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THE TENSOR PRODUCT OF WEIGHTS

By

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1. Introduction

Let φ (resp. ψ) be a normal semi-finite weight on a von Neumann algebra M (resp. N). There exists the maximal weight $\varphi \otimes \psi$ on $M \otimes N$ such that $\varphi \otimes \psi\ (x \otimes y) = \varphi(x)\psi(y)$ for each x in $(m_\varphi)_+$ and y in $(m_\psi)_+$. Furthermore if φ and ψ are faithful in addition, $\varphi \otimes \psi$ is a faithful semi-finite weight on $M \otimes N$ and its one-parameter modular automorphism group is the tensor product of one-parameter modular automorphism groups Σ and Σ^ψ. Let φ_1 (resp. ψ_1) be a normal semi-finite, Σ - invariant weight on M (resp. Σ^ψ, N). By [5] Theorem 5.12 there is a unique positive self-adjoint operator h affiliated with the sub-algebra of fix-points for Σ (resp. k, Σ^ψ) such that $\varphi_1 = \varphi(h\cdot)$ (resp. $\psi_1 = \psi(k\cdot)$) We get $\varphi_1 \otimes \psi_1 = \varphi \otimes \psi\ (h \otimes k\ \cdot)$.

2. The Tensor Product of Unbounded Self-Adjoint Operators

Theorem 2.1. Let H_1 and H_2 be Hilbert spaces, K_1 and K_2 self-adjoint operators on H_1 and H_2 respectively. Then there exists a unique self-adjoint operator $K_1 \otimes K_2$ on the Hilbert space $H_1 \otimes H_2$ such that $D(K_1 \otimes K_2) \supset D(K_1) \otimes_a D(K_2)$ and $K_1 \otimes K_2(\xi_1 \otimes \xi_2) = K_1\xi_1 \otimes K_2\xi_2$ for all $\xi_1 \in D(K_1)$ and $\xi_2 \in D(K_2)$, where $D(K_1) \otimes_a D(K_2) = \{ \sum_{k=1}^{n} \xi_k^1 \otimes \xi_k^2 \in H_1 \otimes H_2 : \xi_k^1 \in D(K_1),\ \xi_k^2 \in D(K_2)$ for $k = 1, \dots, n \}$. Moreover if K_1 and K_2 are positive, $K_1 \otimes K_2$ is positive.

Proof. Let $K_1 = \int_{-\infty}^{\infty} \lambda de(\lambda)$ and $K_2 = \int_{-\infty}^{\infty} \nu dE(\nu)$ be the spectral decompositions of K_1 and K_2 respectively.

Put $D = \bigcup_{n,m=1}^{\infty} R(e_n \otimes E_m)$ where $e_n = e(n) - e(-n)$ and $E_m = E(m) - E(-m)$.

Define an operator $K_1 \otimes K_2$ on D by ;

$$(K_1 \otimes K_2)\xi = (K_1 e_n \otimes K_2 E_m)\xi \ , \quad \text{where } \xi \text{ in } R(e_n \otimes E_m).$$

Then $K_1 \otimes K_2$ is a well-defined and densely defined symmetric operator. Furthermore, it is essentially self-adjoint.

[i] $K_1 \otimes K_2$ is well defined. Suppose that ξ in $R(e_n \otimes E_m)$ and ξ in $R(e_{n_1} \otimes E_{m_1})$.

We may assume $n \leqq n_1$ and $m \leqq m_1$ without the loss of generality.

Then we have

$$\begin{aligned}(K_1 e_{n_1} \otimes K_2 E_{m_2})\xi &= (K_1 e_{n_1} \otimes K_2 E_{m_2})(e_n \otimes E_m)\xi \\ &= (K_1 e_{n_1} e_n \otimes K_2 E_{m_2} E_m)\xi \\ &= (K_1 e_n \otimes K_2 E_m)\xi \ .\end{aligned}$$

[ii] $K_1 \otimes K_2$ is densely defined and symmetric

$D = \bigcup_{n,m=1}^{\infty} R(e_n \otimes E_m)$ is dense in $H_1 \otimes H_2$ since $s - \lim e_n = 1$ and $s - \lim E_m = 1$.

For all ξ in D and η in D we have

$$((K_1 \otimes K_2)\xi \mid \eta) = ((K_1 e_n \otimes K_2 E_m)\xi \mid \eta) \quad \text{for sufficient large } n, m.$$

Since $K_1 e_n \otimes K_2 E_m$ is bounded and self-adjoint, we have

$$((K_1 \otimes K_2)\xi \mid \eta) = (\xi \mid (K_1 e_n \otimes K_2 E_m)\eta)$$
$$= (\xi \mid (K_1 \otimes K_2)\eta) .$$

$K_1 \otimes K_2$ is densely defined and symmetric

[iii] $K_1 \otimes K_2$ is essentilly self-adjoint. Suppose that there exists a constant C such that $|((K_1 \otimes K_2)\xi \mid \eta)| \leq C \|\xi\|$, for all ξ in D

Take $\xi = (K_1 e_n \otimes K_2 E_m)\eta = (e_n \otimes E_m)(K_1 e_n \otimes K_2 E_m)\eta$ in D

then we have

$$\|(K_1 e_n \otimes K_2 E_m)\eta\| \leq C \qquad \text{for all } n, m .$$

Since $\|(K_1 e_n \otimes K_2 E_m)\eta\|^2 = ((K_1^2 e_n \otimes K_2^2 E_m)\eta \mid \eta)$ is monotone increasing with respect to (n, m), there exists $\lim_{(n,m)} \|(K_1 e_n \otimes K_2 E_m)\eta\|^2$.

We have, for $n \leq n_1$, $m \leq m_1$,

$$\|(K_1 \otimes K_2)(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)\eta\|^2$$
$$= \|(K_1 e_{n_1} \otimes K_1 E_{m_1})(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)\eta\|^2$$
$$= \|(K_1 e_{n_1} \otimes K_2 E_{n_1})\eta\|^2 - \|(K_1 e_n \otimes K_2 E_m)\eta\|^2$$
$$\leq \varepsilon \qquad \text{for sufficient large } (n_1, m_1) \geq (n, m).$$

Since $\text{s-}\lim_{(n,m)} (e_n \otimes E_m)\eta = \eta$, and $\text{s-}\lim_{(n,m)} (K_1 \otimes K_2)(e_n \otimes E_m)\eta$ exists, we get η in $D((K_1 \otimes K_2)^{**})$, therefore $(K_1 \otimes K_2)^{**}$ is equal to $(K_1 \otimes K_2)^*$. We denote the closed extension of $K_1 \otimes K_2$ defined above again by $K_1 \otimes K_2$. It is noticed that $(K_1 \otimes K_2)e_n \otimes E_m = K_1 e_n \otimes K_2 E_m$ for all n, m in $\mathbb{N}$.

For each ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$,

$$\| (K_1 \otimes K_2)(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)(\xi_1 \otimes \xi_2) \|^2$$

$$= \| K_1 e_{n_1} \xi_1 \otimes K_2 E_{m_1} \xi_2 - K_1 e_n \xi_1 \otimes K_2 E_m \xi_2 \|^2$$

$$\leq 2\{ \| (K_1 e_{n_1} \xi_1 - K_1 e_n \xi_1) \otimes K_2 E_{m_1} \xi_2 \|^2 + \| K_1 e_n \xi_1 \otimes (K_2 E_{m_1} \xi_2 - K_2 E_m \xi_2) \|^2 \}$$

$$\leq 2\{ \| (K_1 e_{n_1} - K_1 e_n)\xi_1 \|^2 \cdot \| K_2 \xi_2 \|^2 + \| K_1 \xi_1 \|^2 \cdot \| (K_2 E_{m_1} - K_2 E_m)\xi_2 \|^2 \}.$$

We get $\xi_1 \otimes \xi_2$ in $D(K_1 \otimes K_2)$ by the closedness of $K_1 \otimes K_2$, which means that $D(K_1 \otimes K_2) \supset D(K_1) \otimes_a D(K_2)$ and $K_1 \otimes K_2(\xi_1 \otimes \xi_2) = K_1 \xi_1 \otimes K_2 \xi_2$ for all ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$.

Let T be an another self-adjoint operator on $H_1 \otimes H_2$ with the above properties. By $T(e_n \otimes E_m)(\xi_1 \otimes \xi_2) = K_1 e_n \xi_1 \otimes K_2 E_m \xi_2 = (K_1 \otimes K_2)(e_n \otimes E_m)(\xi_1 \otimes \xi_2)$ for all ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$, and the closedness of T, we get $T(e_n \otimes E_m) = (K_1 \otimes K_2)e_n \otimes E_m$. Using the closedness of T again and $\text{s-}\lim_{(n,m)} e_n \otimes E_m = 1$, we have $T \supset K$, therefore $T = K$ by the self-adjointness of T and K, then $K_1 \otimes K_2$ is determined uniquely. If K_1 and K_2 are positive, $K_1 \otimes K_2$ is positive since $(K_1 \otimes K_2)e_n \otimes E_m = K_1 e_n \otimes K_1 E_m$ is a pcsitive bounded operator.

Notice 2.2. Let K_1 and K_2 be bounded positive operators on H_1 and H_2 respectively, $K_1 \otimes K_2$ is a positive (bounded) operator on $H_1 \otimes H_2$.

Remark 2.3. In the Theorem 2.1 if K_1 and K_2 are affiliated with von Neumann algebras M and N respectively, then $K_1 \otimes K_2$ is affiliated with the von Neumann algebra $M \otimes N$.

Definition 2.4. If h and k are positive self-akjoint operators on Hilbert space H and $\varepsilon > 0$ we put $h_\varepsilon = h(1 + \varepsilon h)^{-1}$. We write $h \leqq k$ if $h_\varepsilon \leqq k_\varepsilon$ for some (and hence any) $\varepsilon > 0$. This is equivalent to the two conditions

$$D(h^{\frac{1}{2}}) \supset D(k^{\frac{1}{2}}) \qquad \text{and} \qquad \| h^{\frac{1}{2}} \xi \|^2 \leqq \| k^{\frac{1}{2}} \xi \|^2$$

for each ξ in $D(k^{\frac{1}{2}})$. We say that a net $\{ h_i \}$ of positive self-adjoint operators increases to the self-adjoint operator h, and write $h_i \nearrow h$ if $h_{i\varepsilon} \nearrow h_\varepsilon$. Thus $h_\varepsilon \nearrow h$ when $\varepsilon \searrow 0$.

Lemma 2.5. $K_{1_\delta} \otimes K_{2_\varepsilon} \nearrow K_1 \otimes K_2$ when K_1 and K_2 are positive self-adjoint operators on H_1 and H_2 respectively, $\delta \searrow 0$, $\varepsilon \searrow 0$.

Proof.

$$(K_{1_\delta} \otimes K_{2_\varepsilon})(e_n \otimes E_m) = K_{1_\delta} e_n \otimes K_{2_\varepsilon} E_m , \quad \text{for each } n, m \text{ in } \mathbb{N}$$

$$K_{1_\delta} e_n \leqq K_{1_{\delta'}} e_n \leqq K_1 e_n, \quad K_{2_\varepsilon} E_m \leqq K_{2_{\varepsilon'}} E_m \leqq K_2 E_m$$

where $\delta \geqq \delta'$ and $\varepsilon \geqq \varepsilon'$,

By Notice 2.2, we get

$$(K_{1_\delta} \otimes K_{2_\varepsilon})(e_n \otimes E_m) \leqq (K_{1_{\delta'}} \otimes K_{2_{\varepsilon'}})(e_n \otimes E_m) \leqq (K_1 \otimes K_2)(e_n \otimes E_m)$$

moreover

$$K_{1_\delta} e_n \otimes K_{2_\varepsilon} E_m \nearrow (K_1 \otimes K_2) e_n \otimes E_m .$$

Then

$$(1 + (K_{1_\delta} \otimes K_{2_\varepsilon}))^{-1} e_n \otimes E_m \searrow (1 + (K_1 \otimes K_2))^{-1} e_n \otimes E_m.$$

Since the operator norms of $(1 + K_{1_\delta} \otimes K_{2_\varepsilon})^{-1}$ and $(1 + K_1 \otimes K_2)^{-1}$ are smaller than 1, $s - \lim_{(n,m)} e_n \otimes E_m = 1$, we get

$$(1 + K_{1_\delta} \otimes K_{2_\varepsilon})^{-1} \searrow (1 + K_1 \otimes K_2)^{-1},$$

Then we get

$$(K_{1_\delta} \otimes K_{2_\varepsilon})_1 = 1 - (1 + K_{1_\delta} \otimes K_{2_\varepsilon})^{-1} \nearrow 1 - (1 + K_1 \otimes K_2)^{-1} = (K_1 \otimes K_2)_1.$$

Hence

$$K_{1_\delta} \otimes K_{2_\varepsilon} \nearrow K_1 \otimes K_2.$$

3. The Tensor Product of Normal Semi-Finite Weights.

In this chapter, we often refer to [5] The Radon-Nikodym theorem for von Neumann algebra, and let φ be a faithful normal semi-finite weight on von Neumann algebra M, which gives rise to a one-parameter group Σ of automorphisms of M. The proof of Lemma 3.1 is almost similar to [5] Lemma 5.2.

Lemma 3.1. Let ψ be a normal semi-finite weight on M, if there exists a σ-weakly dense *-subalgebra B in m_φ, invariant under Σ such that $\varphi = \psi$ on B, then we have $\psi \leqq \varphi$, $\dot{\psi}\,|m_\varphi = \dot{\varphi}$, and ψ is faithful.

Proof. If x and y are in B, then $\varphi(x \cdot y)$ and $\psi(x \cdot y)$ are normal functionals on M which agree on B, since B is an algebra. Therefore $\varphi(x \cdot y) = \psi(x \cdot y)$. Since B is a dense *-algebra there is a net $\{ u_\lambda \}$ in B_+ such that u_λ converges $\sigma*$-strongly to 1 and $\| u_\lambda \| \leqq 1$. Put

$$h_\lambda = \pi^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp - t^2 \sigma_t(u_\lambda) dt$$

Since B is invariant under Σ we have

$$\varphi(\sigma_t(u_\lambda)\, x\sigma_s(u_\lambda)) = \psi(\sigma_t(u_\lambda)\, x\sigma_s(u_\lambda))$$

for all s and t and each x in M. It follows from [5] Lemma 3.1, by the polarization identity, that $\varphi(h_\lambda x h_\lambda) = \psi(h_\lambda x h_\lambda)$.

Each h_λ is an analytic element with

$$\sigma(h_\lambda) = \pi^{-\frac{1}{2}}\int \exp(-(t-\alpha)^2)\sigma_t(u_\lambda)dt$$

$$\begin{aligned}\| (1-\sigma_\alpha(h_\lambda))\xi \| &= \| (1-\pi^{-\frac{1}{2}}\int \exp(-(t-\alpha)^2)\sigma_t(u_\lambda)dt)\xi \| \\ &= \| \pi^{-\frac{1}{2}}\int \exp(-(t-\alpha)^2\sigma_t(1-u_\lambda)\xi dt \| \\ &\leq \pi^{-\frac{1}{2}} \int |\exp(-(t-\alpha)^2| \, \| \sigma_t(1-u_\lambda)\xi \| dt \\ &= \pi^{-\frac{1}{2}} \exp(\mathrm{Im}\alpha)^2 \int \exp - (t-\mathrm{Re}\alpha)^2 \| \sigma_t(1-u_\lambda)\xi \| dt\end{aligned}$$

$\lim_\lambda \| \sigma_t(1-u_\lambda)\xi \| = 0$ and. $\| \sigma_t(1-u_\lambda)\xi \| \leq 2\|\xi\|$, for all λ in $\mathbb{C}$, and so by Lebesgue dominated convergence theorem we have

$$\lim_\lambda \| (1-\sigma_\alpha(h_\lambda))\xi \| = 0 \quad \text{ie} \quad s-\lim \sigma_\alpha(h_\lambda) = 1 \quad \text{for all } \alpha \text{ in } \mathbb{C}.$$

Take now x in m_+. Using the σ-weakly lower semi-continuity of ψ and $\Delta^{\frac{1}{2}} h_\lambda \Delta^{-\frac{1}{2}} = \sigma_{-1/2}(h_\lambda)$ on $D(\Delta^{-\frac{1}{2}})$ by [5] Lemma 3.5 we get

$$\begin{aligned}\psi(x) &\leq \underline{\lim}\, \psi(h_\lambda x h_\lambda) = \underline{\lim}\, \varphi(h_\lambda x h_\lambda) = \underline{\lim}\, \| \eta(x^{\frac{1}{2}} h_\lambda) \|^2 \\ &= \underline{\lim}\, \| S h_\lambda S \eta(x^{\frac{1}{2}}) \|^2 = \underline{\lim}\, \| J\Delta^{\frac{1}{2}} h_\lambda \Delta^{\frac{-1}{2}} J \eta(x^{\frac{1}{2}}) \|^2 \\ &= \underline{\lim}\, \| \sigma_{-1/2}(h_\lambda) J\eta(x^{\frac{1}{2}}) \|^2 = \lim \| \sigma_{-1/2}(h_\lambda) J\eta(x^{\frac{1}{2}}) \|^2 \\ &= \| J\eta(x^{\frac{1}{2}}) \|^2 = \| \eta(x^{\frac{1}{2}}) \|^2 = \varphi(x).\end{aligned}$$

Thus $\psi \leq \varphi$.

By [1] Lemma 2.3, there exists T in $\pi_\varphi(M)'$ such that

$$0 \leq T \leq 1 , \quad \psi(y^*x) = (\eta(x) \mid T\,\eta(y))$$

for x, y in n_φ. Then we have

$$\begin{aligned}\psi(h_\lambda x h_\lambda) &= (\eta(x^{\frac{1}{2}} h_\lambda) \mid T\,\eta(x^{\frac{1}{2}} h_\lambda)) \\ &= \| T^{\frac{1}{2}} \eta(x^{\frac{1}{2}} h_\lambda) \|^2 .\end{aligned}$$

By the same arguement above

$$\psi(h_\lambda x\, h_\lambda) = \| T^{\frac{1}{2}} J\sigma_{-i/2}(h)\, J\eta(x^{\frac{1}{2}}) \|^2 .$$

Then we have $\underline{\lim}\ \psi(h_\lambda x\, h_\lambda) = \lim \| T^{\frac{1}{2}} J\sigma_{-i/2}(h_\lambda)\, J\eta(x^{\frac{1}{2}}) \|^2$

$$= \| T^{\frac{1}{2}} J\cdot J\eta(x^{\frac{1}{2}}) \|^2$$

$$= \| T^{\frac{1}{2}}\eta(x^{\frac{1}{2}}) \|^2 = \psi(x).$$

Therefore $\psi(x) = \varphi(x)$ for all x in $(m_\varphi)_+$.

We refer to [5] Lemma 3.1 with respect to the faithfulness of ψ.

Proposition 3.2.([5] proposition 5.9) If ψ is Σ-invariant normal semi-finite weight on M which is equal to φ on a σ-weakly dense Σ-invariant *-subalgebra of m_φ then $\varphi = \psi$.

Proposition 3.3. Let φ and ψ be faithful normal semi-finite weights on von Neumann algebras M, N, σ_t and ρ_t one-parameter groups of automorphisms of φ and ψ, which are denoted by Σ and Σ^ψ respectively. There exists a unique $\Sigma \otimes \Sigma^\psi$-invariant normal semi-finite weight θ on $M \otimes N$ such that

$$m_\theta \supset m_\varphi \otimes_a m_\psi , \quad \theta(x \otimes y) = \varphi(x)\cdot\psi(y)$$

for all x in $(m_\varphi)_+$, y in $(m_\psi)_+$. Moreover let g be a normal semi-finite weight on $M \otimes N$ such that $m_g \supset m_\varphi \otimes_a m_\psi$, $g(x \otimes y) = \varphi(x)\psi(y)$ for x in $(m_\varphi)_+$, y in $(m_\psi)_+$. Then we have $g \leq \theta$, $\dot{g}|m_\theta = \dot{\theta}$ and g is faithful.

Proof. We may assume that $M = \mathcal{L}(\mathcal{U}_{\varphi_0})$, $N = \mathcal{L}(\mathcal{U}_{\psi_0})$ where $\mathcal{U}_{\varphi_0}$ and $\mathcal{U}_{\psi_0}$ are the maximal modular algebras associated with $\mathcal{U}_\varphi$ and $\mathcal{U}_\psi$ in [2] Theorem 2.13 respectively. By [4] Theorem 11.1 $\{\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0};\ \Delta_1(\alpha) \otimes_a \Delta_2(\alpha)$, α in $\mathbb{C}\}$ is also a modular algebra, moreover we get

$$\mathcal{L}(\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0}) = \mathcal{L}(\mathcal{U}_{\varphi_0}) \otimes \mathcal{L}(\mathcal{U}_{\psi_0}).$$

By [4] Lemma 2.1 there exists a unique positive self-adjoint non-singular operator Δ on $H_\varphi \otimes H_\psi$ such that Δ^α is the closure of $\Delta_1(\alpha) \otimes_a \Delta_2(\alpha)$ for all α in $\mathbb{C}$, therefore $\Delta^{it} = \Delta_1^{it} \otimes \Delta_2^{it}$ for all t in $\mathbb{R}$.

For each η_1 in $\mathfrak{U}_{\varphi_0}$ and η_2 in $\mathfrak{U}_{\psi_0}$ we get

$$\begin{aligned}\sigma_t \otimes \rho_t(\pi(\eta_1) \otimes \pi(\eta_2)) &= \sigma_t(\pi(\eta_1)) \otimes \rho_t(\pi(\eta_2)) \\ &= (\Delta_1^{it}\pi(\eta_1)\Delta_1^{-it}) \otimes (\Delta_2^{it}\pi(\eta_2)\Delta_2^{-it}) \\ &= (\Delta_1^{it} \otimes \Delta_2^{it})(\pi(\eta_1) \otimes \pi(\eta_2))(\Delta_1^{-it} \otimes \Delta_2^{-it}) \\ &= \Delta^{it}\pi(\eta_1 \otimes \eta_2)\Delta^{-it}.\end{aligned}$$

Then $\sigma_t \otimes \rho_t$ coincides with the modular automorphism group of $\mathfrak{U}_{\varphi_0} \otimes_a \mathfrak{U}_{\psi_0}$ on a σ-weakly dense sub-algebra $\pi(\mathfrak{U}_{\varphi_0}) \otimes_a \pi(\mathfrak{U}_{\psi_0})$. Therefore $\sigma_t \otimes \rho_t$ is equal to it.

Let θ be the canonical weight of $\mathfrak{U}_{\varphi_0} \otimes_a \mathfrak{U}_{\psi_0}$ defined in [2] Theorem 2.11. By [2] Propsition 4.4 θ is a faithful normal semi-finite K.M.S. weight with respect to $\sigma_t \otimes \rho_t$, $\beta = 1$. Since $\mathfrak{U}_{\varphi_0}$ (resp $\mathfrak{U}_{\psi_0}$) is equivalent to $\mathfrak{U}_\varphi$ (resp. $\mathfrak{U}_\psi$) we have $\xi \otimes \eta$ in $(\mathfrak{U}_{\varphi_0} \otimes_a \mathfrak{U}_{\psi_0})''$ for each ξ in $\mathfrak{U}_\varphi$ and η in $\mathfrak{U}_\psi$ and $\pi(\xi \otimes \eta) = \pi(\xi) \otimes \pi(\eta)$. We get

$$\begin{aligned}\theta((\pi(\xi) \otimes \pi(\eta))^*(\pi(\xi) \otimes \pi(\eta))) &= \theta(\pi(\xi \otimes \eta)^*\pi(\xi \otimes \eta)) \\ &= (\xi \otimes \eta \mid \xi \otimes \eta) \\ &= \|\xi\|^2 \cdot \|\eta\|^2 \\ &= \varphi(\pi(\xi)^*\pi(\xi)) \cdot \psi(\pi(\eta)^*\pi(\eta)).\end{aligned}$$

By [2] Lemma 2.4 for each x in $(m_\varphi)_+$ and y in $(m_\psi)_+$ there exist ξ in $\mathfrak{U}_\varphi$ and η in $\mathfrak{U}_\psi$ such that $x^{\frac{1}{2}} = \pi(\xi)$, $y^{\frac{1}{2}} = \pi(\eta)$, then we have $\theta(x \otimes y) = \varphi(x) \cdot \psi(y)$.

Let g be an another $\Sigma \otimes \Sigma^{\psi}$-invariant normal semi-finite weight on $M \otimes N$ such that ;

$$m_g \supset m_{\varphi} \otimes_a m_{\psi} , \qquad g(x \otimes y) = \varphi(x) \cdot \psi(y)$$

for all x in$(m_{\varphi})_+$, y in$(m_{\psi})_+$. By Proposition 3.2 we have $\theta = g$. The last part of Proposition 3.3 is clear by Lemma 3.1.

Theorem 3.4 . Let φ_1 and ψ_1 be normal semi-finite weights on M and N, p and q the support projections of φ_1 and ψ_1 respectively. There exists a unique normal semi-finite weight θ_1 on $M \otimes N$ such that ;

(i) $m_{\theta_1} \supset m_{\varphi_1} \otimes_a m_{\psi_1}$

(ii) $\theta_1(x \otimes y) = \varphi_1(x) \cdot \psi_1(y)$

for each x in$(m_{\varphi_1})_+$ and y in$(m_{\psi_1})_+$, and that θ_1 is $\Sigma^{\varphi_1} \otimes \Sigma^{\psi_1}$-invariant on the von Neumann algebra $p \otimes q(M \otimes N)p \otimes q$. Furthermore θ_1 is the maximal normal semi-finite weight with the properties (i), (ii) and its support projection is the tensor product $p \otimes q$.

Proof. It follows from Proposition 3.3.

Definition 3.5. The maximal weight above is called the tensor product of φ_1 and ψ_1, which is denoted by $\varphi_1 \otimes \psi_1$.

Corollary 3.6.([3] Proposition 6.2) Let M and N be two von Neumann algebras, ν and μ two normal strictly semi-finite weights on M^+ and N^+, $(f_i)_{i\epsilon I}$ [resp. $(g_j)_{j\epsilon J}$] a family of positive normal linear functionals such that $\Sigma_{i\epsilon I} f_i = \nu$ on M^+ and their supports are mutually orthogonal [resp. $\Sigma_{j\epsilon J} g_j = \mu$, N^+].

(i) $\tau = \Sigma_{i,j} f_i \otimes g_j$ is a strictly semi-finite normal weight on $(M \otimes N)^+$.

This weight does not depend on the choice of $(f_i)_{i\epsilon I}$, $(g_j)_{j\epsilon I}$, and its support is the tensor product of the supports of ν and μ. The algebra m_τ contains $m_\nu \otimes_a m_\mu$ and we have $\dot{\tau}|_{m_\nu \otimes_a m_\mu} = \dot{\nu} \otimes_a \dot{\mu}$. Let θ be an another normal semi-finite weight on $(M \otimes N)^+$ with the above properties. Then we get ;

$$m_\theta \supset m_\tau \quad \text{and} \quad \dot{\tau} = \dot{\theta}|_{m_\tau}$$

(ii) We suppose that ν [resp. μ] is K.M.S. with respect to a one-parameter automorphism group $\{\omega_t\}$ [resp. χ_t], $\beta = 1$. Then there exists a unique normal weight τ on $(M \otimes N)^+$ such that $m_\tau \supset m_\nu \otimes_a m_\mu$, $\dot{\tau}|_{m_\nu \otimes_a m_\mu} = \dot{\nu} \otimes_a \dot{\mu}$ and τ is K.M.S. with respect to $\{\omega_t \otimes \chi_t\}$ on $M \otimes N$, $\beta = 1$. This weight is equal to the weight defined above.

Proof. (i) By the choice of f_i, $f_i(\cdot)$ is equal to $\nu(p_i \cdot p_i)$ where p_i is the support projection of f_i, therefore f_i is Σ^ν-invariant on pMp where p is the support projection of ν. Similarly g_j is Σ^μ-invariant on qNq where q is that of μ. Since $\tau = \Sigma_{i,j}\, f_i \otimes g_j$ is $\Sigma^\nu \otimes \Sigma^\mu$-invariant on $p \otimes q(M \otimes N)p \otimes q$, τ is the maximal weight in Theorem 3.4.

(ii) By the uniqueness of K.M.S. in [2] Proposition 4.8 we have ;

$$p\omega_t(\ \cdot\ )p = \sigma_t^{\nu}$$

$$q\chi_t(\ \cdot\ )q = \sigma_t^{\mu}.$$

Therefore τ is $\Sigma^\nu \otimes \Sigma^\mu$-invariant on $p \otimes q(M \otimes N)p \otimes q$. It follows from Theorem 3.4.

Corollary 3.7. ([2] Corollary 6.5) Let ν and μ be two normal semi-finite traces on von Neumann algebras M and N. $\tau = \nu \otimes \mu$ is a unique normal semi-finite trace of $M \otimes N$ such that $m_\nu \otimes_a m_\mu \subset m_\tau$ and $\dot{\tau}|_{m_\nu \otimes_a m_\mu} = \dot{\nu} \otimes_a \dot{\mu}$.

Remark 3.8. (i) If ν and μ are strictly semi-finite normal weights on M and N, $\nu \otimes \mu$ is a normal strictly semi-finite weight on $M \otimes N$.

(ii) If ν and μ are normal semi-finite traces on M and N, $\nu \otimes \mu$ is a normal semi-finite trace on $M \otimes N$.

Corollary 3.9. (The extension of [2] Corollary 6.4) Let φ and ψ be two faithful normal semi-finite weights on M and N, $\mathcal{U}_\varphi$, $\mathcal{U}_\psi$ and $\mathcal{U}_{\varphi\otimes\psi}$ be the generalized Hilbert algebras defined by φ, ψ and $\varphi\otimes\psi$ respectively. Then $\mathcal{U}_{\varphi\otimes\psi}$ is isomorphic to the acheived ~~generalized~~ generalized Hilbert algebra of $\mathcal{U}_\varphi \otimes_a \mathcal{U}_\psi$. Furthermore the modular operator Δ of $\mathcal{U}_{\varphi\otimes\psi}$ is the tensor product of modular operators of $\mathcal{U}_\varphi$ and $\mathcal{U}_\psi$.

Proof. It has already been proved in Proposition 3.3.

4. The Radon-Nikodym Theorem in the Tensor Product.

Theorem 4.1 Let φ and ψ be faithful normal semi-finite weights on M and N, $\varphi_1 = \varphi(h\ \cdot)$ and $\psi_1 = \psi(k\ \cdot)$ where positive self-adjoint operators h and k are affiliated with M^{Σ^φ} and N^{Σ^ψ} respectively.

Then we get

$$\varphi_1 \otimes \psi_1(\ \cdot\ ) = \varphi \otimes \psi(h \otimes k\ \cdot\ )$$

where $h \otimes k$ has been defined in §2.

That is $\varphi(h\ \cdot) \otimes \psi(k\ \cdot) = \varphi\otimes\psi(h \otimes k\ \cdot)$.

Proof. For each x in$(m_{\varphi_1})_+$ and y in$(m_{\psi_1})_+$

$$\varphi_1(x) = \lim_\varepsilon \varphi(h_\varepsilon \cdot x)$$
$$\psi_1(y) = \lim_\delta \psi(k_\delta \cdot y).$$

By [5] Proposition 4.2 we have

$$h_\varepsilon^{\frac{1}{2}}\, x\, h_\varepsilon^{\frac{1}{2}} \text{ in } m_\varphi$$

$$k_\delta^{\frac{1}{2}}\, y\, k_\delta^{\frac{1}{2}} \text{ in } m_\psi$$

for all $\varepsilon > 0$ $\delta > 0$,

$$\varphi_1 \otimes \psi_1(x \otimes y) = \varphi_1(x)\psi_1(y)$$

$$= \lim_{(\delta,\varepsilon)} \varphi(h_\varepsilon x)\psi(k_\delta y)$$

$$= \lim_{(\delta,\varepsilon)} \varphi \otimes \psi(h_\varepsilon \otimes k_\delta \cdot x \otimes y).$$

By Lemma 1.2 and [5] Proposition 4.2 we get

$$\varphi_1 \otimes \psi_1(x \otimes y) = \varphi \otimes \psi(h \otimes k \cdot x \otimes y)$$

for x $\in (m_{\varphi_1})_+$ y $\in (m_{\psi_1})_+$.

[5] Theorem 4.6 says that

$$\sigma_t^{\varphi_1} = h^{it}\sigma_t^{\varphi}(\cdot)h^{-it}$$

$$\sigma_t^{\psi_1} = k^{it}\sigma_t^{\psi} k^{-it}$$

By the definition of $\varphi_1 \otimes \psi_1$; $\sigma_t^{\varphi_1 \otimes \psi_1} = (h^{it} \otimes k^{it})\sigma_t^{\varphi} \otimes \rho_t^{\psi}(\cdot)(h^{-it} \otimes k^{-it})$.

Since $h^{it} \otimes k^{it} \in M^{\Sigma^\varphi} \otimes N^{\Sigma^\psi}$ and $h \otimes k$ commutes with $h^{it} \otimes k^{it}$ $\varphi \otimes \psi(h \otimes k \cdot)$ is $\sigma_t^{\varphi_1 \otimes \psi_1}$-invariant on $[h] \otimes [k](M \otimes N)[h] \otimes [k]$ where $[h]$ and $[k]$ are the range projections of h and k respectively. By Theorem 3.4 we get $\varphi \otimes \psi(h \otimes k \cdot) = \varphi_1 \otimes \psi_1$.

Corollary 4.2. In Theorem 4.1 we suppose that φ_1 and ψ_1 are K.M.S. weights with respect to σ_t and ρ_t respectively.
Then $\varphi_1 \otimes \psi_1$ is K.M.S. weight with respect to $\sigma_t \otimes \rho_t$.

Proof. It follows from Theorem 4.1 and [5] Corollary 4.1.

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