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Sequences of Stochastic Automata*

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1. Introduction.

Information processing of living brain develops with growth of the life and in another view point of species, evolves into advanced functions with succession of generations. This time depending features of living brain may be introduced to the management systems of scientific information. In accordance with the progress of science, the management systems are improved with new scientific information and their old contents have to be adjusted in connection with the new data. The management systems develop in views of their functions and amount of information.

In this paper we consider some sequences of stochastic automata which are models of the information management systems or living brain. Internal states of the automata correspond to contents of the system and transitions between their states are mutual relationships of the contents. Moreover the number of the states increases with increasing of information or documents.

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2. Stochastic automata with time parameter.

We define stochastic automaton with time parameter which assumes discrete values $t=1,2,\cdots$

Definition 1.

Stochastic automaton with parameter t *is 4-tuple*

$$U_t = (\pi_t, S_t, \{A_t(\sigma), \sigma \in \Sigma\}, \eta^{F_t})$$

where

π_t : *row probability vector of initial states at time* t,

S_t : *finite set of states at time* t, $\{s_i, i=1,2,\cdots, |S_t|,\}$

$\{A_t(\sigma), \sigma \in \Sigma\}$: *set of transition probability matrices, when input symbol is* $\sigma \in \Sigma$ *at time* t, *where* Σ *is finite set of input symbols independent on* t,

η^{F_t} : *column vector whose elements* η_i, $i=1,2,\cdots, |S_t|$, *are given by*

$$\eta_i = 1, \quad s_i \in F_t$$
$$= 0, \quad s_i \notin F_t$$

where $F_t \subset S_t$ *is final states set.*

In this paper we use following notations.

1. If $u=\sigma_1\cdots\sigma_k$, $\sigma_i \in \Sigma$, $i=1,2,\cdots, k$, then $A_t(u)=A_t(\sigma_1)\cdots A_t(\sigma_k)$.
2. If $u \in \Sigma^*$ then $p_t(u)=\pi_t A_t(u)\eta^{F_t}$.

3. Language accepted by U_t with cut point λ is given by

$$T(U_t, \lambda) = \{u \; ; \; p_t(u) > \lambda, \; u \in \Sigma^*\}$$

where $0 < \lambda \leq 1$.

Distance between two stochastic automata U_i and U_j, $dis(U_i, U_j)$, is defined.

<u>*Difinition* 2.</u>

$$dis(U_i, U_j) = \sup_{u \in \Sigma^*} |p_i(u) - p_j(u)|$$

where two sets of input alphabet are same for U_i *and* U_j.

Now we introduce stationarity of the sequence, $\{U_t, t=1, 2, \cdots\}$.

<u>*Definition* 3.</u>

Sequence of stochastic automata $\{U_t\}$ *is said power sequence if*

$$\pi_t = \pi, \quad S_t = S, \quad F_t = F$$

and

$$A_t(\sigma) = A^t(\sigma), \quad \sigma \in \Sigma.$$

Language accepted by limit stochastic automaton is given by following theorem.

Theorem 1.

If transition matrices of power sequence of stochastic automata $\{U_t\}$ *are all ergodic then there exists the limit* *stochastic automaton*

$$\lim_{t\to\infty} U_t = U$$

and for each word $u\in\Sigma^*$, *only the last symbol of* u *decides whether* u *belongs to* $T(u, \lambda)$ *or not.*

Proof.

From the ergodicity of $A(\sigma)$, $\sigma\in\Sigma$ the limit matrices $\lim_{t\to\infty} A^t(\sigma)=A(\sigma)$ exist, and $A(\sigma)$, $\sigma\in\Sigma$ are stochastic matrices hence limit stochastic automaton U is given by

$$U = (\pi, S, \{A(\sigma), \sigma\in\Sigma\}, \eta^F).$$

Since row vectors of $A(\sigma)$ are all equal, say $a(\sigma)$, for any $u=\sigma_1\cdots\sigma_k\in\Sigma^*$,

$$p(u) = \pi A(\sigma_1)\cdots A(\sigma_k)\eta^F = a(\sigma_k)\eta^F$$

This imples that $T(u, \lambda)\ni u$ depends upon the last symbol σ_k only.

Theorem 2.

If transition matrices $\{A(\sigma), \sigma\in\Sigma\}$ *of stochastic automaton*

$$U_t = (\pi, S, \{A^t(\sigma), \sigma\in\Sigma\}, \eta^F)$$

are all periodic and period of $A(\sigma_i)$ *is* r_i, $i=1,\cdots,k$ *for* $u=\sigma_1\cdots\sigma_k$, *then* $p_t(u)$ *is periodic function of* t *and its period* *is the least common multiple* (L. C. M.) *of* r_i, $i=1,\cdots,k$.

Proof.

This is shown from

$$p_t(u) = \pi A^t(\sigma_1)\cdots A^t(\sigma_k)\eta^F.$$

Now we define (ε, δ) direct sum of two Stochastic matrices by generalization of direct sum of two matrices.

Definition 4.

Let $A^{(i)}$, $i=1,2$, *are two stochastic matrices with order* $n\times n$. (ε, δ) *direct sum of* $A^{(i)}$, $i=1,2$, *is defined as*

$$A^{(1)} \underset{(\varepsilon,\ \delta)}{\dot{+}} A^{(2)} = \begin{bmatrix} A^{(1)}-\varepsilon, & \varepsilon \\ \delta\quad , & A^{(2)}-\delta \end{bmatrix}$$

where ε *and* δ *are two square matrices with order* $n\times n$ *and are such that* $A^{(1)} \underset{(\varepsilon,\delta)}{\dot{+}} A^{(2)}$ *is again stochastic matrices.*

If $\varepsilon=\delta=0$ *then* $A^{(1)} \underset{(\varepsilon,\delta)}{\dot{+}} A^{(2)}$ *is direct sum of* $A^{(1)}$ *and* $A^{(2)}$ *and we use the notation* $A^{(1)}\dot{+}A^{(2)}$.

Let α and β be positive real number such that $\alpha+\beta=1$. Using (ε, δ) direct sum we introduce the $(\varepsilon, \delta, \alpha, \beta)$ sum of two stochastic automata.

Definition 5.

$\mathfrak{V}_i=(\pi_i, S_i, \{A_i(\sigma), \sigma\in\Sigma\}, \eta^{F_i})$, $i=1,2$, *be two sto-*

chastic automata with $|S_1|=|S_2|$ *and* $S_1 \cap S_2=\emptyset$. $(\varepsilon, \delta, \alpha, \beta)$ *sum* $U_1 \underset{(\varepsilon,\delta,\alpha,\beta)}{+} U_2$ *of* U_1 *are* U_2, *is defined by stochastic automata* $(\pi, S, \{A(\sigma), \sigma \in \Sigma\}, \eta^F)$,

where

$$\pi = (\alpha\pi_1, \beta\pi_2)$$

$$S = S_1 \cup S_2$$

$$A(\sigma) = A_1(\sigma) \underset{(\varepsilon,\delta)}{\dot{+}} A_2(\sigma)$$

$$F_1 = F_1 \cup F_2$$

In the case $\varepsilon = \delta = 0$, we obtain the following theorem.

Theorem 3.

If

$$T(U_1, \lambda) \supset T(U_2, \lambda)$$

then there exists α, β *such that*

$$T(U, \lambda) \supset T(U_2, \lambda)$$

where

$$U = U_1 \underset{(0,0,\alpha,\beta)}{+} U_2$$

Proof.

After $T(U, \lambda) \supseteq T(U_2, \lambda)$ is proved, we shall show the existence of $u \in T(U, \lambda) - T(U_2, \lambda)$.

For any positive real number $\alpha, \beta, \alpha+\beta=1$, if

$$u \in T(U_2, \lambda) \subset T(U_1, \lambda)$$

then

$$p(u) = \pi A(u)\eta^F = (\alpha\pi_1,\ \beta\pi_2)\begin{bmatrix} A_1(u) & 0 \\ 0 & A_2(u)\end{bmatrix}\begin{bmatrix}\eta^{F_1} \\ \eta^{F_2}\end{bmatrix}$$

$$= \alpha\pi_1 A_1(u)\eta^{F_1}+\beta\pi_2 A_2(u)\eta^{F_2} > \alpha\lambda+\beta\lambda=\lambda.$$

So that we obtain,

$$u \in T(U,\ \lambda).$$

From the assumption there exists a word u such that $u \in T(U_2,\ \lambda)$, $u \notin T(U_1,\ \lambda)$.

We have

$$p_2(u) > \lambda.$$

We can take λ^1, $p_2(u) > \lambda^1 > \lambda$,

Now we put $\alpha=1-\frac{\lambda}{\lambda^1}$, $\beta=\frac{\lambda}{\lambda^1}$.

Then

$$p(u) = \frac{\lambda-\lambda}{\lambda^1}\pi_1 A_1(u)\eta^{F_1}+\frac{\lambda}{\lambda^1}\pi_2 A_2(u)\eta^{F_2}$$

$$> \frac{\lambda-\lambda}{\lambda^1}\pi_1 A_1(u)\eta^{F_1}+\lambda > \lambda,$$

and we have

$$u \in T(u,\ \lambda)$$

which was proved.

This theorem gives a method which enable us to make a expansion U of stochastic automata U_2.

Theorem 4.

Let $U^{(1)}$ *and* $U^{(2)}$ *be stochastic automata with* $S^{(1)} \cap S^{(2)} = \emptyset$.

We put

$$U = U^{(1)} \underset{(\alpha,\beta,\varepsilon,\delta)}{+} U^{(2)}$$

and stochastic automaton U_t *is constructed by*

$$A_t(\alpha) = A^t(\alpha), \quad \alpha \in \Sigma,$$

where $\{A(\sigma), \sigma \in \Sigma\}$ *is a set of transition probability matrices of* U.

If $A_1(\sigma)-\varepsilon(\sigma)$, $A_2(\sigma)-\delta(\sigma)$, $\sigma \in \Sigma$ *are all irreducible then for sufficiently large* t, Σ^* *is divided into following three classes,*

1. $p_t(u)$ *depends upon* σ_k *only,*
2. $p_t(u)$ *depends upon* σ_1 and σ_k, $1 \le i < k$,
3. $p_t(u)$ *is periodic function of* t,

where we put $u = \sigma_1 \cdots \sigma_k$.

Proof. We start with the case of non-peridic matrices $\{A(\sigma), \sigma \in \Sigma\}$. In this case, the matrix $A(\sigma)$ is written as

$$A(\sigma) = \begin{bmatrix} A_1(\sigma)-\varepsilon(\sigma) & \varepsilon(\sigma) \\ \delta(\sigma) & A_2(\sigma)-\delta(\sigma) \end{bmatrix}$$

We put $|S_1|=|S_2|=n$ and $\Theta^{(n,n)}(\alpha)$ be $n\times n$ matrix with equal rows $(\theta(\alpha), \cdots, \theta_n(\sigma))$. Limit matrix $\lim_{t\to\infty} A^t(\sigma)$ has four types corresponding to the forms of $\varepsilon(\sigma)$ and $\delta(\sigma)$.

Type a. $\varepsilon(\sigma)\neq 0$ and $\delta(\sigma)\neq 0$. In this case $A(\sigma)$ becomes irreducible and we have

$$\lim_{t\to\infty} A^t(\sigma) = \Theta^{(2n,2n)}(\sigma).$$

Type b. $\varepsilon(\sigma)=0$ and $\delta(\sigma)\neq 0$. In this case S_2 is transient and we have

$$\lim_{t\to\infty} A^t(\sigma) = \begin{bmatrix} \Theta^{(n,n)}(\sigma) & 0 \\ \Theta^{(n,n)}(\sigma) & 0 \end{bmatrix}.$$

Type c. $\varepsilon(\sigma)\neq 0$ and $\delta(\sigma)=0$. In this case S_1 is transient and we have

$$\lim_{t\to\infty} A^t(\sigma) = \begin{bmatrix} 0 & \Theta^{(n,n)}(\sigma) \\ 0 & \Theta^{(n,n)}(\sigma) \end{bmatrix}.$$

Type d. $\varepsilon(\sigma)=\sigma(\sigma)=0$. In this case S_1 and S_2 are irreducible respectively, and we have

$$\lim_{t\to\infty} A^t(\sigma) = \begin{bmatrix} \Theta_1^{(n,n)}(\sigma) & 0 \\ 0 & \Theta_2^{(n,n)}(\sigma) \end{bmatrix}.$$

Where $\theta_1^{(n,n)}(\sigma)$ and $\theta_2^{(n,n)}(\sigma)$ are equal row matrixes. Then a problem arises. What is the type of product matrix $A^{(\infty)}(\sigma_1)$ $A^{(\infty)}(\sigma_2)$? This is shown by the following table.

$A^{(\infty)}(\sigma_1)$ \ $A^{(\infty)}(\sigma_2)$	a	b	c	d
a	$A^{(\infty)}(\sigma_2)$	$A^{(\infty)}(\sigma_2)$	$A^{(\infty)}(\sigma_2)$	$A^{(\infty)}(\sigma_1)A^{(\infty)}(\sigma_2)$
b	"	"	"	"
c	"	"	"	"
d	"	"	"	$A^{(\infty)}(\sigma_2)$

From this table we can calculate,

$$p_\infty(u) = \pi A^{(\infty)}(\sigma_1)\cdots A^{(\infty)}(\sigma_k)\eta^F$$

where each $A^{(\infty)}(\sigma_i)$ is type a, b, c, or d.

(i) If $A^{(\infty)}(\sigma_k)$ is type a, b or c then $p_\infty(u)=\pi A^{(\infty)}(\sigma_k)\eta^F$

(ii) If $A^{(\infty)}(\sigma_i)$ $i=j,\cdots k$, is type d and $A^{(\infty)}(\sigma_{j-1})$ is type a, b or c then $p_\infty(u)=\pi A^{(\infty)}(\sigma_{j-1})A^{(\infty)}(\sigma_k)$.

(iii) If $A^{(\infty)}(\sigma_i)$, $i=1,\cdots,k$ are all type d then $p^{(\infty)}(u)=\pi A^{(\infty)}(\sigma_k)\eta^F$

Then we obtain results 1 and 2 of the theorem.

If some matrices $A(\sigma_{i_1}),\cdots, A(\sigma_{i_j})$ are periodic, then from Theorem 2, we obtain final result 3 of the Theorem.

In the case where a limit stochastic automata is added by some other stochastic automata, we consider power sequence of the resultant stochastic automata. Then, what is the language acepted by limit stochastic automata of the power sequence?

Theorem 5.

Let $U^{(i)}=(\pi^{(i)}, S^{(i)}, \{A^{(i)}(\sigma), \sigma\in\Sigma\}, \eta^{F^{(i)}})$, $i=1,2,$ *be two stochastic automata where* $\{A^{(1)}(\sigma), \sigma\in\Sigma\}$ *are stochastic matrices of type* a, b, c or d *in theorem* 4, $\{A^{(2)}(\sigma), \sigma\in\Sigma\}$ *are all ergodic and me assume* $|S^{(1)}|=|S^{(2)}|$, $S^{(1)}\cap S^{(2)}=\phi$. *Let* $U_t=(\pi, S, \{A^t(\sigma),\sigma\in\Sigma\}, \eta^F)$ *be power sequence of stochastic automa obtained from* $U^{(1)} \underset{(\alpha,\beta,\varepsilon,\delta)}{+} U^{(2)}$, *where we assume that* $A_1(\sigma)-\varepsilon(\sigma)$, $\sigma\in\Sigma$ *are irreduceble or type* d, $A_2(\sigma)-\delta(\sigma)$, $\sigma\in\Sigma$ *are irreducedle, and* $A(\sigma)$, $\sigma\in\Sigma$ *are not periodic.*

Then there exists $\lim_{t\to\infty} U_t=U_\infty$ *and* Σ^* *is divided into following five sub-classes, where* $\Sigma^*\ni u=\sigma_1\cdots\sigma_k$.

1. $p_\infty(u)$ *depends upon* σ_k *only.*
2. $p_\infty(u)$ *depends upon* σ_{j-1} *and* $\sigma_k, j-1<k$.
3. $p_\infty(u)$ *depends upon* π *and* σ_k.
4. $p_\infty(u)$ *depends upon* $\sigma_{i_1},\cdots, \sigma_{i_n}, \sigma_k$, $1\leq i_1<i_2<\cdots<i_n<k$.
5. $p_\infty(u)$ *depends upon* $u=\sigma_1\cdots\sigma_k$ *and* π.

Proof. It is sufficient to show the types of the limit matrices $\lim_{t\to\infty} A^t(\sigma)=A^{(\infty)}(\sigma)$, $\sigma\in\Sigma$ and of their products, where

$$A(\sigma) = \begin{bmatrix} A_1(\sigma)-\varepsilon(\sigma) & \varepsilon(\sigma) \\ \delta(\sigma) & A_2(\sigma)-\delta(\sigma) \end{bmatrix}$$

If $A_1(\sigma)-\varepsilon(\sigma)$ is irreducible then $A^{(\infty)}(\sigma)$ is type a, b, c or d and our results are obtained from Theorem 4. If $A_1(\sigma)-\varepsilon(\sigma)$ is reducible then we can write

$$A_1(\sigma)-\varepsilon(\sigma) = \begin{bmatrix} \theta_1(\sigma)-\varepsilon_1(\sigma) & 0 \\ 0 & \theta_2(\sigma)-\varepsilon_2(\sigma) \end{bmatrix}$$

and simply we assume the matrices $\theta_1(\sigma)$, $\theta_2(\sigma)$, and $A_2(\sigma)$ have same order. Then we have

$$A(\sigma) = \begin{bmatrix} \theta_1(\sigma)-\varepsilon_1(\sigma) & 0 & \varepsilon_1(\sigma) \\ 0 & \theta_2(\sigma)-\varepsilon_2(\sigma) & \varepsilon_2(\sigma) \\ \delta_1(\sigma) & \delta_2(\sigma) & A_2(\sigma)-\delta_1(\sigma)-\delta_2(\sigma) \end{bmatrix}$$

Corresponding to the forms of $\varepsilon_1(\sigma)$, $\varepsilon_2(\sigma)$, $\delta_1(\sigma)$ and $\delta_2(\sigma)$, there are five cases.

Case 1. $A^{(\infty)}(\sigma)$ is the matrix of same row vectors for seven pairs of $(\varepsilon_1(\sigma), \varepsilon_2(\sigma), \delta_1(\sigma), \delta_2(\sigma))$ given by Table 1, where $*$ and 0 denote non-zero and zero matrices respectively.

$\varepsilon_1(\sigma)$	$\varepsilon_2(\sigma)$	$\delta_1(\sigma)$	$\delta_2(\sigma)$
*	*	*	*
0	*	*	*
*	0	*	*
*	*	0	*
*	*	*	0
*	*	0	0
*	0	0	*

Table 1. Seven pairs of $(\varepsilon_1(\sigma), \varepsilon_2(\sigma), \delta_1(\sigma), \delta_2(\sigma))$ which give $A^{(\infty)}(\sigma)$ of same row vectors.

Case 2. $A^{(\infty)}(\sigma)$ has two kinds of row vectors for six pairs of $(\varepsilon_1(\sigma), \varepsilon_2(\sigma), \delta_1(\sigma), \delta_2(\sigma))$ given by Table 2.

$\varepsilon_1(\sigma)$	$\varepsilon_2(\sigma)$	$\delta_1(\sigma)$	$\delta_2(\sigma)$
0	0	0	*
0	*	0	*
0	*	0	0
*	0	0	0
*	0	*	0
0	0	*	0

Table 2. Six pairs of $(\varepsilon_1(\sigma), \varepsilon_2(\sigma), \delta_1(\sigma), \delta_2(\sigma))$ which give $A^{(\infty)}(\sigma)$ having two kinds of row vectors.

Matrices $A^t(\sigma)$ and $A^{(\infty)}(\sigma)$ in the six pairs are given by

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & \delta_2 & A_2-\delta_2 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & \theta_2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2-\varepsilon_2 & \varepsilon_2 \\ 0 & \delta_2 & A_2-\delta_2 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2' & \theta_3 \\ 0 & \theta_2' & \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_1-\varepsilon_2 & \varepsilon_2 \\ 0 & 0 & A_2 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & 0 & \theta_3 \\ 0 & 0 & \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1-\varepsilon_1 & 0 & \varepsilon_1 \\ 0 & \theta_2 & 0 \\ 0 & 0 & A_2 \end{bmatrix}^t \longrightarrow \begin{bmatrix} 0 & 0 & \theta_3 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1-\varepsilon_1 & 0 & \varepsilon_1 \\ 0 & \theta_2 & 0 \\ \delta_1 & 0 & A_2-\delta_1 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1' & 0 & \theta_3 \\ 0 & \theta_2 & 0 \\ \theta_1' & 0 & \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ \delta_1 & 0 & A_2-\delta_1 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ \theta_1 & 0 & 0 \end{bmatrix}$$

where $\theta_1, \theta_1', \theta_2, \theta_2'$ and θ_3 dentote submatrices of same row vectors.

Products $A^{(\infty)}(\sigma_1)A^{(\infty)}(\sigma_2)$ of any two limit matrices $A^{(\infty)}(\sigma_1)$ and $A^{(\infty)}(\sigma_2)$ in the first three pairs and in the next three pairs of Table 2, are reduced to $A^{(\infty)}(\sigma_2)$ respectively.

Case 3. This is reducible case where $\varepsilon_1(\sigma)=\varepsilon_2(\sigma)=\delta_1(\sigma)=\delta_2(\sigma)=0$ and matrices $A^t(\sigma)$ and $A^{(\infty)}(\sigma)$ are given by

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}$$

Product $A^{(\infty)}(\sigma_1)A^{(\infty)}(\sigma_2)$ of any two limit matrices in the case is also reduced to $A^{(\infty)}(\sigma_2)$.

Case 4. This is the case where subset of states is transient. There are two pairs, one is $\varepsilon_1(\sigma)=\varepsilon_2(\sigma)=0$, $\delta_1(\sigma)>0$, $\delta_2(\sigma)>0$, that is, states of $A_2(\sigma)$ are transient and the other is $\varepsilon_1(\sigma)=\delta_2(\sigma)=0$, $\varepsilon_2(\sigma)>0$, $\delta_1(\sigma)>0$ that is, states of $\theta_2(\sigma)$ and $A_2(\sigma)$ are transient. $A^t(\sigma)$ and $A^{(\infty)}(\sigma)$ in the two pairs are given by

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ \delta_1 & \delta_2 & A_2-\delta_1-\delta_2 \end{bmatrix}^t \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ \delta_1' & \delta_2' & 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2-\varepsilon_2 & \varepsilon_2 \\ \delta_1 & 0 & A_2-\delta_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \theta_1 & 0 & 0 \\ \theta_2' & 0 & 0 \\ \delta_1' & 0 & 0 \end{bmatrix}$$

where δ_1' and δ_2' are sub-matrices of not necessarily same row vectors. In general, product of any two limit matrices in the two pairs has different row vectors.

We had limit matrices of all sixteen pairs $(\varepsilon_1(\sigma), \varepsilon_2(\sigma), \delta_1(\sigma), \delta_2(\sigma))$, from which the results of theorem are obtained.

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