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Vector-valued Hyperfunctions

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§1. Introduction and preliminaries.

It is the purpose of this talk to outline some elementary developments toward a theory of vector-valued hyperfunctions. The motives behind these are two-fold. In the first place vector-valued hyperfunctions are necessary in applications; S. Ouchi has already used hyperfunctions of one variable taking values in a Banach space [17], and one of the authors (P.D.F. I.) is trying to develop a hyperfunctional quantum field theory which requires many variable hyperfunctions. Secondly, to look at hyperfunctions with values in locally convex spaces of some sort is every bit as natural as it is for distributions. For distributions L. Schwartz himself built up an extensive vector-valued theory [20,21,22]. Here we will report on the results we have obtained in the particularly simple case when the space of values is a Fréchet space (Ion and Kawai [25]) and give notice of some further developments.

The Fréchet values case is simple because one may deduce

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many of the facts required directly from the ordinary scalar hyperfunction theory using some general nonsense.

To start with we must have holomorphic functions. Here we will consider for simplicity's sake, only the case $M = \mathbb{R}^n$ imbedded in its complexification $X = \mathbb{C}^n$; the case of a real analytic manifold M in a complexification X presents no more difficulties here than usually. Let E be, for the moment, a quasi-complete separated locally convex topological vector space. Denote by ${}^E\mathcal{E}$ the sheaf of germs of smooth functions on X with values in E, and by ${}^E\mathcal{O}$ the sheaf of germs of E-valued functions holomorphic on X; ${}^E\mathcal{O}$ is a subsheaf of ${}^E\mathcal{E}$ [20]. A property basic for cohomology theory is the following.

<u>Theorem</u> 1.1.

The sheaf ${}^E\mathcal{E}$ is soft.

<u>Proof</u>. This is obvious since ${}^E\mathcal{E}$ is fine for the same reason $\mathcal{E}$ is [9].

Next remark that we have the soft Dolbeault resolution of ${}^E\mathcal{O}$; here the Dolbeault complex is defined as usual.

<u>Theorem</u> 1.2.

The complex $({}^E\mathcal{E}^{(0,\cdot)}, \bar{\partial})$ is exact.

<u>Proof</u>. The proof of this goes exactly like that for the scalar case $E = \mathbb{C}$; since E is quasi-complete so that the closed convex hull of a compact set in E is also compact, one may use Cauchy's integral formula as usual (cf. Grothendieck [6]) to give a Dolbeault-Grothendieck Lemma [9,12].

In fact the vector-valued generalization of Palamodov's splitting theorem holds.

Theorem 1.3.

On a Stein manifold X the vector-valued Dolbeault complex $E\mathcal{E}^{(0,p)}(X)$ splits for $p > 0$, so for $p \geq 2$ $H^p(X, {}^E\mathcal{O}) = 0$.

Proof. Follows directly from Palamodov [18], Prop.5.1. using $E\mathcal{E}^{(0,p)}(X) \cong \mathcal{E}^{(0,p)}(X) \hat{\otimes} E$.

Next we list three key theorems of Grothendieck.

Theorem 1.4.

Let $u_i : E_i \longrightarrow F_i$ be a topological homomorphism of locally convex spaces such that $u_i E_i$ is dense in F_i, for $i=1,2$. Then $u_1 \hat{\otimes} u_2 : E_1 \hat{\otimes} E_2 \longrightarrow F_1 \hat{\otimes} F_2$ is a topological homomorphism of the projectively completed tensor product $E_1 \hat{\otimes} E_2$ onto a dense subspace of $F_1 \hat{\otimes} F_2$. Further if E_1 and E_2 are metrisable then $u_1 \hat{\otimes} u_2$ is actually a surjection.

Proof. This is Grothendieck [8] Chap I, §1, no.2, Prop.3; the last assertion is an immediate consequence of the Banach Open Mapping Theorem.

Theorem 1.5.

Let E_i be a locally convex space, and let F_i be a vector subspace of E_i, for $i=1,2$. If either F_1 or F_2 is a nuclear space then the natural linear map of $F_1 \hat{\otimes} F_2$ into $E_1 \hat{\otimes} E_2$ is a monomorphism of topological vector spaces.

Proof. This is Grothendieck [8] Chap.II, §3, no.1, Prop.10, Cor.

These two theorems may be applied to yield the following, which is stated in categorical language with an eye to its applications.

Theorem 1.6.

Consider the category of Fréchet nuclear spaces where the morphisms are continuous linear maps, and also the category of projectively completed tensor products of Fréchet spaces with metrisable locally convex spaces, where the morphisms are tensor products of a continuous linear map on the Fréchet factor with the identity on the second factor. Let F be a metrisable locally convex space. Then $E \to E \hat{\otimes} F$, and for morphisms $u \to u \hat{\otimes} 1$, defines a covariant functor from the first mentioned category to the second. This functor is exact.

Proof. This theorem is essentially in Grothendieck [7], or again is a variant of Bungart [2] Thm. 5.3.

§2. Pure codimensionality and the hyperfunction sheaf ${}^{E}\mathcal{B}$

It is of course as a result of the pure codimensionality of $\mathbb{R}^n$ with respect to the sheaf ${}^{E}\mathcal{O}$ that one may define vector-valued hyperfunctions [19]. To get this as a deduction from the scalar case using the tensor product properties just mentioned, we shall have to assume E is Fréchet. In order to use Thm. 1.4 and thus Thm. 1.6, E has to be metrisable, and in order to be a good space of values for holomorphic functions E must be quasi-complete. These two assumptions together imply E is Fréchet, [23] Prop. 34.3. First we need two generalizations of results fundamental to the Harvey-Komatsu development of Sato's hyperfunction theory [10,11,13,14].

Theorem 2.1. (Oka-Cartan Theorem B)

If Ω is a Stein submanifold of $\mathbb{C}^n$ and E is a Fréchet space, then, for $p \geqq 1$, $H^p(\Omega, {}^{E}\mathcal{O}) = 0$.

Proof. We can prove this as in the scalar case by use of the Dolbeault resolution; it is a special case of Bishop [1] Thm. 4 or Bungart [2] Thm. B. The authors are grateful to Sunao Ouchi who pointed out a paper of H.Fujimoto in which references to these last papers were found.

Theorem 2.2. (Malgrange)

Let V be an open set in $\mathbb{C}^n$ and E a Fréchet space, then, for $p \geqq n$, $H^p(V, {}^{E}\mathcal{O}) = 0$.

Proof. This follows form Malgrange's scalar theorem [15] and Thm. 1.6 applied to the Dolbeault complex, which by Thm. 1.1 may be used to calculate the cohomology.

This leads us to :

Theorem 2.3.

R^n is purely n-codimensional with respect to the sheaf ${}^E\mathcal{O}$ over $\mathbb{C}^n$.

Proof. As in the scalar case, by the excision theorem and Grauert's Neighbourhood Theorem [5, 24], it is enough to show, $H^p_{R^n \cap V}(V, {}^E\mathcal{O}) = 0$ for $p \neq n$ and V a bounded and Stein open set. Using then the long exact sequence of relative cohomology together with Thm. 2.1 and 2.2, and noting that, since the E-Dolbeault resolution may be used to calculate the cohomology, one may get information from the scalar result by tensoring by $\hat{\otimes} E$, the theorem will follow.

So if E is a Frechet space, the space of E-valued hyperfunctions on an open set Ω in R^n is defined to be

$$ {}^E\mathcal{B}(\Omega) \equiv H^n_\Omega(V, {}^E\mathcal{O}) $$

where V is an open set in C^n containing Ω as a closed set. By the excision theorem this definition does not depend on what particular complex neighbourhood V of Ω is chosen. We can also prove the analogue of Sato's theorem.

Theorem 2.4.

The assignments $\Omega \to {}^E\mathcal{B}(\Omega)$ for Ω open in R^n and the natural restriction maps define a sheaf. This sheaf, ${}^E\mathcal{B}$, is flabby.

Proof. This is an immediate consequence as suggested by Sato [19] of Thm. 2.4 and the facts of general sheaf theory given in Komatsu [13] Thm. 1.8.

§3. Hyperfunctions as boundary values. Embedding of distributions.

Because of the analogue of the Oka-Cartan theorem B, E-valued hyperfunctions may be viewed as boundary values of holomorphic functions in the usual way. In addition from this follows one way of seeing that the distributions may be embedded in the hyperfunctions.

Theorem 3.1.

The distributions ${}^{E}\mathcal{D}'(R^n)$ may be embedded in the hyperfunctions ${}^{E}\mathcal{B}(R^n)$.

Proof. This follows from the representation of hyperfunctions as the sum of 'boundary values' of functions holomorphic in the tubes defined by the 2^n orthants, and the theorem of Ehrenpreis [3,4] which implies every distribution has such a representation. In fact the analogue of this result holds for the vector-valued version of any analytically uniform space verifying the conditions of Ehrenpreis' theorem.

§4. Holomorphic hyperfunctions.

As will by now be obvious most of the Harvey-Komatsu development of the elements of Sato's theory of hyperfunctions generalizes simply to the case of values in a Fréchet space E. For instance:

Theorem 4.1.

Let $R_j = \mathbb{C}^{j-1} \times \mathbb{R} \times \mathbb{C}^{n-j}$ for $j=1,\cdots,n$ and let $R(\emptyset) = \mathbb{C}^n$, and for $\emptyset \neq J \subset \{1,\cdots,n\}$ define $R(J) = \bigcap\{R_j; j \in J\}$. Then if r is the cardinality of J, $R(J)$ is purely r-codimensional with respect to the sheaf $^E\mathcal{O}$ on $\mathbb{C}^n$. Further, if one writes ${}^E\mathcal{B}J\mathcal{O} \equiv \mathcal{H}^r_{R(J)}({}^E\mathcal{O})$ then for $p > 0$ and V a Stein open set in $\mathbb{C}^n$, $H^p(V, {}^E\mathcal{B}J\mathcal{O}) = 0$.

Proof. This can be proven by mimicking the scalar proof of Komatsu [14], just adding presuperscript E where necessary.

This theorem has as a special case Thm.2.3 ($n = r$) and the proof just indicated is in a sense more elementary than that mentioned before.

Heuristically ${}^E\mathcal{B}J\mathcal{O}$ is the sheaf of germs of functions which are hyperfunctional in the J variables and holomorphic in the rest ; it might be written as $\mathcal{O}(C^{n-r})\,\mathcal{B}(R^r)$.

The proof of Martineau's theorem on the independence of boundary values from the coordinate system which is given by Komatsu [14] does not simply generalize, for the complexes involved, which come from the canonical flabby resolution of some sheaves, do not consist of Fréchet spaces; the statement however does.

§5. Microfunctions. Related matters.

The elementary theory of microfunctions generalizes in the natural manner too [27,28]. In the first place all the geometry involved is unchanged and aside from that the elementary theory only involves the basic facts about hyperfunctions and general results in sheaf theory. As far as the flabbiness of ${}^{E}\mathcal{C}$ is concerned it seems that a proof of this may be constructed along the lines of Kashiwara's original proof which is sketched in [26]. The use of pseudodifferential operators presents some difficulties [28].

So a great deal generalizes easily if E is a Frechet space. We hope to extend the theory to more general quasi-complete E. Another question that is natural is how far one may continue with vector-valued hyperfunctions on totally real subsets in the manner of Harvey and Wells [11,24]. The above topics will be treated in our next paper.

A related matter is the theory of vectorisations of coherent analytic sheaves $\mathfrak{F}$. This was in fact the concern of both Bishop [1] and Bungart [2] who showed that almost all the standard results extended to the case of ${}^{E}\mathfrak{F} = E\hat{\otimes}\mathfrak{F}$ (avec abus de notation). We can add that this is also true of Siu's completion of Malgrange's Vanishing Theorem [28].

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