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Some results on totally real submanifolds

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1. Introduction. Let $\bar{M}$ be a Kaehler manifold with almost complex structure $\bar{J}$ and Hermitian metric $\bar{g}$. Let M be a submanifold isometrically immersed in $\bar{M}$ and for $x \in M$ let M_x denote the tangent space to M at x and N_x denote the normal space at x. Chen and Ogiue [1] have called M totally real if $M_x \cap \bar{J}M_x = \{0\}$ for all $x \in M$ while Okumura [3] has called M anti-holomorphic if $N_x \cap \bar{J}N_x = \{0\}$ for all $x \in M$. If M is a real curve immersed in $\bar{M}$ then M is totally real while if M is a real hypersurface then M is anti-holomorphic. Of course, if the real dimension of M is equal to the complex dimension of $\bar{M}$, then the two definitions coincide and perhaps this is one of the most interesting cases.

The purpose of this talk is to discuss some properties of totally real submanifolds.

2. Basic results. Let M be a totally real submanifold of the Kaehler manifold $\bar{M}$. If X is a vector field tangent to M, then $\bar{J}X$ is a normal vector field to M and if ξ is a normal vector field to M, then we can put $\bar{J}\xi = P\xi + Q\xi$, where $P\xi$ is tangent to M and $Q\xi$ is normal to M. (Note that if the real dimension of M equals the complex dimension of M, then Q = 0.) From this, we obtain

$$(1) \qquad \begin{cases} PQ\xi = 0, & Q^2\xi = -\xi - \bar{J}P\xi, \\ P\bar{J}X = -X, & Q\bar{J}X = 0, \end{cases}$$

where ξ is an arbitrary normal vector to M and X is an arbitrary tangent vector. If we write the Gauss and Weingarten equations as

$$(2)\qquad \begin{cases} \bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \\ \bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \end{cases}$$

where $\bar{\nabla}$ is the Riemannian connection of $\bar{g}$, then ∇ is the Riemannian connection of the metric g induced on M, σ is the second fundamental form, $\nabla^\perp$ is the Riemannian connection induced on the normal bundle and $g(A_\xi X, Y) = \bar{g}(\sigma(X,Y), \xi)$. Applying (2) to the above equations, we obtain

$$(3)\qquad \begin{cases} -A_{\bar{J}Y}X = P\,\sigma(X,Y), \quad \nabla_X^\perp(\bar{J}Y) = \bar{J}\nabla_X Y + Q\,\sigma(X,Y), \\ P\nabla_X^\perp \xi = \nabla_X(P\xi) - A_{Q\xi}X, \quad -\bar{J}A_\xi X + Q\nabla_X^\perp \xi = \sigma(X,P\xi) \\ \qquad\qquad\qquad\qquad + \nabla_X^\perp(Q\xi). \end{cases}$$

<u>Proposition</u> 1. If $\dim_R M = \dim_C \bar{M} > 1$ and M is totally umbilical, then M is totally geodesic.

<u>Proposition</u> 2 ([1]) If $\bar{M}$ is a complex space form of non-zero constant holomorphic curvature then a submanifold M is invariant under the curvature operator $\bar{R}(X,Y)$, where X and Y are tangent to M, if and only if M is holomorphic or totally real.

<u>Proposition</u> 3 ([1]) If M is a totally real, totally geodesic submanifold of a complex space form $\bar{M}$, then M is of constant curvature.

<u>Theorem</u> 1. Let M be as in the above proposition and $\dim_R M = \dim_C \bar{M}$. If $[A_\xi, A_\zeta] = 0$ for any normal vectors ξ and ζ then M is of constant curvature. If in addition M is minimal, then M is totally geodesic.

<u>Theorem</u> 2. If M is as above, then $R \equiv 0$ if and only if $R^N \equiv 0$, where $R^N(X,Y) = [\nabla_X^\perp, \nabla_Y^\perp] - \nabla^\perp_{[X,Y]}$.

3. Pinching Theorem. Applying the formula for the Laplacian of $\|\sigma\|^2$ and the technique of Chern-do Carmo-Kobayashi [2] , we can obtain the following theorem.

Theorem 3. Let M be a totally real minimal submanifold of dimension n of a complex space form $\bar{M}(\bar{c})$, $\bar{c} > 0$. If

$$\|\sigma\|^2 \leqq \frac{n}{2-\frac{1}{p}} \frac{\bar{c}}{4},$$

then M is totally geodesic, where $p = 2m - n$ and $n = \dim_R M$ and $m = \dim_C \bar{M}$.

Corollary 1. ([1]) Let M and $\bar{M}$ be as in the theorem and n = m. If

$$\|\sigma\|^2 < \frac{n(n+1)}{4(2n-1)} \bar{c},$$

then M is totally geodesic.

References.

[1] B.Y.Chen and K.Ogiue, On totally real submanifolds, to appear in Trans. of AMS.,192 (1974).

[2] S.S.Chern, M.P.do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields, (Proc. Conf. for M.Stone, Univ. Chicago, Chicago, Ill. 1968), Springer, New York, 1970, 59-75.

[3] M.Okumura, Submanifolds of real codimension p of a complex projective space, to appear.