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# On Strict Ergodicity

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## Introduction

Having split the category of all ergodic normalized dynamical systems $(\Omega, \mathscr{L}, m, T)$ in the measure-theoretical sense into isomorphy classes, one would next try to establish a system of nice representatives in these classes. The definition of "nice" is, of course, partly a matter of taste, but a rather natural approach is offered by topological dynamics. In that theory one considers systems $(\Omega, T)$ where $\Omega$ is a compact metric space and $T$ a homeomorphism of $\Omega$ onto itself. The natural analogon of ergodicity in topological dynamics is minimality, which means that there is no non-empty proper subset of $\Omega$ which is closed and $T$-invariant. It is well known (see Oxtoby [17], Keane [14], Jacobs-Keane [11]) that minimal invariant sets, carrying always at least one T-invariant normalized measure by the Markoff-Kakutani fixed point theorem, may quite well carry several ones. If one, however, imposes the condition that there be exactly one T-invariant normalized measure $m$ living on the minimal compact metric $\Omega$, one arrives at the definition of a <u>strictly ergodic</u> $(\Omega, T)$, and these give rise to well-defined measure-theoretical dynamical systems $(\Omega, \mathscr{L}, m, T)$ which are, in addition, ergodic, and thus candidates for the task of being "nice" representatives in the isomorphy classes mentioned above. The question "Is every ergodic $(\Omega, \mathcal{L}, m, T)$ measure-theoretically isomorphic to some strictly ergodic system?" or, more suggestively "Can every ergodic $(m, T)$ live in a strictly ergodic $(\Omega, T)$?" is at present practically definitely answered, and it is the purpose of this article to give a survey of the answers obtained so far. The time-continuous analogon of the problem and its solutions will be included here. I am, however, not to go into a description of the combinatorial devices which have been invented in order to obtain even mechanizable constructions of strictly ergodic representatives of various kinds in shift space (see Kakutani [13], Hahn-Katznelson [7], Keane [14], Jacobs-Keane [11], Grillenberger [5], [6], compare also Jacobs [10]).

## §1. Strictly ergodic generators.

It was R. Jewett [12] who proved first the following

Theorem 1.1. Let $(\Omega, \mathcal{L}, m, T)$ be any weakly mixing dynamical system such that $(\Omega, \mathcal{L}, m)$ is a Lebesque measure space with $m(\Omega)=1$. Then there exists an isomorphic dynamical system living in a strictly ergodic $(\Omega', T')$ where $\Omega'$ is a totally disconnected compact metric space.

The following improvement is due to W. Krieger [16] (see also Denker [1]).

Theorem 1.2. Let $(\Omega, \mathcal{L}, m, T)$ be any ergodic dynamical system such that $(\Omega, \mathscr{L}, m)$ is a Lebesgue measure space with $m(\Omega)=1$.

1) If the entropy $h(m, T)$ is finite, then there is a strictly ergodic invariant subset of the shift space with $[2^{h(m,T)}]+1$ symbols, which carries a dynamical system isomorphic to the given one.

2) If $h(m,T)=\infty$, then there is a strictly ergodic compact minimal invariant subset of the shift space with countably many symbols, which carries a dynamical system isomorphic to the given one.

This theorem may equivalently be stated as a theorem on the existence of "strictly ergodic generators" in the original $(\Omega, \mathscr{L}, m, T)$ together with a bound of the length of the generator in the case of finite entropy.

The following theorem of Denker [2] more or less brings the theory to a definite conclusion.

<u>Theorem 1.3</u>. Let $(\Omega, \mathscr{L}, m,T)$ be as in theorem 1.2. Then for $h(m,T)<\infty$ the finite strictly ergodic generators are dense (with respect to the entropy metric) in the set $\{\alpha \mid \alpha$ a finite measurable partition of $\Omega$, $h(\alpha, m, T)= h(m, T)\}$ of all partitions with maximal mean entropy (this set is closed).

## §2. The continuous case.

In the time-continuous case one considers a one-parameter group $(T_t)_{t \text{ real}}$ of automorphisms of a given Lebesgue space $(\Omega, \mathcal{L}, m, T)$. Under mild regularity assumptions this system $(\Omega, \mathcal{L}, m, (T_t)_{t \text{ real}})$ can always be viewed as a flow under a function, and this representation is one of the most effective tools in squeezing it isomorphically into a strictly ergodic system $(\Omega', (T'_t)_{t \text{ real}})$.

In the time-discrete case we chose (two-sided) shift space as an appropriate topological system in which subsequently a strictly ergodic system was to be establised. One of the most important question in the time-continuous case is what is to be the most natural time-continuous analogon of shift space. Physical considerations lead to the choice of

$$\mathrm{Lip}_1 = \{\omega \mid \omega:\mathbb{R} \to \langle 0,1\rangle,\ |\omega(s)-\omega(t)| \leq |s-t|\}$$

of all unit-interval-valued functions on the line which have Lipschitz constant 1. It is obviously a compact metric space with the compact-uniform topology, and the shifts $T_t$ (t real) defined by

$$(T_t\, \omega)(s) = \omega(s + t) \quad (s \text{ real})$$

form a nice one-parameter group of homeomorphisms of $\mathrm{Lip}_1$. The points of $\mathrm{Lip}_1$ may be viewed as the possible outcomes of a time-continuous experiment watched by instruments which have a certain inertia.

A time-continuous generator theorem by Krengel [15] (and likewise another generator theorem by Eberlein [4]) allow to embed every reasonable flow into $Lip_1$ with its shifts. The task of making these embeddings strictly ergodic requires a formidable arsenal of techniques and has been settled in the form of the following theorem by Denker-Eberlein [3] which strongly improves my result [9].

<u>Theorem 2.1</u>. Let $(\Omega, \mathscr{L}, m, (T_t)_{t \text{ real}})$ be a time-continuous ergodic dynamical system representable as a flow under a function, such that $(\Omega, \mathscr{L}, m)$ is a Lebesgue space with $m(\Omega)=1$. Then there is an isomorphic strictly ergodic subflow of $Lip_1$, with its shifts as transformations.

It seems to be difficult to even formulate an analogon of the discrete-time density theorem 1.3, for the time-continuous case. Nothing is known about time-continuous analoga of the combinatorial constructions in [13], [14], [11], [7], [5], [6].

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