb{D}^d - \bar{D}$ as a Fourier hyperfunction$\}$

for any bounded convex open set D of $\mathbb{R}^d$.

Now, we can prove our main theorem.

Theorem

For each bounded convex open set D of $\mathbb{R}^d$, $\mathbb{X}$ has the Markov property.

To prove this theorem, we need a lemma, which is shown later.

Lemma

For any positive integer n, there exists some integer $m=m(n) > n$ such that

$$\{u \in \mathcal{H} \mid p(i\partial)u = 0 \quad \text{on } \mathbb{D}^d \setminus (\partial D)_m^-\} \subset \mathcal{H}((\partial D)n) ,$$

where D is any bounded convex open set.

Proof of Theorem:

Noting (2.1), (2.2) can be reduced to the following statement:

(3.5) For each positive integer n and $m=m(n)$ of Lemma

$$\mathcal{P}_-\mathcal{H}((D^{-c})_m) \subset \mathcal{H}((\partial D)_n)$$

But observing that $\mathcal{H}(D)$ is generated by functions of $R(\cdot - x)$, $x \in D$, it is enough to show (3.5) only for such functions, and this follows from usual technics of Hilbert space, Lemma and (3.4).

Proof of Lemma:

For n, let us fix small m which is determined later and $u \in \mathcal{H}$ such that

(3.6) $$p(i\partial)u = 0 \quad \text{on} \quad \mathbb{R}^d \setminus (\partial D)_m^- .$$

To show $u \in \mathcal{H}((\partial D)_n)$, it suffices to prove

(3.7) $$(v,u)_{\mathcal{H}} = 0 \quad \text{for any} \quad v \in (\mathcal{H}((\partial D)_n))^{\perp} .$$

$v \in \mathcal{H}((\partial D)_n)^{\perp}$ implies

(3.8) $$v = 0 \quad \text{on} \quad (\partial D)_n .$$

However, supposing by (3.2) $u = \frac{\hat{f}}{p}$ and $v = \frac{\hat{g}}{p}$, we obtain the following equality:

$$(v,u)_{\mathcal{H}} = \int_{\mathbb{R}^d} \frac{f\bar{g}}{p} = \widehat{\frac{f\bar{g}}{p}}(0)$$

and

(3.9) $$\widehat{\frac{f\bar{g}}{p}} = (p(i\partial)u) * \check{v} \quad (\check{v}(x) = \overline{v(-x)}) .$$

On the other hand by (3.6), $p(i\partial)u$ has a compact support in $\mathbb{R}^d$ and, if we regard (3.9) as equality in $\mathcal{B}$, we then can justify this convolution by usual integration in the theory of hyperfunction [6].

Noting (3.6) and (3.8) we can find small m such that

$$(p(i\partial)u) * \check{v} = 0 \qquad \text{on a neighbourhood of } 0 ,$$

so that, by (3.9), we have

$$\widehat{\frac{f\bar{g}}{p}} = 0 \qquad \text{on a neighbourhood of } 0 .$$

But, since $\frac{f\bar{g}}{p} \in L^1(dx)$, $\widehat{\frac{f\bar{g}}{p}}$ is continuous, so $\widehat{\frac{f\bar{g}}{p}}(0) = 0$ and this shows (3.7). Consequently, our theorem was proved.

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