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Vector Valued Pseudodifferential Operators and their Applications

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§1. Introduction and the result. In this note we introduce the vector valued pseudodifferential operators where the vector space depends on a parameter (cf. Sjöstrand [3], §4) and take this opportunity to construct global parametrix-like operators for the following operator in R^n:

$$P(x,t,D_x,D_t) = D_t - it^k a(x,t,D_x,D_t) + b(x,t,D_x,D_t),$$

where $(x,t)\in R^n$ with $x\in R^{n-1}$, $t\in R$ and $k\in Z^+$ <u>is odd</u>, and $a(x,t,D_x,D_t)$ and $b(x,t,D_x,D_t)$ are properly supported classical pseudodifferential operators of order 1 and order 0 respectively, and the principal symbol $a_1(x,t,\xi,\tau)$ of $a(x,t,D_x,D_t)$ is positively homogeneous of degree 1 and

(A) $\qquad \operatorname{Re} a_1(x,t,\xi,\tau) \neq 0$

for $(x,t)\in R^n$, $(\xi,\tau) \neq (0,0)$, $\xi\in R^{n-1}$, $\tau\in R$ (cf. Sjöstrand [2]). Introducing the vector valued pseudodifferential operators, we can construct the parametrix-like operators in the above non-elliptic case completely analogous to the elliptic case.

Theorem. (cf. [2], Theorem 1.) <u>Assume that for all</u> ℓ, m $Z^+\cup\{0\}$ <u>and multiindices</u> α,β <u>there exists a constant</u> C =

$C(\alpha, \ell, \beta, m)$ such that

(B) $$\left|D_x^{\alpha} D_t^{\ell} D_{\xi}^{\beta} D_{\tau}^{m} a_1(x,0,\xi,\tau)\right| \leq C(1 + |\xi| + |\tau|)^{1-|\beta|-m}$$

for $x \in R^{n-1}$, $(\xi, \tau) \neq (0,0)$.

I. (i) If $\operatorname{Re} a_1(x,t,\xi,\tau) > 0$ for $(x,t) \in R^n$ and $(\xi,\tau) \neq (0,0)$, then there exist properly supported operators

$$\mathcal{P}_1 = \begin{pmatrix} P \\ R^+ \end{pmatrix} : \mathcal{D}'(R^n) \longrightarrow \begin{matrix} \mathcal{D}'(R^n) \\ \oplus \\ \mathcal{D}'(R^{n-1}) \end{matrix}$$

$$\mathcal{G}_1 = (G_1, G^+) : \begin{matrix} \mathcal{D}'(R^n) \\ \oplus \\ \mathcal{D}'(R^{n-1}) \end{matrix} \longrightarrow \mathcal{D}'(R^n)$$

such that $\mathcal{G}_1 \cdot \mathcal{P}_1 - I$ and $\mathcal{P}_1 \cdot \mathcal{G}_1 - I$ have C^{∞} kernels.

(ii) If $\operatorname{Re} a_1(x,t,\xi,\tau) < 0$ for $(x,t) \in R^n$ and $(\xi,\tau) \neq (0,0)$, then there exist properly supported operators

$$\mathcal{P}_2 = (P, R^-) : \begin{matrix} \mathcal{D}'(R^n) \\ \oplus \\ \mathcal{D}'(R^{n-1}) \end{matrix} \longrightarrow \mathcal{D}'(R^n)$$

$$\mathcal{G}_2 = \begin{pmatrix} G_2 \\ G^- \end{pmatrix} : \mathcal{D}'(R^n) \longrightarrow \begin{matrix} \mathcal{D}'(R^n) \\ \oplus \\ \mathcal{D}'(R^{n-1}) \end{matrix}$$

such that $\mathcal{G}_2 \cdot \mathcal{P}_2 - I$ and $\mathcal{P}_2 \cdot \mathcal{G}_2 - I$ have C^{∞} kernels.

II. For all $s \in R$,

$$G_1, G_2 : H_s^{loc}(R^n) \longrightarrow H^{loc}_{s + \frac{1}{1+k}}(R^n),$$

$$G^+ : H_s^{loc}(R^{n-1}) \longrightarrow H^{loc}_{s + \frac{1}{1+k}}(R^n),$$

$$G^-: H_s^{loc}(R^n) \longrightarrow H_s^{loc}(R^{n-1})$$

<u>are continuous</u>.

III. $WF'(G_1)$, $WF'(G_2) \subset \{((x,t,\xi,\tau),(x,t,\xi,\tau)) \in (T^*(R^n)\setminus 0) \times (T^*(R^n)\setminus 0)\}$;
$WF'(R^-)$, $WF'(G^+) \subset \{((x,0,\xi,0),(x,\xi)) \in (T^*(R^n)\setminus 0)\times(T^*(R^{n-1})\setminus 0)\}$;
$WF'(R^+)$, $WF'(G^-) \subset \{((x,\xi),(x,0,\xi,0))\ (T^*(R^{n-1})\setminus 0)\times(T^*(R^n)\setminus 0)\}$.

§2. Vector valued pseudodifferential operators. (cf. Treves [4], Theorem 4.1.) Let H_1 and H_2 be complex Hilbert spaces and let $\mathcal{L}(H_1,H_2)$ be the Banach space of bounded linear operators $H_1 \longrightarrow H_2$. We define $S^m(R^n \times R^n; H_1,H_2)$ as the space of C^∞ functions $p(x,\xi)$ on $R^n \times R^n$ with values in $\mathcal{L}(H_1,H_2)$ such that for all $K \subset\subset R^n$ and multiindices α,β there exists a constant $C = C(\alpha,\beta, K)$ such that

$$\| D_x^\alpha D_\xi^\beta p(x,\xi)\|_{\mathcal{L}(H_1,H_2)} \leq C(1 + |\xi|)^{m-|\beta|}$$

for all $(x,\xi) \in K \times R^n$. With such symbols we define $L^m(R^n; H_1,H_2)$ to be the space of pseudodifferential operators $P(x,D_x): C_0^\infty(R^n; H_1) \longrightarrow C^\infty(R^n; H_2)$.

We shall consider the case that H_1 or H_2 is equal to the space $D_\xi^k(R)$ with $k \in Z^+$, $\xi \in R^{n-1}$, which is a subspace of $H^1(R)$, given by the norm:

$$\|u\|^2_{D_\xi^k} = (1 + |\xi|)^{\frac{2}{1+k}} \int_{-\infty}^{\infty} |u(t)|^2 dt \ +$$

(continued)

$$(1+|\xi|)^2 \int_{-\infty}^{\infty} t^{2k} |u(t)|^2 dt + \int_{-\infty}^{\infty} |D_t u(t)|^2 dt.$$

(cf. [3], §4.)

In this case the norm $\| \ \|_{\mathcal{L}(H_1,H_2)}$ <u>depends on</u> $\xi \in R^{n-1}$, but all the calculus for scalar operators (cf. Hörmander [1]) extends to the vector valued case, in particular we have the usual composition formula and the results about H_s-continuity because we have the inequality:

$$\|u\|_{D^k} \le \|u\|_{D^k_\xi} \le (1+|\xi|) \|u\|_{D^k}$$

and hence

$$L^m(R^{n-1};\ H_1, D^k_\xi(R)) \subset L^m(R^{n-1};\ H_1, D^k(R)),$$

$$L^m(R^{n-1};\ D^k_\xi(R), H_2) \subset L^{m+1}(R^{n-1};\ D^k(R), H_2),$$

where $D^k(R)$ is the space $D^k_\xi(R)$ with $\xi = 0$.

We define $T^m(R^n)$ to be the space of "pseudodifferential operators" $a(x,t,D_x): C_0^\infty(R^n) \longrightarrow C^\infty(R^n)$ where $a(x,t,\xi) \in S^m(R^n \times R^{n-1})$ (see [2], Appendix).

Under the assumptions (A), (B), we can reduce the proof of Theorem to the proof of

Proposition. (cf. [2], Proposition 3.6.) <u>Let</u> $L(x,t,D_x,D_t) = D_t - it^k r(x,t,D_x) + s(x,t,D_x)$, <u>where</u> $r(x,t,D_x) \in T^1(R^n)$ (<u>resp</u>. $s(x,t,D_x) \in T^0(R^n)$) <u>is properly supported and its symbol</u> $r(x,t,\xi)$ <u>is positively homogeneous of degree</u> 1 <u>and</u> $r(x,t,\xi)$ (<u>resp</u>.

s(x,t,ξ)) _is equal to_ r(x,0,ξ) (_resp._ s(x,0,ξ)) _when_ $|t| \geq C$ _for some constant_ $C > 0$ _and_ $\mathrm{Re}\, r(x,t,\xi) \neq 0$ _for_ $(x,t)\in R^n$ _and_ $\xi \neq 0$.

(i) _If_ $\mathrm{Re}\, r(x,t,\xi) > 0$ _for_ $(x,t)\in R^n$ _and_ $\xi \neq 0$, _then there exist properly supported operators_

$$\mathcal{L}_1(x,D_x) = \begin{pmatrix} L(x,t,D_x,D_t) \\ \\ R^+(x,D_x) \end{pmatrix} \in L^0(R^{n-1};\ D^k_\xi(R), L^2(R)\oplus C),$$

$$\mathcal{E}_1(x,D_x) = (E_1(x,D_x),\ E^+(x,D_x)) \in L^0(R^{n-1};\ L^2(R)\oplus C, D^k_\xi(R))$$

such that

$$\mathcal{L}_1(x,D_x)\cdot\mathcal{E}_1(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ L^2(R)\oplus C, L^2(R)\oplus C),$$

$$\mathcal{E}_1(x,D_x)\cdot\mathcal{L}_1(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ D^k_\xi(R), D^k_\xi(R)).$$

(ii) _If_ $\mathrm{Re}\, r(x,t,\xi) < 0$ _for_ $(x,t)\in R^n$ _and_ $\xi \neq 0$, _then there exist properly supported operators_

$$\mathcal{L}_2(x,D_x) = (L(x,t,D_x,D_t),\ R^-(x,D_x)) \in L^0(R^{n-1};\ D^k_\xi(R)\oplus C, L^2(R)),$$

$$\mathcal{E}_2(x,D_x) = \begin{pmatrix} E_2(x,D_x) \\ \\ E^-(x,D_x) \end{pmatrix} \in L^0(R^{n-1};\ L^2(R), D^k_\xi(R)\oplus C)$$

such that

$$\mathcal{L}_2(x,D_x)\cdot\mathcal{E}_2(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ L^2(R), L^2(R)),$$

$$\mathcal{E}_2(x,D_x)\cdot\mathcal{L}_2(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ D^k_\xi(R)\oplus C, D^k_\xi(R)\oplus C).$$

§3. Sketch of the proof of Proposition.

Lemma 1. Let $L(x,\xi) = L_o(x,\xi) + L_1(x,\xi)$, where $L_o(x,\xi) = D_t - it^k\, r(x,t,\xi)$, $L_1(x,\xi) = s(x,t,\xi)$. Then we have

$$L_o(x,\xi) \in S^o(R^{n-1}\times R^{n-1};\ D^k_\xi(R), L^2(R)),$$

$$L_1(x,\xi) \in S^{-\frac{1}{1+k}}(R^{n-1}\times R^{n-1};\ D^k_\xi(R), L^2(R)).$$

In particular, we can regard $L(x,t,D_x,D_t)$ as an element of $L^o(R^{n-1};\ D^k_\xi(R), L^2(R))$.

The next lemma is the essential step in our proof of Proposition.

Lemma 2. Let $B(x,t,s,\xi) = -\int_s^t \theta^k\, r(x,\theta,\xi)d\theta.$

(i) When $\mathrm{Re}\ r(x,t,\xi) > 0$ for $(x,t)\in R^n$, $\xi \neq 0$, we define the kernel $K_1(x,t,s,\xi)$ by

$$K_1(x,t,s,\xi) = \begin{cases} i\ \exp\left[B(x,t,s,\xi)\right] & 0 \leq s \leq t, \\ -i\ \exp\left[B(x,t,s,\xi)\right] & t \leq s \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) When $\mathrm{Re}\ r(x,t,\xi) < 0$ for $(x,t)\in R^n$, $\xi \neq 0$, we define the kernel $K_2(x,t,s,\xi)$ by

$$K_2(x,t,s,\xi) = \begin{cases} -i\ \exp\left[B(x,t,s,\xi)\right] & 0 \leq t \leq s, \\ i\ \exp\left[B(x,t,s,\xi)\right] & s \leq t \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $K_j(x,t,s,\xi)$ $(j=1,2)$ we have the following estimates:

$$(1)\ \sup_t \int_{-\infty}^{\infty} |K_j(x,t,s,\xi)|\,ds = O(|\xi|^{-\frac{1}{1+k}}),\quad \xi \longrightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |K_j(x,t,s,\xi)|\,dt = O(|\xi|^{-\frac{1}{1+k}}),\quad \xi \longrightarrow \infty,$$

uniformly when x belongs to any compact subset of R^{n-1}.

$$(2)\ \sup_t \int_{-\infty}^{\infty} |t|^k\, |K_j(x,t,s,\xi)|\,ds = O(|\xi|^{-1}),\quad \xi \longrightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |t|^k\, |K_j(x,t,s,\xi)|\,dt = O(|\xi|^{-1}),\quad \xi \longrightarrow \infty,$$

uniformly when x belongs to any compact subset of R^{n-1}.

Lemma 2 follows from the following two facts (cf. Treves [4], Lemma C.1):

(a) There exists a constant $C_1 > 0$ such that

$$|t - s|^{k+1} \leq C_1\, |t^{k+1} - s^{k+1}|$$

for all $t\cdot s \geq 0$ when $k \in Z^+$ is odd.

(b) There exists a constant $C_2 > 0$ such that

$$|t|^k\, |t - s| \leq C_2\, |t^{k+1} - s^{k+1}|$$

for all $t\cdot s \geq 0$ when $k \in Z^+$ is odd.

Combining Corollary in [4], p. 94 and Lemma 2, we can prove

Lemma 3. Let $\operatorname{Re} r(x,t,\xi) > 0$. We define for $|\xi| \geq 1$

$$R^+(x,\xi) : D^k_\xi(R) \longrightarrow C,$$

$$E_0^+(x,\xi):C \longrightarrow D_\xi^k(R),$$

$$E_{1o}(x,\xi):L^2(R) \longrightarrow D_\xi^k(R),$$

by

$$R^+(x,\xi)u = |\xi|^{\frac{1}{1+k}} \int_{-\infty}^{\infty} u(t)\,\overline{\varphi(x,\xi,t)}\,dt,$$

$$E_0^+(x,\xi)z = |\xi|^{-\frac{1}{1+k}} \varphi(x,\xi,t)z,$$

$$E_{1o}(x,\xi)f = \int_{-\infty}^{\infty} K_1(x,t,s,\xi)f(s)ds - E_0^+R^+K_1f,$$

respectively, where

$$\varphi(x,\xi,t) = \exp\left[-\int_0^t \theta^k r(x,\theta,\xi)d\theta\right] \Big/ \left(\int_{-\infty}^{\infty} \exp\left[-2\int_0^t \theta^k \mathrm{Re}\ r(x,\theta,\xi)d\theta\right] dt\right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ,

$$\mathcal{L}_{1o}(x,\xi) = \begin{pmatrix} L_o(x,\xi) \\ \\ R^+(x,\xi) \end{pmatrix} \in S^o(R^{n-1}\times R^{n-1};\ D_\xi^k(R), L^2(R)\oplus C),$$

$$\mathcal{E}_{1o}(x,\xi) = (E_{1o}(x,\xi),\ E_0^+(x,\xi)) \in S^o(R^{n-1}\times R^{n-1};\ L^2(R)\oplus C, D_\xi^k(R))$$

and $\mathcal{E}_{1o}(x,\xi)$ is the inverse of $\mathcal{L}_{1o}(x,\xi)$ for $|\xi|\geq 1$.

Lemma 4. Let $\mathrm{Re}\ r(x,t,\xi) < 0$. We define for $|\xi|\geq 1$

$$R^-(x,\xi):C \longrightarrow L^2(R),$$

$$E_0^-(x,\xi):L^2(R) \longrightarrow C,$$

$$E_{2o}(x,\xi):L^2(R) \longrightarrow D^k_\xi(R),$$

by

$$R^-(x,\xi)z = \psi(x,\xi,t)z,$$

$$E_o^-(x,\xi)f = \int_{-\infty}^{\infty} f(t)\,\overline{\psi(x,\xi,t)}\,dt,$$

$$E_{2o}(x,\xi)f = \int_{-\infty}^{\infty} K_2(x,t,s,\xi)\,(f(s) - R^-E_o^-f(s))ds,$$

respectively, where

$$\psi(x,\xi,t) = \exp\left[\int_o^t \theta^k\overline{r(x,\theta,\xi)}\,d\theta\right] \Big/ \left(\int_{-\infty}^{\infty} \exp\left[2\int_o^t \theta^k \mathrm{Re}\ \overline{r(x,\theta,\xi)}\,d\theta\right] dt\right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ,

$$\mathcal{L}_{2o}(x,\xi) = (L_o(x,\xi),\ R^-(x,\xi)) \in S^o(R^{n-1}\times R^{n-1};\ D^k_\xi(R)\oplus C, L^2(R)),$$

$$\mathcal{E}_{2o}(x,\xi) = \begin{pmatrix} E_{2o}(x,\xi) \\ E_o^-(x,\xi) \end{pmatrix} \in S^o(R^{n-1}\times R^{n-1};\ L^2(R), D^k_\xi(R)\oplus C)$$

and $\mathcal{E}_{2o}(x,\xi)$ is the inverse of $\mathcal{L}_{2o}(x,\xi)$ for $|\xi|\geq 1$.

By Lemma 3 and Lemma 4 the construction of $\mathcal{E}_j(x,D_x)$ $(j=1,2)$ in Proposition is formally the same as the construction of a parametrix of an elliptic operator in the scalar case.

References

[1] Hörmander, L.: Pseudodifferential operators and hypoelliptic equations, Amer. Math. Soc. Symp. on Singular Integral Operators, 138-183 (1966).

[2] Sjöstrand, J.: Operators of principal type with interior boundary conditions, Acta Math., 130, 1-51 (1973).

[3] Sjöstrand, J.: Parametrices for pseudodifferential operators with multiple characteristics (to appear).

[4] Treves, F.: A new method of proof of the subelliptic estimates, Comm. Pure Appl. Math., 24, 71-115 (1971).